Characterization of Almost Interior Ideals and Their Fuzzifications in Ternary Semigroups

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Abstract—**The ternary algebraic structure was given by Lehmer in 1932. Grosek and Stako studied almost ideals in semigroups in 1980. Later, in 2019, S. Suebsung et al. introduced almost ideals in ternary semigroups. This paper aims to define almost interior ideals in ternary semigroups and fuzzy almost interior ideals in ternary semigroups. We discussed the union of almost interior ideals, including almost interior ideals in ternary semigroups. In class, fuzzifications are the same. Finally, we connect almost interior ideals and fuzzy almost interior ideals in ternary semigroups.**

Index Terms—**almost interior ideals, fuzzy almost interior ideals, almost weakly interior ideals, fuzzy almost weakly interior ideals**

I. Introduction

LEHMER gave the concept of a ternary semigroup in 1932 [1], but Kanser studied earlier such structures in 1904 [3]. In 2010, Santiago and Sri Bala [3] discussed regularity conditions in a ternary semigroup. The classical of fuzzy sets was proposed in 1965 by Zadeh [4]. These concepts were applied in many areas, such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, and topology. Rosenfeld used the concepts of fuzzy sets to fuzzy subgroups and fuzzy ideals. Kuroki studied the fuzzy semigroups in 1981. Satko and Grosek defined the concept of an almost-ideal (A-ideal) in a semilattice in 1981 [5]. In 1981, S. Bogdanovic [6] gave the concept of almost bi-ideals in semigroups. In 2019, S. Suebsung et al. [7], [8] investigated almost ideals and fuzzy almost ideals in ternary semigroups. In 2020, Chinram et al. [9] discussed almost interior ideals and weakly almost interior ideals in semigroups and studied the relationship between almost interior ideals and weakly almost interior ideals in semigroups. The research of almost ideals studied in semihypergroups such that in 2021, P. Muangdoo et al. [10] studied almost bi-hyper ideals and their fuzzification of semihypergroups. W. Nakkhasen et al. [11] discussed fuzzy, almost interior ideal hyper ideals of semihypergroups. In 2022, T. Gaketem and P. Khamrot [12] explored the concept of almost ideals within the framework of bipolar fuzzy sets, specifically focusing on bipolar fuzzy almost bi-ideals in semigroups. In 2023, T. Gaketem and P. Khamrot [13] studied bipolar fuzzy almost interior ideals in semigroups. In 2024, T. Gaketem and P. Khamrot [14] discussed bipolar fuzzy almost ideals in semigroups. Recently, P. Khamrot et al. [15] presented almost n-interior ideals and their fuzzifications in semigroups. In addition, almost ideal's work also has many studies, such as almost ideals in ordered semigroup [16], almost ternary semigroup [17], [18], almost ideals in semirings [19], almost ideals in ternary semiring [20], etc.

This paper defines almost interior ideals in ternary semigroups and fuzzy almost interior ideals in ternary semigroups. We discussed the union of almost interior ideals, including almost interior ideals in ternary semigroups. In class, fuzzifications are the same. Moreover, we connect almost interior ideals and fuzzy almost interior ideals in ternary semigroups.

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II. Preliminaries

Now, we discussed the concept of ternary semigroups, fuzzy set, types of fuzzy ideal in ternary semigroups, and basic properties for the study of the next sections.

A non-empty set $\mathcal{T}$ together with a ternary operation defined on $\mathcal{T}$ is said to be a *ternary semigroups* (TSG) if it satisfies the associative law, that is; $(\mathfrak{v}\mathfrak{w}(\mathfrak{x}\mathfrak{y}\mathfrak{z})) = (\mathfrak{v}(\mathfrak{w}\mathfrak{x}\mathfrak{y})\mathfrak{z}) = \mathfrak{v}\mathfrak{w}(\mathfrak{x}\mathfrak{y}\mathfrak{z})$ for all $\mathfrak{v}, \mathfrak{w}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{T}$ [8].

For non-empty subsets V, W, X of TSG $\mathcal{T}$, defined $\mathcal{V}\mathcal{W}\mathcal{X} = \{\mathfrak{v}\mathfrak{w}\mathfrak{x} \mid \mathfrak{v} \in \mathcal{V}, \mathfrak{w} \in \mathcal{W}, \mathfrak{x} \in \mathcal{X}\}$.

Example 2.1. *Let $2\mathbb{Z}^-$ be a set of all even numbers. Then the usual ternary multiplication of negative numbers. Thus, $2\mathbb{Z}^-$ is a TSG.*

Definition 2.2. *[8] A nonempty subset Θ of a TSG $\mathcal{T}$ is called*

(1) *A ternary subsemigroup (TSSG) Θ of $\mathcal{T}$ if $\Theta\Theta\Theta \subseteq \Theta$.*
(2) *A left ideal (LD) Θ of $\mathcal{T}$ if $\mathcal{T}\mathcal{T}\Theta \subseteq \Theta$.*
(3) *A middle ideal (MD) Θ of $\mathcal{T}$ if $\mathcal{T}\Theta\mathcal{T} \subseteq \Theta$.*
(4) *A right ideal (RD) Θ of $\mathcal{T}$ if $\Theta\mathcal{T}\mathcal{T} \subseteq \Theta$.*
(5) *An ideal (ID) Θ of $\mathcal{T}$ if it is a LD, an MD, and a RD of $\mathcal{T}$.*
(6) *An interior ideal (IID) Θ of $\mathcal{T}$ if Θ is a TSSG and $\mathcal{T}\mathcal{T}\Theta\mathcal{T}\mathcal{T} \subseteq \Theta$.*
(7) *An weakly interior ideal (WID) Θ of $\mathcal{T}$ if $\mathcal{T}\mathcal{T}\Theta\mathcal{T}\mathcal{T} \subseteq \Theta$.*
(8) *A left almost ideal (LAD) Θ of $\mathcal{T}$ if $\mathfrak{t}\mathfrak{t}\Theta \cap \Theta \neq \emptyset$, for all $\mathfrak{t} \in \mathcal{T}$.*
(9) *A middle almost ideal (MAD) Θ of $\mathcal{T}$ if $\mathfrak{t}\Theta\mathfrak{t} \cap \Theta \neq \emptyset$, for all $\mathfrak{t} \in \mathcal{T}$.*
(10) *A right almost ideal (RAD) Θ of $\mathcal{T}$ if $\Theta\mathfrak{t}\mathfrak{t} \cap \Theta \neq \emptyset$, for all $\mathfrak{t} \in \mathcal{T}$.*
(11) *An almost ideal (AID) Θ of $\mathcal{T}$ if it is an LAD, an MAD and a RAD of $\mathcal{T}$.*

For any $\mathfrak{h}_i \in [0,1]$, $i \in \mathcal{F}$, define

$$\underset{i\in\mathcal{F}}{\vee}\mathfrak{h}_i := \sup_{i\in\mathcal{F}}\{\mathfrak{h}_i\} \quad \text{and} \quad \underset{i\in\mathcal{F}}{\wedge}\mathfrak{h}_i := \inf_{i\in\mathcal{F}}\{\mathfrak{h}_i\}.$$

We see that for any $\mathfrak{h}, \mathfrak{r} \in [0,1]$, we have

$$\mathfrak{h} \vee \mathfrak{r} = \max\{\mathfrak{h}, \mathfrak{r}\} \quad \text{and} \quad \mathfrak{h} \wedge \mathfrak{r} = \min\{\mathfrak{h}, \mathfrak{r}\}.$$

A *fuzzy set* (FS) Υ in a nonempty set $\mathcal{T}$ is a function from $\mathcal{T}$ into the unit closed interval $[0,1]$ of real numbers, i.e., $\Upsilon : \mathcal{T} \to [0,1]$.

For any two FSs Υ_1 and Υ_2 of a nonempty set $\mathcal{T}$, define the symbol as follows:

(1) $\Upsilon_1 \leq \Upsilon_2 \Leftrightarrow \Upsilon_1(\mathfrak{h}) \leq \Upsilon_2(\mathfrak{h})$ for all $\mathfrak{h} \in \mathcal{T}$,
(2) $\Upsilon_1 = \Upsilon_2 \Leftrightarrow \Upsilon_1 \leq \Upsilon_2$ and $\Upsilon_2 \leq \Upsilon_1$,
(3) $(\Upsilon_1 \wedge \Upsilon_2)(h) = \min\{\Upsilon_1(\mathfrak{h}), \Upsilon_2(\mathfrak{h})\} = \Upsilon_1(\mathfrak{h}) \wedge \Upsilon_2(\mathfrak{h})$ for all $\mathfrak{h} \in \mathcal{T}$,
(4) $(\Upsilon_1 \vee \Upsilon_2)(\mathfrak{h}) = \max\{\Upsilon_1(\mathfrak{h}), \Upsilon_2(\mathfrak{h})\} = \Upsilon_1(\mathfrak{h}) \vee \Upsilon_2(\mathfrak{h})$ for all $\mathfrak{h} \in \mathcal{T}$,
(5) the *support* of Υ instead of $\text{supp}(\Upsilon_1) = \{\mathfrak{h} \in \mathcal{T} \mid \Upsilon_1(\mathfrak{h}) \neq 0\}$.

For the symbol $\Upsilon_1 \geq \Upsilon_2$, we mean $\Upsilon_2 \leq \Upsilon_1$.

If $\emptyset \neq \Theta \subseteq \mathcal{T}$, then the characteristic function Λ_Θ of $\mathcal{T}$ is a function from $\mathcal{T}$ into $\{0,1\}$ defined as follows:

$$\Lambda_\Theta(\mathfrak{h}) = \begin{cases} 1 & \text{if } \mathfrak{h} \in \Theta \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.3. *Let $\mathcal{T}$ be a TSG and $\Xi_\mathfrak{u}$ be a non-empty subset of $\mathcal{T}$, we define the set $\Xi_\mathfrak{u}$ by*

$$\Xi_\mathfrak{u} := \{(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \in \mathcal{T} \times \mathcal{T} \times \mathcal{T} \mid \mathfrak{u} = \mathfrak{x}\mathfrak{y}\mathfrak{z}\}.$$

Definition 2.4. *[8] Let Υ_1, Υ_2 and Υ_3 be FSs of a TSG $\mathcal{T}$. The product of FSs Υ, Υ_2 and η of $\mathcal{T}$ is defined as follow, for all $\mathfrak{u} \in T$*

$$(\Upsilon_1 \circ \Upsilon_2 \circ \Upsilon_3)(u) = \begin{cases} \bigvee\limits_{(\mathfrak{x},\mathfrak{y},\mathfrak{z})\in\Xi_\mathfrak{u}} \{\Upsilon_1(\mathfrak{x}) \wedge \Upsilon_2(\mathfrak{y}) \wedge \Upsilon_3(\mathfrak{z})\} & \text{if } \Xi_\mathfrak{u} \neq \emptyset, \\ 0 & \text{if } \Xi_\mathfrak{u} = \emptyset. \end{cases}$$

Lemma 2.5. *[2] Let Θ_1, Θ_2 and Θ_3 be non-empty subsets of a TSG $\mathcal{T}$. Then the following holds.*

1) $\Lambda_{\Theta_1} \wedge \Lambda_{\Theta_2} \wedge \Lambda_{\Theta_3} = \Lambda_{\Theta_1 \cap \Theta_2 \cap \Theta_3}$.
2) $\Lambda_{\Theta_1} \circ \Lambda_{\Theta_2} \circ \Lambda_{\Theta_3} = \Lambda_{\Theta_1\Theta_2\Theta_3}$.

For $\mathfrak{u} \in \mathcal{T}$ and $\mathfrak{t} \in (0,1]$, a fuzzy point (FP) $\hbar_\mathfrak{t}$ of a set $\mathcal{T}$ is a FS of $\mathcal{T}$ defined by

$$\hbar_\mathfrak{t}(\mathfrak{e}) = \begin{cases} \mathfrak{t} & \text{if } \mathfrak{e} = \mathfrak{u}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.6. *A FS Ψ of a TSG $\mathcal{T}$ is called*

(1) *a fuzzy subsemigroup (FSSG) of $\mathcal{T}$ if $\Psi(\mathfrak{a}\mathfrak{b}\mathfrak{c}) \leq \Psi(\mathfrak{a}) \wedge \Psi(\mathfrak{b}) \wedge \Psi(\mathfrak{c})$ for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{T}$,*
(2) *a fuzzy left ideal (FLID) of $\mathcal{T}$ if $\Psi(\mathfrak{a}\mathfrak{b}\mathfrak{c}) \leq \Psi(\mathfrak{c})$ for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{T}$,*
(3) *a fuzzy middle ideal (FMID) of $\mathcal{T}$ if $\Psi(\mathfrak{a}\mathfrak{b}\mathfrak{c}) \leq \Psi(\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{T}$,*
(4) *a fuzzy right ideal (FRID) of $\mathcal{T}$ if $\Psi(\mathfrak{a}\mathfrak{b}\mathfrak{c}) \leq \Psi(\mathfrak{a})$ for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{T}$,*
(5) *a fuzzy ideal (FID) of $\mathcal{T}$ if it is both a FLID, FMID and FRID of $\mathcal{T}$,*
(6) *a fuzzy interior ideal (FIID) of $\mathcal{T}$ if it is a FSSG and $\Psi(\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}\mathfrak{e}) \leq \Psi(\mathfrak{c})$ for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e} \in \mathcal{T}$.*
(7) *a fuzzy weakly interior ideal (FWID) of $\mathcal{T}$ if $\Psi(\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}\mathfrak{e}) \leq \Psi(\mathfrak{c})$ for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e} \in \mathcal{T}$.*
(8) *a* fuzzy left almost ideal *(FLAID) of $\mathcal{T}$ if $(\hbar_\mathfrak{t} \circ \hbar_\mathfrak{t} \circ \Psi) \wedge \Psi \neq 0$ for all FP $\hbar_\mathfrak{t} \in \mathcal{T}$.*
(9) *a* fuzzy middle almost ideal *(FMAID) of Θ if $(\hbar_\mathfrak{t} \circ \Psi \circ \hbar_\mathfrak{t}) \wedge \Psi \neq 0$ for all FP $\hbar_\mathfrak{t} \in \mathcal{T}$.*
(10) *a* fuzzy right almost ideal *(FRAID) of Θ if $(\Psi \circ \hbar_\mathfrak{t} \circ \hbar_\mathfrak{t}) \wedge \Psi \neq 0$ for all FP $\hbar_\mathfrak{t} \in \mathcal{T}$.*
(11) *a* fuzzy almost ideal *(FAID) of Θ if it is both a FLAID, FMAID and FRAID of $\mathcal{T}$.*

III. Almost interior ideal and fuzzy almost interior ideal

In this section, we define the almost interior ideal and fuzzy almost interior ideal in TSG. We study basic some interesting properties of almost interior ideal and fuzzy almost interior ideal in the TSG.

Definition 3.1. *A non-empty subset Θ on a TSG $\mathcal{T}$ is called a* almost interior ideal *(AIID) of $\mathcal{T}$ if $\mathfrak{t}_1\mathfrak{t}_2\Theta\mathfrak{t}_1\mathfrak{t}_2 \cap \Theta \neq \emptyset$ for all $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}$.*

Example 3.2. *Let $\mathbb{Z}_6$ be a TSG under the addition on $\mathbb{Z}_6$ and let $\mathcal{K} = \{\bar{1}, \bar{3}, \bar{4}\}$. Then $\mathcal{K}$ is an almost interior ideal of $\mathbb{Z}_6$.*
If $t = \bar{0}$, then $(\bar{0} + \bar{0} + \mathcal{K} + \bar{0} + \bar{0}) \cap \mathcal{K} = \mathcal{K}$.
If $t = \bar{1}$, then $(\bar{1} + \bar{1} + \mathcal{K} + \bar{1} + \bar{1}) \cap \mathcal{K} = \{\bar{1}, \bar{3}\}$.
If $t = \bar{2}$, then $(\bar{2} + \bar{2} + \mathcal{K} + \bar{2} + \bar{2}) \cap \mathcal{K} = \{\bar{1}, \bar{3}\}$.
If $t = \bar{3}$, then $(\bar{3} + \bar{3} + \mathcal{K} + \bar{1} + \bar{3}) \cap \mathcal{K} = \{\bar{3}, \bar{3}\}$.
If $t = \bar{4}$, then $(\bar{4} + \bar{4} + \mathcal{K} + \bar{4} + \bar{4}) \cap \mathcal{K} = \{\bar{1}, \bar{3}\}$.
If $t = \bar{5}$, then $(\bar{5} + \bar{5} + \mathcal{K} + \bar{5} + \bar{5}) \cap \mathcal{K} = \{\bar{1}, \bar{3}\}$.
But it is not an interior ideal of $\mathbb{Z}_6$ because $\bar{5}+\bar{5}+\bar{4}+\bar{5}+\bar{5} = \bar{0} \nsubseteq \mathcal{K}$.

Theorem 3.3. *Every IID of a TSG $\mathcal{T}$ is an AIID of $\mathcal{T}$.*

Proof: Assume that Θ is an IID of a TSG $\mathcal{T}$ and let $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}$. Then $\mathfrak{t}_1\mathfrak{t}_2\Theta\mathfrak{t}_1\mathfrak{t}_2 \subseteq \mathcal{T}\mathcal{T}\Theta\mathcal{T}\mathcal{T}\Theta$. Thus, $\mathfrak{t}_1\mathfrak{t}_2\Theta\mathfrak{t}_1\mathfrak{t}_2 \cap \Theta \neq \emptyset$. We conclude that Θ is an AIID of $\mathcal{T}$. ■

Theorem 3.4. *Let Θ and Ω be two non-empty subsets of a TSG $\mathcal{T}$ such that $\Theta \subseteq \Omega$. If Θ is an AIID of $\mathcal{T}$, then Ω is also an AIID of $\mathcal{T}$.*

Proof: Let Ω be a subset of $\mathcal{T}$ with it containing Θ and let $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}$. Then $\mathfrak{t}_1\mathfrak{t}_2\Theta\mathfrak{t}_1\mathfrak{t}_2 \subseteq \mathfrak{t}_1\mathfrak{t}_2\Omega\mathfrak{t}_1\mathfrak{t}_2$ Thus, $\mathfrak{t}_1\mathfrak{t}_2\Omega\mathfrak{t}_1\mathfrak{t}_2 \cap \Omega \neq \emptyset$. Hence, Ω is an AIID of $\mathcal{T}$. ■

The following result is an obvious of Theorem 3.4.

Corollary 3.5. *Let Θ_1 and Θ_2 be AIIDs of a TSG $\mathcal{T}$. Thus $\Theta_1 \cup \Theta_2$ is also an AIID of $\mathcal{T}$.*

Proof: Since $\Theta_1 \subseteq \Theta_1 \cup \Theta_2$, by Theorem 3.4, $\Theta_1 \cup \Theta_2$ is an AIID of $\mathcal{T}$. ■

Theorem 3.6. *Let Θ_1 and Θ_2 be nonempty subsets of a TSG $\mathcal{T}$. If Θ_1 is an AIID of $\mathcal{T}$, then $\Theta_1 \cup \Theta_2$ is an AIID of $\mathcal{T}$.*

Proof: By Theorem 3.4, and $\Theta_1 \subseteq \Theta_1 \cup \Theta_2$. Thus, $\Theta_1 \cup \Theta_2$ is an AIID of $\mathcal{T}$. ■

Corollary 3.7. *The finite union of AIIDs of a TSG $\mathcal{T}$ is an AIID of $\mathcal{T}$.*

Example 3.8. *Let $\mathbb{Z}_6$ be a TSG under the addition on $\mathbb{Z}_6$ and let $\mathcal{K}_1 = \{\bar{1}, \bar{3}, \bar{4}\}$ and $\mathcal{K}_2 = \{\bar{0}, \bar{3}, \bar{5}\}$. By Theorem 3.2, $\mathcal{K}_1$ and $\mathcal{K}_2$ are AIIDs of $\mathbb{Z}_6$. But $\mathcal{K}_1 \cap \mathcal{K}_2 = \{3\}$ is not AIID of $\mathcal{Z}_6$.*

Definition 3.9. *A FS Υ on a TSG $\mathcal{T}$ is called a* fuzzy almost interior ideal *(FAIID) of $\mathcal{T}$ if $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon \neq 0$. for any FP $\hbar_{\mathfrak{t}}, \flat_{\mathfrak{t}}, \hbar_{\mathfrak{t}'}, \flat_{\mathfrak{t}'} \in \mathcal{T}$.*

Theorem 3.10. *If Υ_1 is a FAIID of a TSG $\mathcal{T}$ and Υ_2 is a FS of $\mathcal{T}$ such that $\Upsilon_1 \leq \Upsilon_2$, then Υ_2 is a FAIID of $\mathcal{T}$.*

Proof: Suppose that Υ_1 is a FAIID of a TSG $\mathcal{T}$ and Υ_2 is a FS of $\mathcal{T}$ such that $\Upsilon_1 \leq \Upsilon_2$. Then for any FPs $\hbar_{\mathfrak{t}}, \flat_{\mathfrak{t}}, \hbar_{\mathfrak{t}'}, \flat_{\mathfrak{t}'} \in \mathcal{T}$, we obtain that $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon_1 \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon_1 \neq 0$. Thus,

$$(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon_1 \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon_1 \leq (\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon_2 \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon_2 \neq 0.$$

Hence $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon_2 \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon_2 \neq 0$. Therefore, Υ_2 is a FAIID of $\mathcal{T}$. ∎

The following result is an obvious of Theorem 3.10.

Theorem 3.11. *Let Υ_1 and Υ_2 be FAIIDs of a TSG $\mathcal{T}$. Then $\Upsilon_1 \vee \Upsilon_2$ is also a FAIID of $\mathcal{T}$.*

Proof: Since $\Upsilon_1 \leq \Upsilon_1 \vee \Upsilon_2$, by Theorem 3.10, $\Upsilon_1 \vee \Upsilon_2$ is a FAIID of $\mathcal{T}$. ∎

Theorem 3.12. *If Υ_1 is a FAIID of a TSG $\mathcal{T}$ and Υ_2 is a FS, then $\Upsilon_1 \vee \Upsilon_2$ is a FAIID of $\mathcal{T}$.*

Proof: By Theorem 3.10, and $\Upsilon_1 \leq \Upsilon_1 \vee \Upsilon_2$. Thus, $\Upsilon_1 \vee \Upsilon_2$ is a FAIID of $\mathcal{T}$. ∎

Corollary 3.13. *Let $\mathcal{T}$ be a TSG. Then the finite maximum of FAIIDs of $\mathcal{T}$ is a FAIID of $\mathcal{T}$.*

Example 3.14. *Define $\Upsilon : \mathbb{Z}_6 \to [0,1]$ by $\Upsilon(\bar{0}) = 0$ $\Upsilon(\bar{1}) = 0.5$ $\Upsilon(\bar{2}) = 0$ $\Upsilon(\bar{3}) = 0$ $\Upsilon(\bar{4}) = 0.4$ $\Upsilon(\bar{5}) = 0.3$, and $\psi : \mathbb{Z}_6 \to [0,1]$ by $\psi(\bar{0}) = 0$ $\psi(\bar{1}) = 0.6$ $\psi(\bar{2}) = 0.1$ $\psi(\bar{3}) = 0$ $\psi(\bar{4}) = 0$ $\psi(\bar{5}) = 0.3$. Then Υ and ψ are FAIIDs of $\mathbb{Z}_6$. Thus, $\Upsilon \vee \psi$ is a FAIID of $\mathbb{Z}_6$. But $\Upsilon \wedge \psi$ is not a FAIID of $\mathbb{Z}_6$.*

Theorem 3.15. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then Θ is an AIID of $\mathcal{T}$ if and only if Λ_Θ is a FAIID of $\mathcal{T}$.*

Proof: Suppose that Θ is an AIID of $\mathcal{T}$. Then $\mathfrak{x}\mathfrak{y}\Theta\mathfrak{x}\mathfrak{y} \cap \Theta \neq \emptyset$ for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{T}$. Thus, there exists $\mathfrak{c} \in \mathcal{T}$ such that $\mathfrak{c} \in \mathfrak{x}\mathfrak{y}\Theta\mathfrak{x}\mathfrak{y}$ and $\mathfrak{c} \in \Theta$. Let $\hbar_{\mathfrak{t}}, \flat_{\mathfrak{t}}, \hbar_{\mathfrak{t}'}, \flat_{\mathfrak{t}'} \in \mathcal{T}$ and $\mathfrak{t}, \mathfrak{t}' \in (0,1]$. Then $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_\Theta \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'})(\mathfrak{c}) \neq 0$ and $\Lambda_\Theta(\mathfrak{c}) \neq 0$. Thus, $((\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_\Theta \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Lambda_\Theta)(\mathfrak{c}) \neq 0$. So $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_\Theta \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Lambda_\Theta \neq 0$. Hence, Λ_Θ is a FAIID of $\mathcal{T}$.

Conversely, suppose that Λ_Θ is a FAIID of $\mathcal{T}$ and let $\hbar_{\mathfrak{t}}, \flat_{\mathfrak{t}}, \hbar_{\mathfrak{t}'}, \flat_{\mathfrak{t}'} \in \mathcal{T}$ and $\mathfrak{t}, \mathfrak{t}' \in (0,1]$. Then $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_\Theta \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Lambda_\Theta \neq 0$.. Thus, there exists $\mathfrak{c} \in \Theta$ such that $((\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_\Theta \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Lambda_\Theta)(\mathfrak{c}) \neq 0$. It implies that $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_\Theta \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'})(\mathfrak{c}) \neq 0$ and $\Lambda_\Theta(\mathfrak{c}) \neq 0$.Hence, $\mathfrak{c} \in \mathfrak{x}\mathfrak{y}\Theta\mathfrak{x}\mathfrak{y}$ and $\mathfrak{c} \in \Theta$. So $\mathfrak{x}\mathfrak{y}\Theta\mathfrak{x}\mathfrak{y} \cap \Theta \neq \emptyset$ for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{T}$. We conclude that Θ is an AIID of $\mathcal{T}$. ∎

Theorem 3.16. *Let Υ be a fuzzy subset of a TSG $\mathcal{T}$. Then Υ is a FAIID of $\mathcal{T}$ if and only if* $\mathrm{supp}(\Upsilon)$ *is an AIID of $\mathcal{T}$.*

Proof: Assume that Υ is a FAIID of a TSG $\mathcal{T}$ and let $\hbar_{\mathfrak{t}}, \flat_{\mathfrak{t}}, \hbar_{\mathfrak{t}'}, \flat_{\mathfrak{t}'} \in \mathcal{T}$ and $\mathfrak{t}, \mathfrak{t}' \in (0,1]$. Then $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon \neq 0$. Thus, there exists $\mathfrak{z} \in \mathcal{T}$ such that $((\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon)(\mathfrak{k}) \neq 0$. So $((\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'})(\mathfrak{k}) \neq 0$ and $\Upsilon(\mathfrak{k}) \neq 0$ Thus, there exists $\mathfrak{k} \in \mathcal{T}$ such that such that $\mathfrak{k} = \mathfrak{x}\mathfrak{b}\mathfrak{c}\mathfrak{y}$ and $\Upsilon(\mathfrak{k}) \neq 0$. So, $((\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_{\mathrm{supp}(\Upsilon)} \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Lambda_{\mathrm{supp}(\Upsilon)})(\mathfrak{k}) \neq 0$. Hence, $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_{\mathrm{supp}(\Upsilon)} \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Lambda_{\mathrm{supp}(\Upsilon)} \neq 0$. Therefore, $\Lambda_{\mathrm{supp}(\Upsilon)}$ is a FAIID of $\mathcal{T}$. By Theorem 3.15, $\mathrm{supp}(\Upsilon)$ is an AIID of T.

Conversely, suppose that $\mathrm{supp}(\Upsilon)$ is an AIID of $\mathcal{T}$. By Theorem 3.15, $\Lambda_{\mathrm{supp}(\Upsilon)}$ is a FAIID of $\mathcal{T}$. Then for any fuzzy point $\hbar_{\mathfrak{t}}, \flat_{\mathfrak{t}}, \hbar_{\mathfrak{t}'}, \flat_{\mathfrak{t}'} \in \mathcal{T}$ and $\mathfrak{t}, \mathfrak{t}' \in (0,1]$, we have $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_{\mathrm{supp}(\Upsilon)} \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Lambda_{\mathrm{supp}(\Upsilon)} \neq 0$.. Thus, there exists $\mathfrak{k} \in \mathcal{T}$ such that $((\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_{\mathrm{supp}(\Upsilon)} \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Lambda_{\mathrm{supp}(\Upsilon)})(\mathfrak{k}) \neq 0$. Hence, $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Lambda_{\mathrm{supp}(\Upsilon)} \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'})(\mathfrak{k}) = 0$, $\Lambda_{\mathrm{supp}(\Upsilon)}(\mathfrak{k}) \neq 0$. Then there exists $\mathfrak{k} \in \mathcal{T} \in \mathrm{supp}(\Upsilon)$ such that $\mathfrak{k} = \mathfrak{x}\mathfrak{b}\mathfrak{c}\mathfrak{y}$ Thus, $\Upsilon(\mathfrak{k}) \neq 0$. So $((\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon)(\mathfrak{k}) \neq 0$. Hence, $(\hbar_{\mathfrak{t}} \circ \flat_{\mathfrak{t}} \circ \Upsilon \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon \neq 0$. Therefore, Υ is a FAIID of $\mathcal{T}$. ∎

Next, we investigate relationships between minimal and maximal almost interior ideals and minimal and maximal fuzzy almost interior ideals of TSGs.

Definition 3.17. *An AIID Θ of a TSG $\mathcal{T}$ is called*

(1) *a* minimal almost interior ideal *(MiAIID) if for any AIID Ω of $\mathcal{T}$ if whenever $\Omega \subseteq \Theta$, then $\Omega = \Theta$,*
(2) *a* maximal almost interior ideal *(MaAIID) if for any AIID Ω of $\mathcal{T}$ if whenever $\Theta \subseteq \Omega$, then $\Omega = \Theta$.*

Definition 3.18. *A FAIID Υ of a TSG $\mathcal{T}$ is called*

(1) *a* minimal fuzzy almost interior ideal *(MiFAIID) if for any FAIID Υ_2 of $\mathcal{T}$ if whenever $\Upsilon_2 \leq \Upsilon_1$, then* $\mathrm{supp}(\Upsilon_2) = \mathrm{supp}(\Upsilon_1)$,
(2) *a* maximal fuzzy almost interior ideal *(MaFAIID) if for any FAIID Υ_2 of $\mathcal{T}$ if whenever $\Upsilon_1 \leq \Upsilon_2$, then* $\mathrm{supp}(\Upsilon_2) = \mathrm{supp}(\Upsilon_1)$.

Theorem 3.19. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then*

(1) *Θ is a MiAIID of $\mathcal{T}$ if and only if Λ_Θ is a MiFAIID of $\mathcal{T}$.*
(2) *Θ is a MaAIID of $\mathcal{T}$ if and only if Λ_Θ is a MaFAIID of $\mathcal{T}$.*

Proof:

(1) Assume that Θ is a MiAIID of $\mathcal{T}$. Then Θ is an AIID of a TSG $\mathcal{T}$. Thus by Theorem 3.15, Λ_Θ is a FAIID of $\mathcal{T}$. Let Υ be a FAIID of $\mathcal{T}$ such that $\Upsilon \leq \Lambda_\Theta$. Then by Theorem 3.16, $\mathrm{supp}(\Upsilon)$ is an AIID of $\mathcal{T}$ such that $\mathrm{supp}(\Upsilon) \subseteq \mathrm{supp}(\Lambda_\Theta) = \Theta$. Thus, $\mathrm{supp}(\Upsilon) \subseteq \Theta$. Since Θ is a MiAIID of $\mathcal{T}$ we have $\mathrm{supp}(\Upsilon) = \Theta = \mathrm{supp}(\Lambda_\Theta)$. Therefore, Λ_Θ is a MiFAIID of $\mathcal{T}$.
Conversely, suppose that Λ_Θ is a MiFAIID of $\mathcal{T}$. Then Λ_Θ is a FAIID of $\mathcal{T}$. Thus by Theorem 3.15, Θ is an AIID of a TSG $\mathcal{T}$. Let Ω be an AIID of $\mathcal{T}$ such that $\Omega \subseteq \Theta$. Then Λ_Ω is a FAIID of $\mathcal{T}$ such that $\Lambda_\Omega \leq \Lambda_\Theta$. Thus, $\mathrm{supp}(\Lambda_\Omega) \subseteq \mathrm{supp}(\Lambda_\Theta)$. So, $\Omega = \mathrm{supp}(\Lambda_\Omega) \subseteq \mathrm{supp}(\Lambda_\Theta) = \Theta$ implies that $\Omega \subseteq \Theta$. Since Λ_Θ is a MiFAIID of $\mathcal{T}$ we have $\mathrm{supp}(\Lambda_\Omega) = \mathrm{supp}(\Lambda_\Theta)$. Thus, $\Omega = \mathrm{supp}(\chi_\Omega) = \mathrm{supp}(\Lambda_\Theta) = \Theta$. Hence, Θ is a MiAIID of $\mathcal{T}$.
(2) Assume that Θ is a MaAIID of $\mathcal{T}$. Then Θ is an AIID of a TSG $\mathcal{T}$. Thus by Theorem 3.15, Λ_Θ is a FAIID of $\mathcal{T}$. Let Υ be a FAIID of $\mathcal{T}$ such that $\Lambda_\Theta \leq \Upsilon$. Then by Theorem 3.16, $\mathrm{supp}(\Upsilon)$ is an AIID of $\mathcal{T}$ such that $\Theta =$

$\mathrm{supp}(\Lambda_\Theta) \subseteq \mathrm{supp}(\Upsilon)$. Since Θ is a MaAIID of $\mathcal{T}$ we have $\mathrm{supp}(\Upsilon) = \mathrm{supp}(\Lambda_\Theta)$. Hence, Λ_Θ is a MaFAIID of $\mathcal{T}$.

Conversely, suppose that Λ_Θ is a MaFAIID of $\mathcal{T}$. Then Λ_Θ is a FAIID of $\mathcal{T}$. Thus by Theorem 3.15, Θ is an AIID of a TSG $\mathcal{T}$. Let Ω be an AIID of $\mathcal{T}$ such that $\Theta \subseteq \Omega$. Then Λ_Ω is a FAIID of $\mathcal{T}$ such that $\Lambda_\Theta \leq \Lambda_\Omega$. Thus, $\mathrm{supp}(\Lambda_\Theta) \subseteq \mathrm{supp}(\Lambda_\Omega)$. So, $\Theta = \mathrm{supp}(\Lambda_\Theta) \subseteq \mathrm{supp}(\Lambda_\Omega) = \Omega$. It implies that $\mathrm{supp}(\Lambda_\Theta) \subseteq \mathrm{supp}(\Lambda_\Omega)$. Since Λ_Θ is a MAFAIID of $\mathcal{T}$ we have $\mathrm{supp}(\Lambda_\Omega) = \mathrm{supp}(\Lambda_\Theta)$. Thus, $\Omega = \mathrm{supp}(\chi_\Omega) = \mathrm{supp}(\Lambda_\Theta) = \Theta$. Hence, Θ is a MaAIID of $\mathcal{T}$.

■

Corollary 3.20. *Let $\mathcal{T}$ be a TSG. Then $\mathcal{T}$ has no proper AIID if and only if $\mathrm{supp}(\Upsilon) = \mathcal{T}$ for every FAIID Υ of $\mathcal{T}$.*

Next, we give definitions of prime (resp., semiprime, strongly prime) AIIDs, and prime (resp., semiprime strongly prime) FAIIDs. We study the relationships between prime (resp., semiprime strongly prime) AIIDs and their fuzzification of TSGs.

Definition 3.21. *Let Θ be an AIID of a TSG $\mathcal{T}$. Then we called*

(1) *Θ is a* ***prime almost interior ideal*** *(PAIID) if for any three AIIDs Ω_1, Ω_2 and Ω_3 of $\mathcal{T}$ such that $\Omega_1\Omega_2\Omega_3 \subseteq \Theta$ implies that $\Omega_1 \subseteq \Theta$ or $\Omega_2 \subseteq \Theta$ or $\Omega_3 \subseteq \Theta$.*
(2) *Θ is a* ***semiprime almost interior ideal*** *(SPAIID) if for any AIID Ω of $\mathcal{T}$ such that $\Omega^3 \subseteq \Theta$ implies that $\Omega \subseteq \Theta$.*
(3) *Θ is a* ***strongly prime almost interior ideal*** *(StPAIID) if for any AIIDs Ω_1, Ω_2 and Ω_3 of $\mathcal{T}$ such that $\Omega_1\Omega_2\Omega_3 \cap \Omega_3\Omega_2\Omega_1 \subseteq \Theta$ implies that $\Omega_1 \subseteq \Theta$ or $\Omega_2 \subseteq \Theta$ or $\Omega_3 \subseteq \Theta$.*

Definition 3.22. *A FAIID Υ on a TSG $\mathcal{T}$. Then we called*

(1) *Υ is a* ***prime fuzzy almost interior ideal*** *(PFAIID) if for any three FAIIDs Ψ_1, Ψ_2 and Ψ_3 of $\mathcal{T}$ such that $\Psi_1 \circ \Psi_2 \circ \Psi_3 \leq \Upsilon$ implies that $\Psi_1 \leq \Upsilon$ or $\Psi_2 \leq \Upsilon$ or $\Psi_3 \leq \Upsilon$.*
(2) *Υ is a* ***semiprime fuzzy almost interior ideal*** *(SPFAIID) if for any FAIID Ψ of $\mathcal{T}$ such that $\Psi \circ \Psi \circ \Psi \leq \Upsilon$ implies that $\Psi \leq \Upsilon$.*
(3) *Υ is a* ***strongly prime fuzzy almost interior ideal*** *(StPFAIID) if for any three FAIIDs Ψ_1, Ψ_2 and Ψ_3 of $\mathcal{T}$ such that $(\Psi_1 \circ \Psi_2 \circ \Psi_3) \wedge (\Psi_3 \circ \Psi_2 \circ \Psi_1) \leq \Upsilon$ implies that $\Psi_1 \leq \Upsilon$ or $\Psi_2 \leq \Upsilon$ or $\Psi_3 \leq \Upsilon$.*

It is clear, every StPFAIID of a TSG is a PFAIID and every PFAIIDs of a TSG is a SPFAIIDs.

Theorem 3.23. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then Θ is a PAIID of $\mathcal{T}$ if and only if Λ_Θ is a PFAIID of $\mathcal{T}$.*

Proof: Suppose that Θ is a PAIID of $\mathcal{T}$. Then Θ is an AIID of $\mathcal{T}$. Thus by Theorem 3.15, Λ_Θ is a FAIID of $\mathcal{T}$. Let Ψ_1, Ψ_2 and Ψ_3 be FAIIDs of $\mathcal{T}$. such that $\Psi_1 \circ \Psi_2 \circ \Psi_3 \leq \Lambda_\Theta$. Assume that $\Psi_1 \nleq \Lambda_\Theta$ and $\Psi_2 \nleq \Lambda_\Theta$ and $\Psi_3 \nleq \Lambda_\Theta$. Then there exist $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \mathfrak{T}$ such that $\Psi_1(\mathfrak{h}) \neq 0\, \Psi_2(\mathfrak{r}) \neq 0$ and $\Psi_3(\mathfrak{d}) \neq 0$. While $\Lambda_\Theta(\mathfrak{h}) = 0$, $\Lambda_\Theta(\mathfrak{r}) = 0$ and $\Lambda_\Theta(\mathfrak{r}) = 0$. Thus, $\mathfrak{h} \in \mathrm{supp}(\Psi_1)$, $\mathfrak{r} \in \mathrm{supp}(\Psi_2)$ and $\mathfrak{d} \in \mathrm{supp}(\Psi_3)$, but $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \notin \Theta$. So $\mathrm{supp}(\Psi_1) \nsubseteq \Theta\, \mathrm{supp}(\Psi_2) \nsubseteq \Theta$ and $\mathrm{supp}(\Psi_3) \nsubseteq \Theta$. Since Ψ_1, Ψ_2 and Ψ_3 is FAIIDs of $\mathcal{T}$ we have $\mathrm{supp}(\Psi_1)\, \mathrm{supp}(\Psi_2)$ and $\mathrm{supp}(\Psi_3)$ are AIIDs of $\mathcal{T}$. Thus, $\mathrm{supp}(\Psi_1)\,\mathrm{supp}(\Psi_2)\,\mathrm{supp}(\Psi_3) \nsubseteq \Theta$. So, there exists $\mathfrak{m} = \mathfrak{p}\mathfrak{q}\mathfrak{b}$ for some $\mathfrak{p} \in \mathrm{supp}(\Psi_1)$, $\mathfrak{q} \in \mathrm{supp}(\Psi_2)$ and $\mathfrak{b} \in \mathrm{supp}(\Psi_3)$ such that $\mathfrak{m} \notin \Theta$. Hence, $\Lambda_\Theta(\mathfrak{m}) = 0$ implies that $(\Psi_1 \circ \Psi_2 \circ \Psi_3)(\mathfrak{m}) = 0$. Since $\Psi_1 \circ \Psi_2 \circ \Psi_3 \leq \Lambda_\Theta$ such that $\mathfrak{p} \in \mathrm{supp}(\Psi_1)$, $\mathfrak{q} \in \mathrm{supp}(\Psi_2)$ and $\mathfrak{b} \in \mathrm{supp}(\Psi_3)$ we have, $\Psi_1(\mathfrak{p}) \neq 0$, $\Psi_2(\mathfrak{q}) \neq 0$, and $\Psi_3(\mathfrak{b}) \neq 0$. Thus,

$$(\Psi_1 \circ \Psi_2 \circ \Psi_3)(\mathfrak{m}) = \bigvee_{(\mathfrak{p}\mathfrak{q}\mathfrak{b}) \in \Xi_\mathfrak{m}} \{\Psi_1(\mathfrak{p}) \wedge \Psi_2(\mathfrak{q}) \wedge \Psi_3(\mathfrak{b})\} \neq 0$$

It is a contradiction so $\Psi_1 \leq \Lambda_\Theta$ or $\Psi_2 \leq \Lambda_\Theta$ or $\Psi_3 \leq \Lambda_\Theta$. Therefore, Λ_Θ is a PFAIID of $\mathcal{T}$.

Conversely, suppose that Λ_Θ is a PFAIID of $\mathcal{T}$. Then Λ_Θ is a FAIID of $\mathcal{T}$. Thus by Theorem 3.15, Θ is an AIID of $\mathcal{T}$. Let Ω_1, Ω_2 and Ω_3 be AIIDs of $\mathcal{T}$ such that $\Omega_1\Omega_2\Omega_3 \subseteq \Theta$. Then $\Lambda_{\Omega_1}\, \Lambda_{\Omega_2}$ and Λ_{Ω_3} are FAIIDs of $\mathcal{T}$. By Lemma 2.5 $\Lambda_{\Omega_1} \circ \Lambda_{\Omega_2} \circ \Lambda_{\Omega_3} = \Lambda_{\Omega_1\Omega_2\Omega_3} \leq \Lambda_\Theta$. By assumption, $\Lambda_{\Omega_1} \leq \Lambda_\Theta$ or $\Lambda_{\Omega_2} \leq \Lambda_\Theta$ or $\Lambda_{\Omega_3} \leq \Lambda_\Theta$. Thus, $\Omega_1 \subseteq \Theta$ or $\Omega_2 \subseteq \Theta$ or $\Omega_3 \subseteq \Theta$. We conclude that Θ is a PAIID of $\mathcal{T}$. ■

Theorem 3.24. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then Θ is a SPAIID of $\mathcal{T}$ if and only if Λ_Θ is a SPFAIID of $\mathcal{T}$.*

Proof: Suppose that Θ is a SPAIID of $\mathcal{T}$. Then Θ is an AIID of $\mathcal{T}$. Thus by Theorem 3.15, Λ_Θ is a FAIID of $\mathcal{T}$. Let Ψ be a FAIID of $\mathcal{T}$. such that $\Psi^3 = \Psi \circ \Psi \circ \Psi \leq \Lambda_\Theta$. Assume that $\Psi \nleq \Lambda_\Theta$. Then there exist $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \mathfrak{T}$ such that $\Psi(\mathfrak{h}) \neq 0\, \Psi(\mathfrak{r}) \neq 0$ and $\Psi(\mathfrak{d}) \neq 0$. While $\Lambda_\Theta(\mathfrak{h}) = 0$, $\Lambda_\Theta(\mathfrak{r}) = 0$ and $\Lambda_\Theta(\mathfrak{r}) = 0$. Thus, $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \mathrm{supp}(\Psi)$, but $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \notin \Theta$. So $\mathrm{supp}(\Psi) \nsubseteq \Theta$. Since Ψ is a FAIID of $\mathcal{T}$ we have $\mathrm{supp}(\Psi)$ is an AIIDs of $\mathcal{T}$. Thus, $\mathrm{supp}(\Psi)\,\mathrm{supp}\,\Psi)\,\mathrm{supp}(\Psi) \nsubseteq \Theta$. So, there exists $\mathfrak{m} = \mathfrak{p}\mathfrak{q}\mathfrak{b}$ for some $\mathfrak{p}, \mathfrak{q}, \mathfrak{b} \in \mathrm{supp}(\Psi)$ such that $\mathfrak{m} \notin \Theta$. Hence, $\Lambda_\Theta(\mathfrak{m}) = 0$ implies that $(\Psi \circ \Psi \circ \Psi)(\mathfrak{m}) = 0$. Since $\Psi^3 = \Psi \circ \Psi \circ \Psi \leq \Lambda_\Theta$ such that $\mathfrak{p}, \mathfrak{q}, \mathfrak{b} \in \mathrm{supp}(\Psi)$ we have, $\Psi(\mathfrak{p}) \neq 0$, $\Psi(\mathfrak{q}) \neq 0$, and $\Psi(\mathfrak{b}) \neq 0$. Thus,

$$(\Psi \circ \Psi \circ \Psi)(\mathfrak{m}) = \bigvee_{(\mathfrak{p}\mathfrak{q}\mathfrak{b}) \in \Xi_\mathfrak{m}} \{\Psi(\mathfrak{p}) \wedge \Psi(\mathfrak{q}) \wedge \Psi(\mathfrak{b})\} \neq 0$$

It is a contradiction so $\Psi \leq \Lambda_\Theta$. Therefore, Λ_Θ is a SPFAIID of $\mathcal{T}$.

Conversely, suppose that Λ_Θ is a SPFAIID of $\mathcal{T}$. Then Λ_Θ is a FAIID of $\mathcal{T}$. Thus by Theorem 3.15, Θ is an AIID of $\mathcal{T}$. Let Ω be an AIID of $\mathcal{T}$ such that $\Omega^3 \subseteq \Theta$. Then Λ_Ω is a FAIID of $\mathcal{T}$. By Lemma 2.5 $\Lambda_\Omega^3 = \Lambda_{\Omega\Omega\Omega} \leq \Lambda_\Theta$. By assumption, $\Lambda_\Omega \leq \Lambda_\Theta$. Thus, $\Omega \subseteq \Theta$. We conclude that Θ is a SPAIID of $\mathcal{T}$. ■

Theorem 3.25. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then Θ is a StPAIID of $\mathcal{T}$ if and only if Λ_Θ is a StPFAIID of $\mathcal{T}$.*

Proof: Suppose that Θ is a StPAIID of $\mathcal{T}$. Then Θ is an AIID of $\mathcal{T}$. Thus by Theorem 3.15, Λ_Θ is a FAIID of $\mathcal{T}$. Let Ψ_1, Ψ_2 and Ψ_3 be FAIIDs of $\mathcal{T}$ such that $(\Psi_1 \circ \Psi_2 \circ \Psi_3) \wedge (\Psi_3 \circ \Psi_2 \circ \Psi_1) \leq \Lambda_\Theta$. Assume that $\Psi_1 \nleq \Lambda_\Theta$ and $\Psi_2 \nleq \Lambda_\Theta$ or $\Psi_3 \nleq \Lambda_\Theta$. Then there exist $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \mathfrak{T}$ such that $\Psi(\mathfrak{h}) \neq 0\, \Psi(\mathfrak{r}) \neq 0$ and $\Psi(\mathfrak{d}) \neq 0$. While $\Lambda_\Theta(\mathfrak{h}) = 0$, $\Lambda_\Theta(\mathfrak{r}) = 0$ and $\Lambda_\Theta(\mathfrak{r}) = 0$. Thus, $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \mathrm{supp}(\Psi)$, but $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \notin \Theta$. So $\mathrm{supp}(\Psi_1) \nsubseteq \Theta\, \mathrm{supp}(\Psi_2) \nsubseteq \Theta$ and $\mathrm{supp}(\Psi_3) \nsubseteq \Theta$. Since Ψ_1, Ψ_2 and Ψ_3 is FAIIDs of $\mathcal{T}$ we have $\mathrm{supp}(\Psi_1)\, \mathrm{supp}(\Psi_2)$ and $\mathrm{supp}(\Psi_3)$ are AIIDs of $\mathcal{T}$. Thus, $\mathrm{supp}(\Psi_1)\,\mathrm{supp}(\Psi_2)\,\mathrm{supp}(\Psi_3) \nsubseteq \Theta$. and

$\mathrm{supp}(\Psi_3)\,\mathrm{supp}(\Psi_2)\,\mathrm{supp}(\Psi_1) \nsubseteq \Theta$. So, there exists $\mathfrak{m} = \mathfrak{p}\mathfrak{q}\mathfrak{b}$ for some $\mathfrak{p} \in \mathrm{supp}(\Psi_1)$, $\mathfrak{q} \in \mathrm{supp}(\Psi_2), \mathfrak{b} \in \mathrm{supp}(\Psi_3)$ and $\mathfrak{m} = \mathfrak{g}\mathfrak{f}\mathfrak{k}$ for some $\mathfrak{g} \in \mathrm{supp}(\Psi_3)$, $\mathfrak{f} \in \mathrm{supp}(\Psi_2), \mathfrak{k} \in \mathrm{supp}(\Psi_1)$ such that $\mathfrak{m} \notin \Theta$. Hence, $\Lambda_\Theta(\mathfrak{m}) = 0$ implies that $(\Psi_1 \circ \Psi_2 \circ \Psi_3)(\mathfrak{m}) = 0$ and $(\Psi_3 \circ \Psi_2 \circ \Psi_1)(\mathfrak{m}) = 0$. Since $\Psi_1 \circ \Psi_2 \circ \Psi_3 \leq \Lambda_\Theta$ and $\Psi_3 \circ \Psi_2 \circ \Psi_1 \leq \Lambda_\Theta$ such that $\mathfrak{p} \in \mathrm{supp}(\Psi_1)$, $\mathfrak{q} \in \mathrm{supp}(\Psi_2)$ $\mathfrak{b} \in \mathrm{supp}(\Psi_3)$ and $\mathfrak{g} \in \mathrm{supp}(\Psi_3)$, $\mathfrak{f} \in \mathrm{supp}(\Psi_2), \mathfrak{k} \in \mathrm{supp}(\Psi_1)$ we have, $\Psi_1(\mathfrak{p}) \neq 0$, $\Psi_2(\mathfrak{q}) \neq 0$, $\Psi_3(\mathfrak{b}) \neq 0$ and $\Psi_1(\mathfrak{k}) \neq 0$, $\Psi_2(\mathfrak{f}) \neq 0$, $\Psi_3(\mathfrak{g}) \neq 0$ Thus,

$$(\Psi_1 \circ \Psi_2 \circ \Psi_3)(\mathfrak{m}) = \bigvee_{(\mathfrak{p}\mathfrak{q}\mathfrak{b}) \in \Xi_\mathfrak{m}} \{\Psi_1(\mathfrak{p}) \wedge \Psi_2(\mathfrak{q}) \wedge \Psi_3(\mathfrak{b})\} \neq 0$$

and

$$(\Psi_3 \circ \Psi_2 \circ \Psi_1)(\mathfrak{m}) = \bigvee_{(\mathfrak{g}\mathfrak{f}\mathfrak{k}) \in \Xi_\mathfrak{m}} \{\Psi_3(\mathfrak{g}) \wedge \Psi_2(\mathfrak{f}) \wedge \Psi_1(\mathfrak{k})\} \neq 0$$

It is a contradiction so $\Psi_1 \leq \Lambda_\Theta$ or $\Psi_2 \leq \Lambda_\Theta$ or $\Psi_3 \leq \Lambda_\Theta$. Therefore, Λ_Θ is a StPFAIID of $\mathcal{T}$.

Conversely, suppose that Λ_Θ is a StPFAIID of $\mathcal{T}$. Then Λ_Θ is a FAIID of $\mathcal{T}$. Thus, by Theorem 3.15, Θ is an AIID of $\mathcal{T}$. Let Ω_1, Ω_2 and Ω_3 be AIIDs of $\mathcal{T}$ such that $\Omega_1\Omega_2\Omega_3 \cap \Omega_3\Omega_2\Omega_1 \subseteq \Theta$. Then Λ_{Ω_1}, Λ_{Ω_2} and Λ_{Ω_3} are FAIIDs of $\mathcal{T}$. By Lemma 2.5 $\Lambda_{\Omega_1\Omega_2\Omega_3} = \Lambda_{\Omega_1} \circ \Lambda_{\Omega_2} \circ \Lambda_{\Omega_3}$ and $\Lambda_{\Omega_3\Omega_2\Omega_1} = \Lambda_{\Omega_3} \circ \Lambda_{\Omega_2} \circ \Lambda_{\Omega_1}$. Thus, $(\Lambda_{\Omega_1\Omega_2\Omega_3}) \wedge (\Lambda_{\Omega_1} \circ \Lambda_{\Omega_2} \circ \Lambda_{\Omega_3}) = \Lambda_{\Omega_1\Omega_2\Omega_3} \wedge \Lambda_{\Omega_3\Omega_2\Omega_1} = \Lambda_{\Omega_1\Omega_2\Omega_3 \cap \Omega_3\Omega_2\Omega_1} \leq \Lambda_\Theta$. By assumption, $\Lambda_{\Omega_1} \leq \Lambda_\Theta$, $\Lambda_{\Omega_2} \leq \Lambda_\Theta$ and $\Lambda_{\Omega_3} \geq \Lambda_\Theta$. Thus, $\Omega_1 \subseteq \Theta$ or $\Omega_2 \subseteq \Theta$ or $\Omega_3 \subseteq \Theta$. We conclude that Θ is a StPAIID of $\mathcal{T}$. ■

IV. Almost weakly interior ideal and fuzzy almost weakly interior ideal

In this section, we define the weakly almost interior ideal and fuzzy weakly almost interior ideal in TSG. We study basic some interesting properties of almost interior ideal and fuzzy almost interior ideal in the TSG.

Definition 4.1. *A non-empty subset Θ on a TSG $\mathcal{T}$ is called a* weakly almost interior ideal *(WAIID) of $\mathcal{T}$ if $\mathfrak{t}_1\mathfrak{t}_1\Theta\mathfrak{t}_1\mathfrak{t}_1 \cap \Theta \neq \emptyset$ for all $\mathfrak{t}_1 \in \mathcal{T}$.*

Theorem 4.2. *Every WIID of a TSG $\mathcal{T}$ is an WAIID of $\mathcal{T}$.*

Proof: Assume that Θ is a WIID of a TSG $\mathcal{T}$ and let $\mathfrak{t}_1 \in \mathcal{T}$. Then $\mathfrak{t}_1\mathfrak{t}_1\Theta\mathfrak{t}_1\mathfrak{t}_1 \subseteq \mathcal{T}\mathcal{T}\Theta\mathcal{T}\mathcal{T}\Theta$. Thus $\mathfrak{t}_1\mathfrak{t}_1\Theta\mathfrak{t}_1\mathfrak{t}_1 \cap \Theta \neq \emptyset$. We conclude that Θ is a WAIID of $\mathcal{T}$. ■

Theorem 4.3. *Let Θ and Ω be two non-empty subsets of a TSG $\mathcal{T}$ such that $\Theta \subseteq \Omega$. If Θ is a WAIID of $\mathcal{T}$, then Ω is also a WAIID of $\mathcal{T}$.*

Proof: Let Ω be a subset of $\mathcal{T}$ with it containing Θ and let $\mathfrak{t}_1 \in \mathcal{T}$. Then $\mathfrak{t}_1\mathfrak{t}_1\Theta\mathfrak{t}_1\mathfrak{t}_1 \subseteq \mathfrak{t}_1\mathfrak{t}_1\Omega\mathfrak{t}_1\mathfrak{t}_1$ Thus, $\mathfrak{t}_1\mathfrak{t}_1\Omega\mathfrak{t}_1\mathfrak{t}_1 \cap \Omega \neq \emptyset$. Hence, Ω is a WAIID of $\mathcal{T}$. ■

The following result is an obvious of Theorem 4.3.

Corollary 4.4. *Let Θ_1 and Θ_2 be WAIIDs of a TSG $\mathcal{T}$. Thus $\Theta_1 \cup \Theta_2$ is also a WAIID of $\mathcal{T}$.*

Proof: Since $\Theta_1 \subseteq \Theta_1 \cup \Theta_2$, by Theorem 4.3, $\Theta_1 \cup \Theta_2$ is a WAIID of $\mathcal{T}$. ■

Theorem 4.5. *Let Θ_1 and Θ_2 be nonempty subsets of a TSG $\mathcal{T}$. If Θ_1 is a WAIID of $\mathcal{T}$, then $\Theta_1 \cup \Theta_2$ is a WAIID of $\mathcal{T}$.*

Proof: By Theorem 4.3, and $\Theta_1 \subseteq \Theta_1 \cup \Theta_2$. Thus, $\Theta_1 \cup \Theta_2$ is a WAIID of $\mathcal{T}$. ■

Corollary 4.6. *The finite union of WAIIDs of a TSG $\mathcal{T}$ is a WAIID of $\mathcal{T}$.*

Definition 4.7. *A FS Υ on a TSG $\mathcal{T}$ is called a* fuzzy weakly almost interior ideal *(FWAIID) of $\mathcal{T}$ if $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Upsilon \neq 0$. for any FP $\hbar_\mathfrak{t}, \flat_\mathfrak{t} \in \mathcal{T}$.*

Theorem 4.8. *If Υ_1 is a FWAIID of a TSG $\mathcal{T}$ and Υ_2 is a FS of $\mathcal{T}$ such that $\Upsilon_1 \leq \Upsilon_2$, then Υ_2 is a FWAIID of $\mathcal{T}$.*

Proof: Suppose that Υ_1 is a FWAIID of a TSG $\mathcal{T}$ and Υ_2 is a FS of $\mathcal{T}$ such that $\Upsilon_1 \leq \Upsilon_2$. Then for any FPs $\hbar_\mathfrak{t}, \flat_\mathfrak{t} \in \mathcal{T}$, we obtain that $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon_1 \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Upsilon_1 \neq 0$. Thus,

$$(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon_1 \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Upsilon_1 \leq (\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon_2 \circ \hbar_{\mathfrak{t}'} \circ \flat_{\mathfrak{t}'}) \wedge \Upsilon_2 \neq 0.$$

Hence $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon_2 \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Upsilon_2 \neq 0$. Therefore, Υ_2 is a FWAIID of $\mathcal{T}$. ■

The following result is an obvious of Theorem 4.8.

Theorem 4.9. *Let Υ_1 and Υ_2 be FWAIIDs of a TSG $\mathcal{T}$. Then $\Upsilon_1 \vee \Upsilon_2$ is also a FWAIID of $\mathcal{T}$.*

Proof: Since $\Upsilon_1 \leq \Upsilon_1 \vee \Upsilon_2$, by Theorem 3.10, $\Upsilon_1 \vee \Upsilon_2$ is a FWAIID of $\mathcal{T}$. ■

Theorem 4.10. *If Υ_1 is a FWAIID of a TSG $\mathcal{T}$ and Υ_2 is a FS, then $\Upsilon_1 \vee \Upsilon_2$ is a FWAIID of $\mathcal{T}$.*

Proof: By Theorem 4.8, and $\Upsilon_1 \leq \Upsilon_1 \vee \Upsilon_2$. Thus, $\Upsilon_1 \vee \Upsilon_2$ is a FWAIID of $\mathcal{T}$. ■

Corollary 4.11. *Let $\mathcal{T}$ be a TSG. Then the finite maximum of FWAIIDs of $\mathcal{T}$ is a FWAIID of $\mathcal{T}$.*

Theorem 4.12. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then Θ is a WAIID of $\mathcal{T}$ if and only if Λ_Θ is a FWAIID of $\mathcal{T}$.*

Proof: Suppose that Θ is a WAIID of $\mathcal{T}$. Then $\mathfrak{x}\mathfrak{x}\Theta\mathfrak{x}\mathfrak{x} \cap \Theta \neq \emptyset$ for all $\mathfrak{x} \in \mathcal{T}$. Thus, there exists $\mathfrak{c} \in \mathcal{T}$ such that $\mathfrak{c} \in \mathfrak{x}\mathfrak{x}\Theta\mathfrak{x}\mathfrak{x}$ and $\mathfrak{c} \in \Theta$. Let $\hbar_\mathfrak{t}, \flat_\mathfrak{t}, \in \mathcal{T} \in (0,1]$. Then $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_\Theta \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t})(\mathfrak{c}) \neq 0$ and $\Lambda_\Theta(\mathfrak{c}) \neq 0$. Thus, $((\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_\Theta \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Lambda_\Theta)(\mathfrak{c}) \neq 0$. So $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_\Theta \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Lambda_\Theta \neq 0$. Hence, Λ_Θ is a FWAIID of $\mathcal{T}$.

Conversely, suppose that Λ_Θ is a FWAIID of $\mathcal{T}$ and let $\hbar_\mathfrak{t}, \flat_\mathfrak{t} \in \mathcal{T}$ and $\mathfrak{t} \in (0,1]$. Then $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_\Theta \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Lambda_\Theta \neq 0$. Thus, there exists $\mathfrak{c} \in \Theta$ such that $((\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_\Theta \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Lambda_\Theta)(\mathfrak{c}) \neq 0$.. It implies that $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_\Theta \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t})(\mathfrak{c}) \neq 0$ and $\Lambda_\Theta(\mathfrak{c}) \neq 0$.Hence, $\mathfrak{c} \in \mathfrak{x}\mathfrak{x}\Theta\mathfrak{x}\mathfrak{x}$ and $\mathfrak{c} \in \Theta$. So $\mathfrak{x}\mathfrak{y}\Theta\mathfrak{x}\mathfrak{y} \cap \Theta \neq \emptyset$ for all $\mathfrak{x} \in \mathcal{T}$. We conclude that Θ is a WAIID of $\mathcal{T}$. ■

Theorem 4.13. *Let Υ be a fuzzy subset of a TSG $\mathcal{T}$. Then Υ is a FWAIID of $\mathcal{T}$ if and only if $\mathrm{supp}(\Upsilon)$ is a WAIID of $\mathcal{T}$.*

Proof: Assume that Υ is a FWAIID of a TSG $\mathcal{T}$ and let $\hbar_\mathfrak{t}, \flat_\mathfrak{t}, \in \mathcal{T}$ and $\mathfrak{t} \in (0,1]$. Then $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Upsilon \neq 0$. Thus, there exists $\mathfrak{z} \in \mathcal{T}$ such that $((\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Upsilon)(\mathfrak{k}) \neq 0$. So $((\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Upsilon)(\mathfrak{k}) \neq 0$ and $\Upsilon(\mathfrak{k}) \neq 0$ Thus, there exists $\mathfrak{k} \in \mathcal{T}$ such that such that $\mathfrak{k} = \mathfrak{x}\mathfrak{b}\mathfrak{c}\mathfrak{y}$ and $\Upsilon(\mathfrak{k}) \neq 0$. So, $((\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_{\mathrm{supp}(\Upsilon)} \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Lambda_{\mathrm{supp}(\Upsilon)})(\mathfrak{k}) \neq 0$. Hence, $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_{\mathrm{supp}(\Upsilon)} \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Lambda_{\mathrm{supp}(\Upsilon)} \neq 0$. Therefore, $\Lambda_{\mathrm{supp}(\Upsilon)}$ is a FWAIID of $\mathcal{T}$. By Theorem 4.12, $\mathrm{supp}(\Upsilon)$ is a WAIID of T.

Conversely, suppose that $\operatorname{supp}(\Upsilon)$ is a WAIID of $\mathcal{T}$. By Theorem 4.12, $\Lambda_{\operatorname{supp}(\Upsilon)}$ is a FWAIID of $\mathcal{T}$. Then for any fuzzy point $\hbar_\mathfrak{t}, \flat_\mathfrak{t} w \in \mathcal{T}$ and $\mathfrak{t} \in (0,1]$, we have $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_{\operatorname{supp}(\Upsilon)} \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Lambda_{\operatorname{supp}(\Upsilon)} \neq 0$.. Thus, there exists $\mathfrak{z} \in \mathcal{T}$ such that $((\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_{\operatorname{supp}(\Upsilon)} \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Lambda_{\operatorname{supp}(\Upsilon)})(\mathfrak{k}) \neq 0$. Hence, $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Lambda_{\operatorname{supp}(\Upsilon)} \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t})(\mathfrak{k}) = 0$, $\Lambda_{\operatorname{supp}(\Upsilon)}(\mathfrak{k}) \neq 0$. Then there exists $\mathfrak{k} \in \mathcal{T} \in \operatorname{supp}(\Upsilon)$ such that $\mathfrak{k} = \mathfrak{x}\mathfrak{b}\mathfrak{c}\mathfrak{y}$ Thus, $\Upsilon(\mathfrak{k}) \neq 0$. So $((\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Upsilon)(\mathfrak{k}) \neq 0$. Hence, $(\hbar_\mathfrak{t} \circ \flat_\mathfrak{t} \circ \Upsilon \circ \hbar_\mathfrak{t} \circ \flat_\mathfrak{t}) \wedge \Upsilon \neq 0$. Therefore, Υ is a FWAIID of $\mathcal{T}$. ■

Next, we investigate relationships between minimal and maximal weakly almost interior ideals and minimal and maximal fuzzy weakly almost interior ideals of TSGs.

Definition 4.14. *A WAIID Θ of a TSG $\mathcal{T}$ is called*

(1) *a* minimal weakly almost interior ideal *(MiWAIID) if for any WAIID Ω of $\mathcal{T}$ if whenever $\Omega \subseteq \Theta$, then $\Omega = \Theta$,*
(2) *a* maximal weakly almost interior ideal *(MaAIID) if for any WAIID Ω of $\mathcal{T}$ if whenever $\Theta \subseteq \Omega$, then $\Omega = \Theta$.*

Definition 4.15. *A FAIID Υ of a TSG $\mathcal{T}$ is called*

(1) *a* minimal fuzzy weakly almost interior ideal *(MiFWAIID) if for any FWAIID Υ_2 of $\mathcal{T}$ if whenever $\Upsilon_2 \leq \Upsilon_1$, then $\operatorname{supp}(\Upsilon_2) = \operatorname{supp}(\Upsilon_1)$,*
(2) *a* maximal fuzzy weakly almost interior ideal *(MaFWAIID) if for any FWAIID Υ_2 of $\mathcal{T}$ if whenever $\Upsilon_1 \leq \Upsilon_2$, then $\operatorname{supp}(\Upsilon_2) = \operatorname{supp}(\Upsilon_1)$.*

Theorem 4.16. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then*

(1) *Θ is a MiWAIID of $\mathcal{T}$ if and only if Λ_Θ is a MiFWAIID of $\mathcal{T}$.*
(2) *Θ is a MaWAIID of $\mathcal{T}$ if and only if Λ_Θ is a MaFWAIID of $\mathcal{T}$.*

Proof:

1) Assume that Θ is a MiWAIID of $\mathcal{T}$. Then Θ is a WAIID of a TSG $\mathcal{T}$. Thus by Theorem 4.12, Λ_Θ is a FWAIID of $\mathcal{T}$. Let Υ be a FWAIID of $\mathcal{T}$ such that $\Upsilon \leq \Lambda_\Theta$. Then by Theorem 4.13, $\operatorname{supp}(\Upsilon)$ is a WAIID of $\mathcal{T}$ such that $\operatorname{supp}(\Upsilon) \subseteq \operatorname{supp}(\Lambda_\Theta) = \Theta$. Thus, $\operatorname{supp}(\Upsilon) \subseteq \Theta$. Since Θ is a MiWAIID of $\mathcal{T}$ we have $\operatorname{supp}(\Upsilon) = \Theta = \operatorname{supp}(\Lambda_\Theta)$. Therefore, Λ_Θ is a MiFWAIID of $\mathcal{T}$.
Conversely, suppose that Λ_Θ is a MiFWAIID of $\mathcal{T}$. Then Λ_Θ is a FWAIID of $\mathcal{T}$. Thus by Theorem 4.12, Θ is a WAIID of a TSG $\mathcal{T}$. Let Ω be a WAIID of $\mathcal{T}$ such that $\Omega \subseteq \Theta$. Then Λ_Ω is a FWAIID of $\mathcal{T}$ such that $\Lambda_\Omega \leq \Lambda_\Theta$. Thus, $\operatorname{supp}(\Lambda_\Omega) \subseteq \operatorname{supp}(\Lambda_\Theta)$. So, $\Omega = \operatorname{supp}(\Lambda_\Omega) \subseteq \operatorname{supp}(\Lambda_\Theta) = \Theta$ implies that $\Omega \subseteq \Theta$. Since Λ_Θ is a MiFWAIID of $\mathcal{T}$ we have $\operatorname{supp}(\Lambda_\Omega) = \operatorname{supp}(\Lambda_\Theta)$. Thus, $\Omega = \operatorname{supp}(\chi_\Omega) = \operatorname{supp}(\Lambda_\Theta) = \Theta$. Hence, Θ is a MiWAIID of $\mathcal{T}$.
2) Assume that Θ is a MaWAIID of $\mathcal{T}$. Then Θ is a WAIID of a TSG $\mathcal{T}$. Thus by Theorem 4.12, Λ_Θ is a FWAIID of $\mathcal{T}$. Let Υ be a FWAIID of $\mathcal{T}$ such that $\Lambda_\Theta \leq \Upsilon$. Then by Theorem 4.13, $\operatorname{supp}(\Upsilon)$ is a WAIID of $\mathcal{T}$ such that $\Theta = \operatorname{supp}(\Lambda_\Theta) \subseteq \operatorname{supp}(\Upsilon)$. Since Θ is a MaWAIID of $\mathcal{T}$ we have $\operatorname{supp}(\Upsilon) = \operatorname{supp}(\Lambda_\Theta)$. Hence, Λ_Θ is a MaFWAIID of $\mathcal{T}$.
Conversely, suppose that Λ_Θ is a MaFWAIID of $\mathcal{T}$. Then Λ_Θ is a FWAIID of $\mathcal{T}$. Thus by Theorem 4.12, Θ is a WAIID of a TSG $\mathcal{T}$. Let Ω be a WAIID of $\mathcal{T}$ such that $\Theta \subseteq \Omega$. Then Λ_Ω is a FWAIID of $\mathcal{T}$ such that $\Lambda_\Theta \leq \Lambda_\Omega$. Thus, $\operatorname{supp}(\Lambda_\Theta) \subseteq \operatorname{supp}(\Lambda_\Omega)$. So, $\Theta = \operatorname{supp}(\Lambda_\Theta) \subseteq \operatorname{supp}(\Lambda_\Omega) = \Omega$. It implies that $\operatorname{supp}(\Lambda_\Theta) \subseteq \operatorname{supp}(\Lambda_\Omega)$. Since Λ_Θ is a MAFWAIID of $\mathcal{T}$ we have $\operatorname{supp}(\Lambda_\Omega) = \operatorname{supp}(\Lambda_\Theta)$. Thus, $\Omega = \operatorname{supp}(\chi_\Omega) = \operatorname{supp}(\Lambda_\Theta) = \Theta$. Hence, Θ is a MaWAIID of $\mathcal{T}$.

■

Corollary 4.17. *Let $\mathcal{T}$ be a TSG. Then $\mathcal{T}$ has no proper WAIID if and only if $\operatorname{supp}(\Upsilon) = \mathcal{T}$ for every FWAIID Υ of $\mathcal{T}$.*

Next, we give definitions of prime (resp., semiprime, strongly prime) WAIIDs, and prime (resp., semiprime strongly prime) WFAIIDs. We study the relationships between prime (resp., semiprime strongly prime) WAIIDs and their fuzzification of TSGs.

Definition 4.18. *Let Θ be a WAIID of a TSG $\mathcal{T}$. Then we called*

(1) *Θ is a* ***prime weakly almost interior ideal*** *(PWAIID) if for any three WAIIDs Ω_1, Ω_2 and Ω_3 of $\mathcal{T}$ such that $\Omega_1\Omega_2\Omega_3 \subseteq \Theta$ implies that $\Omega_1 \subseteq \Theta$ or $\Omega_2 \subseteq \Theta$ or $\Omega_3 \subseteq \Theta$.*
(2) *Θ is a* ***semiprime weakly almost interior ideal*** *(SP-WAIID) if for any WAIID Ω of $\mathcal{T}$ such that $\Omega^3 \subseteq \Theta$ implies that $\Omega \subseteq \Theta$.*
(3) *Θ is a* ***strongly prime weakly almost interior ideal*** *(StPWAIID) if for any WAIIDs Ω_1, Ω_2 and Ω_3 of $\mathcal{T}$ such that $\Omega_1\Omega_2\Omega_3 \cap \Omega_3\Omega_2\Omega_1 \subseteq \Theta$ implies that $\Omega_1 \subseteq \Theta$ or $\Omega_2 \subseteq \Theta$ or $\Omega_3 \subseteq \Theta$.*

Definition 4.19. *A FAIID Υ on a TSG $\mathcal{T}$. Then we called*

(1) *Υ is a* ***prime fuzzy weakly almost interior ideal*** *(PFWAIID) if for any three FWAIIDs Ψ_1, Ψ_2 and Ψ_3 of $\mathcal{T}$ such that $\Psi_1 \circ \Psi_2 \circ \Psi_3 \leq \Upsilon$ implies that $\Psi_1 \leq \Upsilon$ or $\Psi_2 \leq \Upsilon$ or $\Psi_3 \leq \Upsilon$.*
(2) *Υ is a* ***semiprime fuzzy weakly almost interior ideal*** *(SPFWAIID) if for any FWAIID Ψ of $\mathcal{T}$ such that $\Psi \circ \Psi \circ \Psi \leq \Upsilon$ implies that $\Psi \leq \Upsilon$.*
(3) *Υ is a* ***strongly prime fuzzy weakly almost interior ideal*** *(StPFWAIID) if for any three FWAIIDs Ψ_1, Ψ_2 and Ψ_3 of $\mathcal{T}$ such that $(\Psi_1 \circ \Psi_2 \circ \Psi_3) \wedge (\Psi_3 \circ \Psi_2 \circ \Psi_1) \leq \Upsilon$ implies that $\Psi_1 \leq \Upsilon$ or $\Psi_2 \leq \Upsilon$ or $\Psi_3 \leq \Upsilon$.*

It is clear, every StPFWAIID of a TSG is a PFWAIID and every PFWAIIDs of a TSG is a SPFWAIIDs.

Theorem 4.20. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then Θ is a PWAIID of $\mathcal{T}$ if and only if Λ_Θ is a PFWAIID of $\mathcal{T}$.*

Proof: Suppose that Θ is a PWAIID of $\mathcal{T}$. Then Θ is a WAIID of $\mathcal{T}$. Thus by Theorem 4.12, Λ_Θ is a FWAIID of $\mathcal{T}$. Let Ψ_1, Ψ_2 and Ψ_3 be FWAIIDs of $\mathcal{T}$. such that $\Psi_1 \circ \Psi_2 \circ \Psi_3 \leq \Lambda_\Theta$. Assume that $\Psi_1 \nleq \Lambda_\Theta$ and $\Psi_2 \nleq \Lambda_\Theta$ and $\Psi_3 \nleq \Lambda_\Theta$. Then there exist $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \mathfrak{T}$ such that $\Psi_1(\mathfrak{h}) \neq 0\, \Psi_2(\mathfrak{r}) \neq 0$ and $\Psi_3(\mathfrak{d}) \neq 0$. While $\Lambda_\Theta(\mathfrak{h}) = 0$, $\Lambda_\Theta(\mathfrak{r}) = 0$ and $\Lambda_\Theta(\mathfrak{r}) = 0$. Thus, $\mathfrak{h} \in \operatorname{supp}(\Psi_1)$, $\mathfrak{r} \in \operatorname{supp}(\Psi_2)$ and $\mathfrak{d} \in \operatorname{supp}(\Psi_3)$, but $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \notin \Theta$. So $\operatorname{supp}(\Psi_1) \nsubseteq \Theta$ $\operatorname{supp}(\Psi_2) \nsubseteq \Theta$ and $\operatorname{supp}(\Psi_3) \nsubseteq \Theta$. Since Ψ_1, Ψ_2 and Ψ_3 is FWAIIDs of $\mathcal{T}$ we have $\operatorname{supp}(\Psi_1)$ $\operatorname{supp}(\Psi_2)$ and $\operatorname{supp}(\Psi_3)$ are WAIIDs of $\mathcal{T}$. Thus, $\operatorname{supp}(\Psi_1)\operatorname{supp}(\Psi_2)\operatorname{supp}(\Psi_3) \nsubseteq \Theta$. So, there exists $\mathfrak{m} = \mathfrak{p}\mathfrak{q}\mathfrak{b}$ for some $\mathfrak{p} \in \text{supp}(\Psi_1)$, $\mathfrak{q} \in \text{supp}(\Psi_2)$ and $\mathfrak{b} \in \text{supp}(\Psi_3)$ such that $\mathfrak{m} \notin \Theta$. Hence, $\Lambda_\Theta(\mathfrak{m}) = 0$ implies that $(\Psi_1 \circ \Psi_2 \circ \Psi_3)(\mathfrak{m}) = 0$. Since $\Psi_1 \circ \Psi_2 \circ \Psi_3 \leq \Lambda_\Theta$ such that $\mathfrak{p} \in \text{supp}(\Psi_1)$, $\mathfrak{q} \in \text{supp}(\Psi_2)$ and $\mathfrak{b} \in \text{supp}(\Psi_3)$ we have, $\Psi_1(\mathfrak{p}) \neq 0$, $\Psi_2(\mathfrak{q}) \neq 0$, and $\Psi_3(\mathfrak{b}) \neq 0$. Thus,

$$((\Psi_1 \circ \Psi_2 \circ \Psi_3)(\mathfrak{m}) = \bigvee_{(\mathfrak{p}\mathfrak{q}\mathfrak{b}) \in \Xi_\mathfrak{m}} \{\Psi_1(\mathfrak{p}) \wedge \Psi_2(\mathfrak{q}) \wedge \Psi_3(\mathfrak{b})\} \neq 0$$

It is a contradiction so $\Psi_1 \leq \Lambda_\Theta$ or $\Psi_2 \leq \Lambda_\Theta$ or $\Psi_3 \leq \Lambda_\Theta$. Therefore, Λ_Θ is a PFWAIID of $\mathcal{T}$.

Conversely, suppose that Λ_Θ is a PFWAIID of $\mathcal{T}$. Then Λ_Θ is a FWAIID of $\mathcal{T}$. Thus by Theorem 4.12, Θ is a WAIID of $\mathcal{T}$. Let Ω_1, Ω_2 and Ω_3 be WAIIDs of $\mathcal{T}$ such that $\Omega_1\Omega_2\Omega_3 \subseteq \Theta$. Then Λ_{Ω_1} Λ_{Ω_2} and Λ_{Ω_3} are FWAIIDs of $\mathcal{T}$. By Lemma 2.5 $\Lambda_{\Omega_1} \circ \Lambda_{\Omega_2} \circ \Lambda_{\Omega_3} = \Lambda_{\Omega_1\Omega_2\Omega_3} \leq \Lambda_\Theta$. By assumption, $\Lambda_{\Omega_1} \leq \Lambda_\Theta$ or $\Lambda_{\Omega_2} \leq \Lambda_\Theta$ or $\Lambda_{\Omega_3} \leq \Lambda_\Theta$. Thus, $\Omega_1 \subseteq \Theta$ or $\Omega_2 \subseteq \Theta$ or $\Omega_3 \subseteq \Theta$. We conclude that Θ is a PWAIID of $\mathcal{T}$. ■

Theorem 4.21. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then Θ is a SPWAIID of $\mathcal{T}$ if and only if Λ_Θ is a SPFWAIID of $\mathcal{T}$.*

Proof: Suppose that Θ is a SPWAIID of $\mathcal{T}$. Then Θ is a WAIID of $\mathcal{T}$. Thus by Theorem 4.12, Λ_Θ is a FWAIID of $\mathcal{T}$. Let Ψ be a FWAIID of $\mathcal{T}$. such that $\Psi^3 = \Psi \circ \Psi \circ \Psi \leq \Lambda_\Theta$. Assume that $\Psi \nleq \Lambda_\Theta$. Then there exist $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \mathfrak{T}$ such that $\Psi(\mathfrak{h}) \neq 0\, \Psi(\mathfrak{r}) \neq 0$ and $\Psi(\mathfrak{d}) \neq 0$. While $\Lambda_\Theta(\mathfrak{h}) = 0$, $\Lambda_\Theta(\mathfrak{r}) = 0$ and $\Lambda_\Theta(\mathfrak{r}) = 0$. Thus, $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \text{supp}(\Psi)$, but $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \notin \Theta$. So $\text{supp}(\Psi) \nsubseteq \Theta$. Since Ψ is a FWAIID of $\mathcal{T}$ we have $\text{supp}(\Psi)$ is a WAIIDs of $\mathcal{T}$. Thus, $\text{supp}(\Psi)\,\text{supp}(\Psi)\,\text{supp}(\Psi) \nsubseteq \Theta$. So, there exists $\mathfrak{m} = \mathfrak{p}\mathfrak{q}\mathfrak{b}$ for some $\mathfrak{p}, \mathfrak{q}, \mathfrak{b} \in \text{supp}(\Psi)$ such that $\mathfrak{m} \notin \Theta$. Hence, $\Lambda_\Theta(\mathfrak{m}) = 0$ implies that $(\Psi \circ \Psi \circ \Psi)(\mathfrak{m}) = 0$. Since $\Psi^3 = \Psi \circ \Psi \circ \Psi \leq \Lambda_\Theta$ such that $\mathfrak{p}, \mathfrak{q}, \mathfrak{b} \in \text{supp}(\Psi)$ we have, $\Psi(\mathfrak{p}) \neq 0$, $\Psi(\mathfrak{q}) \neq 0$, and $\Psi(\mathfrak{b}) \neq 0$. Thus,

$$(\Psi \circ \Psi \circ \Psi)(\mathfrak{m}) = \bigvee_{(\mathfrak{p}\mathfrak{q}\mathfrak{b}) \in \Xi_\mathfrak{m}} \{\Psi(\mathfrak{p}) \wedge \Psi(\mathfrak{q}) \wedge \Psi(\mathfrak{b})\} \neq 0$$

It is a contradiction so $\Psi \leq \Lambda_\Theta$. Therefore Λ_Θ is a SPFWAIID of $\mathcal{T}$.

Conversely, suppose that Λ_Θ is a SPFWAIID of $\mathcal{T}$. Then Λ_Θ is a FWAIID of $\mathcal{T}$. Thus by Theorem 3.15, Θ is a WAIID of $\mathcal{T}$. Let Ω be a WAIID of $\mathcal{T}$ such that $\Omega^3 \subseteq \Theta$. Then Λ_Ω is a FWAIID of $\mathcal{T}$. By Lemma 2.5 $\Lambda_\Omega^3 = \Lambda_{\Omega\Omega\Omega} \leq \Lambda_\Theta$. By assumption, $\Lambda_\Omega \leq \Lambda_\Theta$. Thus $\Omega \subseteq \Theta$. We conclude that Θ is a SPWAIID of $\mathcal{T}$. ■

Theorem 4.22. *Let Θ be a nonempty subset of a TSG $\mathcal{T}$. Then Θ is a StPWAIID of $\mathcal{T}$ if and only if Λ_Θ is a StPFWAIID of $\mathcal{T}$.*

Proof: Suppose that Θ is a StPWAIID of $\mathcal{T}$ Then Θ is a WAIID of $\mathcal{T}$. Thus by Theorem 4.12, Λ_Θ is a FWAIID of $\mathcal{T}$. Let Ψ_1, Ψ_2 and Ψ_3 be FWAIIDs of $\mathcal{T}$ such that $(\Psi_1 \circ \Psi_2 \circ \Psi_3) \wedge (\Psi_3 \circ \Psi_2 \circ \Psi_1) \leq \Lambda_\Theta$. Assume that $\Psi_1 \nleq \Lambda_\Theta$ and $\Psi_2 \nleq \Lambda_\Theta$ or $\Psi_3 \nleq \Lambda_\Theta$. Then there exist $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \mathfrak{T}$ such that $\Psi(\mathfrak{h}) \neq 0\, \Psi(\mathfrak{r}) \neq 0$ and $\Psi(\mathfrak{d}) \neq 0$. While $\Lambda_\Theta(\mathfrak{h}) = 0$, $\Lambda_\Theta(\mathfrak{r}) = 0$ and $\Lambda_\Theta(\mathfrak{r}) = 0$. Thus, $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \in \text{supp}(\Psi)$, but $\mathfrak{h}, \mathfrak{r}, \mathfrak{d} \notin \Theta$. So $\text{supp}(\Psi_1) \nsubseteq \Theta$ $\text{supp}(\Psi_2) \nsubseteq \Theta$ and $\text{supp}(\Psi_3) \nsubseteq \Theta$. Since Ψ_1, Ψ_2 and Ψ_3 is FWAIIDs of $\mathcal{T}$ we have $\text{supp}(\Psi_1)$ $\text{supp}(\Psi_2)$ and $\text{supp}(\Psi_3)$ are WAIIDs of $\mathcal{T}$. Thus, $\text{supp}(\Psi_1)\,\text{supp}(\Psi_2)\,\text{supp}(\Psi_3) \nsubseteq \Theta$. and $\text{supp}(\Psi_3)\,\text{supp}(\Psi_2)\,\text{supp}(\Psi_1) \nsubseteq \Theta$. So, there exists $\mathfrak{m} = \mathfrak{p}\mathfrak{q}\mathfrak{b}$ for some $\mathfrak{p} \in \text{supp}(\Psi_1)$, $\mathfrak{q} \in \text{supp}(\Psi_2), \mathfrak{b} \in \text{supp}(\Psi_3)$ and $\mathfrak{m} = \mathfrak{g}\mathfrak{f}\mathfrak{k}$ for some $\mathfrak{g} \in \text{supp}(\Psi_3)$, $\mathfrak{f} \in \text{supp}(\Psi_2), \mathfrak{k} \in \text{supp}(\Psi_1)$ such that $\mathfrak{m} \notin \Theta$. Hence, $\Lambda_\Theta(\mathfrak{m}) = 0$ implies that $(\Psi_1 \circ \Psi_2 \circ \Psi_3)(\mathfrak{m}) = 0$ and $(\Psi_3 \circ \Psi_2 \circ \Psi_1)(\mathfrak{m}) = 0$. Since $\Psi_1 \circ \Psi_2 \circ \Psi_3 \leq \Lambda_\Theta$ and $\Psi_3 \circ \Psi_2 \circ \Psi_1 \leq \Lambda_\Theta$ such that $\mathfrak{p} \in \text{supp}(\Psi_1)$, $\mathfrak{q} \in \text{supp}(\Psi_2)$, $\mathfrak{b} \in \text{supp}(\Psi_3)$ and $\mathfrak{g} \in \text{supp}(\Psi_3)$, $\mathfrak{f} \in \text{supp}(\Psi_2), \mathfrak{k} \in \text{supp}(\Psi_1)$ we have, $\Psi_1(\mathfrak{p}) \neq 0$, $\Psi_2(\mathfrak{q}) \neq 0$, $\Psi_3(\mathfrak{b}) \neq 0$ and $\Psi_1(\mathfrak{k}) \neq 0$, $\Psi_2(\mathfrak{f}) \neq 0$, $\Psi_3(\mathfrak{g}) \neq 0$ Thus,

$$(\Psi_1 \circ \Psi_2 \circ \Psi_3)(\mathfrak{m}) = \bigvee_{(\mathfrak{p}\mathfrak{q}\mathfrak{b}) \in \Xi_\mathfrak{m}} \{\Psi_1(\mathfrak{p}) \wedge \Psi_2(\mathfrak{q}) \wedge \Psi_3(\mathfrak{b})\} \neq 0$$

and

$$(\Psi_3 \circ \Psi_2 \circ \Psi_1)(\mathfrak{m}) = \bigvee_{(\mathfrak{g}\mathfrak{f}\mathfrak{k}) \in \Xi_\mathfrak{m}} \{\Psi_3(\mathfrak{g}) \wedge \Psi_2(\mathfrak{f}) \wedge \Psi_1(\mathfrak{k})\} \neq 0$$

It is a contradiction so $\Psi_1 \leq \Lambda_\Theta$ or $\Psi_2 \leq \Lambda_\Theta$ or $\Psi_3 \leq \Lambda_\Theta$. Therefore, Λ_Θ is a StPFWAIID of $\mathcal{T}$.

Conversely, suppose that Λ_Θ is a StPFWAIID of $\mathcal{T}$. Then Λ_Θ is a FWAIID of $\mathcal{T}$. Thus, by Theorem 4.12, Θ is a WIID of $\mathcal{T}$. Let Ω_1, Ω_2 and Ω_3 be WAIIDs of $\mathcal{T}$ such that $\Omega_1\Omega_2\Omega_3 \cap \Omega_3\Omega_2\Omega_1 \subseteq \Theta$. Then Λ_{Ω_1}, Λ_{Ω_2} and Λ_{Ω_3} are FWAIIDs of $\mathcal{T}$. By Lemma 2.5 $\Lambda_{\Omega_1\Omega_2\Omega_3} = \Lambda_{\Omega_1} \circ \Lambda_{\Omega_2} \circ \Lambda_{\Omega_3}$ and $\Lambda_{\Omega_3\Omega_2\Omega_1} = \Lambda_{\Omega_3} \circ \Lambda_{\Omega_2} \circ \Lambda_{\Omega_1}$. Thus, $(\Lambda_{\Omega_1\Omega_2\Omega_3}) \wedge (\Lambda_{\Omega_1} \circ \Lambda_{\Omega_2} \circ \Lambda_{\Omega_3}) = \Lambda_{\Omega_1\Omega_2\Omega_3} \wedge \Lambda_{\Omega_3\Omega_2\Omega_1} = \Lambda_{\Omega_1\Omega_2\Omega_3 \cap \Omega_3\Omega_2\Omega_1} \leq \Lambda_\Theta$. By assumption, $\Lambda_{\Omega_1} \leq \Lambda_\Theta$, $\Lambda_{\Omega_2} \leq \Lambda_\Theta$ and $\Lambda_{\Omega_3} \geq \Lambda_\Theta$. Thus, $\Omega_1 \subseteq \Theta$ or $\Omega_2 \subseteq \Theta$ or $\Omega_3 \subseteq \Theta$. We conclude that Θ is a StPWAIID of $\mathcal{T}$. ■

V. Conclusion

The aim paper gives the concept of almost interior ideals in ternary semigroups. The union of two almost interior ideals is also an almost interior ideal in ternary semigroups, and the results in class fuzzifications are the same. In Theorems 3.15, 3.16, 3.19, 4.20, and 4.21, we prove the relationship between almost interior ideals and class fuzzifications. In future work, we can study other kinds of almost ideals and their fuzzifications in ordered ternary semigroups.

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