Numerical Eigenvalue Bounds

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Abstract—**In this paper, we derive expressions for bounding intervals of the eigenvalues of real symmetric matrices. These bounds are relatively good provided approximations to the eigenvectors are known. We show how the problem is reduced to that of finding the spectrum of matrices of order two. We also prove the existence of a non optimal vector parameter.**

Index Terms—**symmetric, eigenvalues, bounds, trace**

I. Introduction

THE eigenvalue distribution of a matrix $\mathbf{A}$ is indispensable in almost all branches of science and engineering. For real symmetric matrices, this distribution is limited to $\mathbb{R}$. Some recent applications have been to search engines [5] and crypto correlation matrices [3]. The latter facilitates the timing of investment into a particular crypto asset, based on the rally of its accompanying correlated crypto asset. The conditioning of a symmetric linear system depends on the $\frac{|\lambda_n|}{|\lambda_1|}$, λ_n is the eigenvalue of largest absolute magnitude and λ_1 is the eigenvalue of least absolute magnitude. Accurate location of the eigenvalues is usually accompanied by the computation of the associated eigenvectors. However, such computations are rarely accurate, being limited by machine precision. Some simple, yet effective methods based on matrix entries are the Gerschgorin disks and ovals of Cassini [2]. These methods provide crucial regions to search for eigenvalues. Bounds based on only the traces of the matrix and the traces of its powers, have been studied in great detail [8]–[10], [13]. Singh et al [12] have shown how to optimally derive inner bounds for the extremal eigenvalues. Furthermore for positive definite matrices, they have shown [11] that it is possible to obtain bounding intervals for the extremal eigenvalues, by considering the minimal polynomial.. The power methods and its variants, together with the Rayleigh quotient [6] can locate the dominant eigenpairs effectively. Usually iterative methods are employed to efficiently compute the eigenpairs. Earlier numerical bounds due to Kato are discussed by Hayes [14], while bounds due to Weinstein are discussed by Cohen and Fedmann [15]. Here we shall find bounding intervals for each eigenvalue.

Manuscript received January 13, 2025; revised March 29, 2025.

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II. Theory

Let $\boldsymbol{\lambda} = (\lambda_i), i = 1, 2, \cdots, n$ denote the eigenvalues of a real symmetric matrix $\mathbf{A}$ arranged in increasing order

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \tag{1}$$

The corresponding eigenbasis is denoted by the set

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}, \tag{2}$$

where $\|\mathbf{u}_i\|_2$=1. Let $(\tilde{\lambda}, \tilde{\mathbf{u}})$ be an approximate eigenpair of $\mathbf{A}$, where $\|\tilde{\mathbf{u}}\|_2$=1, then the residual $\boldsymbol{\delta}\tilde{\mathbf{u}}$ is defined by

$$\boldsymbol{\delta}\tilde{\mathbf{u}} = \mathbf{A}\tilde{\mathbf{u}} - \tilde{\lambda}\tilde{\mathbf{u}} \neq \mathbf{0} \tag{3}$$

Theorem 1: Let $\mathbf{A}$ be a real symmetric matrix with the eigenvalues arranged as in (1), then there is an eigenvalue of $\mathbf{A}$ in the interval $[\tilde{\lambda} - \|\boldsymbol{\delta}\tilde{\mathbf{u}}\|, \tilde{\lambda} - \|\boldsymbol{\delta}\tilde{\mathbf{u}}\|]$. There is also an eigenvalue of $\mathbf{A}$ in the interval $(-\infty, \tilde{\lambda} - \|\boldsymbol{\delta}\tilde{\mathbf{u}}\|]$ or $[\tilde{\lambda} + \|\boldsymbol{\delta}\tilde{\mathbf{u}}\|, \infty)$.

Proof: Let $\mathbf{Q}$ be the orthogonal matrix that diagonalizes $\mathbf{A}$, then $\mathbf{Q^t A Q} = \mathbf{D}$, where $\mathbf{D}$ is diagonal. Since $\tilde{\lambda} \notin \sigma(\mathbf{A})$, it follows that $\mathbf{A} - \tilde{\lambda}\mathbf{I}$ is invertible so that

$$\begin{aligned} \tilde{\mathbf{u}} &= (\mathbf{A} - \tilde{\lambda}\mathbf{I})^{-1}\boldsymbol{\delta}\tilde{\mathbf{u}} \\ &= (\mathbf{QDQ^t} - \tilde{\lambda}\mathbf{QQ^t})^{-1}\boldsymbol{\delta}\tilde{\mathbf{u}} \\ &= \mathbf{Q}(\mathbf{D} - \tilde{\lambda}\mathbf{I})^{-1}\mathbf{Q^t}\boldsymbol{\delta}\tilde{\mathbf{u}} \end{aligned}$$

Thus, we get

$$\|\tilde{\mathbf{u}}\|_2 \le \|\mathbf{Q}\|_2 \|\mathbf{Q^t}\|_2 \|(\mathbf{D} - \tilde{\lambda}\mathbf{I})^{-1}\|_2 \|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2, \tag{4}$$

so that

$$\begin{aligned} 1 &\le \max_{i=1,2,\ldots,n} |\lambda_i - \tilde{\lambda}|^{-1} \|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2 \\ &= \frac{1}{|\lambda_p - \tilde{\lambda}|} \|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2, \end{aligned} \tag{5}$$

where λ_p is the closest eigenvalue to $\tilde{\lambda}$. Thus

$$|\lambda_p - \tilde{\lambda}| \le \|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2, \tag{6}$$

from which the result follows. From (3) it follows that

$$\begin{aligned} \boldsymbol{\delta}\tilde{\mathbf{u}} &= \mathbf{QDQ^t}\tilde{\mathbf{u}} - \tilde{\lambda}\mathbf{QQ^t}\tilde{\mathbf{u}} \\ &= \mathbf{Q}(\mathbf{D} - \tilde{\lambda}\mathbf{I})\mathbf{Q^t}\tilde{\mathbf{u}} \end{aligned} \tag{7}$$

Thus, we get

$$\|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2 \le \|\mathbf{Q}\|_2 \|\mathbf{Q^t}\|_2 \|\mathbf{D} - \tilde{\lambda}\mathbf{I}\|_2 \|\tilde{\mathbf{u}}\|_2, \tag{8}$$

so that

$$\|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2 \le |\lambda_q - \tilde{\lambda}|, \tag{9}$$

where λ_q is the furthest eigenvalue from $\tilde{\lambda}$. The result then follows from (9) ■

The bounds in Theorem 1 are known as Weinstein bounds.

Theorem 2: Let $\tilde{\mathbf{u}}$ be an approximate eigenvector of $\mathbf{A}$ and $\tilde{\lambda} = \langle \mathbf{A}\tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ be the corresponding approximate eigenvalue. If $\mu, \nu \in \mathbb{R}$ are known such that

$$\lambda_{k-1} \le \mu < \tilde{\lambda} < \nu \le \lambda_{k+1}, \tag{10}$$

then

$$\tilde{\lambda} - \frac{\|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2^2}{\nu - \tilde{\lambda}} \le \lambda_k \le \tilde{\lambda} + \frac{\|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2^2}{\tilde{\lambda} - \mu}. \tag{11}$$

Proof: We shall prove the left hand side of (11). The matrix

$$\mathbf{B} = \mathbf{A}^2 - (\lambda_k + \nu)\mathbf{A} + \lambda_k \nu \mathbf{I} \tag{12}$$

is positive semi-definite. This is easy to deduce as the eigenvalues of $\mathbf{B}$ are

$$\begin{aligned} \lambda^2 - (\lambda_k + \nu)\lambda + \lambda_k \nu &= (\lambda - \lambda_k)(\lambda - \nu) \\ &\ge 0. \end{aligned} \tag{13}$$

Thus it follows that $\langle \mathbf{B}\tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \ge 0$. Hence

$$\begin{aligned} &\langle (\mathbf{A}^2 - (\lambda_k + \nu)\mathbf{A} + \lambda_k \nu \mathbf{I})\tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \\ &= \|\mathbf{A}\tilde{\mathbf{u}}\|_2^2 - (\lambda_k + \nu)\langle \mathbf{A}\tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle + \lambda_k \nu \\ &= \|\mathbf{A}\tilde{\mathbf{u}}\|_2^2 - (\lambda_k + \nu)\tilde{\lambda} + \lambda_k \nu \ge 0 \end{aligned} \tag{14}$$

From (3) we have that

$$\begin{aligned} \|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2^2 &= \langle \mathbf{A}\tilde{\mathbf{u}} - \tilde{\lambda}\tilde{\mathbf{u}}, \mathbf{A}\tilde{\mathbf{u}} - \tilde{\lambda}\tilde{\mathbf{u}} \rangle \\ &= \|\mathbf{A}\tilde{\mathbf{u}}\|_2^2 - 2\tilde{\lambda}\langle \mathbf{A}\tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle + \tilde{\lambda}^2 \\ &= \|\mathbf{A}\tilde{\mathbf{u}}\|_2^2 - \tilde{\lambda}^2. \end{aligned} \tag{15}$$

Substituting (15) into (14) results in

$$\begin{aligned} \tilde{\lambda}^2 - (\lambda_k + \nu)\tilde{\lambda} + \lambda_k \nu &\ge -\|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2^2 \\ (\lambda_k - \tilde{\lambda})(\nu - \tilde{\lambda}) &\ge -\|\boldsymbol{\delta}\tilde{\mathbf{u}}\|_2^2. \end{aligned} \tag{16}$$

Solving for λ_k from (16) yields the result. If ν is replaced by μ in the above proof, the right hand side of (11) is obtained. ■

The bounds in Theorem 2 are attributed to Kato.

Corollary 1: If $k = 1$ in the left hand side of (11) then we obtain

$$\frac{\tilde{\lambda}\nu - (\tilde{\lambda}^2 + \|\delta\tilde{\mathbf{u}}\|_2^2)}{\nu - \tilde{\lambda}} \le \lambda_1 \tag{17}$$

or by using (15)

$$\frac{\nu\langle \mathbf{A}\tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle - \langle \mathbf{A}\tilde{\mathbf{u}}, \mathbf{A}\tilde{\mathbf{u}} \rangle^2}{\nu - \langle \mathbf{A}\tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle} \le \lambda_1. \tag{18}$$

Equation (18) represents the well known Temple bounds. Let $S_m = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$, where $m \le n$, $\|\mathbf{v}_i\|_2 = 1$ and $\mathbf{v}_i \perp \mathbf{v}_j$. Define the matrix $\mathbf{V}$ by

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m],$$

then $\mathbf{A}\,\mathrm{span}(S_m) = R(\mathbf{AV}) \subseteq R(\mathbf{A})$, where $R(.)$ denotes the range of an operator. The matrix $\mathbf{B}_m = \mathbf{V}^{\mathbf{t}}\mathbf{AV}$ is $m \times m$ and symmetric, $\mathbf{V}^{\mathbf{t}}\mathbf{V} = \mathbf{I}_{\mathbf{m}}$ and $\mathbf{V}\mathbf{V}^{\mathbf{t}}$ is $n \times n$. Thus $\mathbf{V}$ is semi-orthogonal. In fact $\mathbf{V}\mathbf{V}^{\mathbf{t}}$ is a projector from $\mathbb{R}^n$ to the subspace $\mathrm{span}(S_m)$. When $\mathbf{v}_i = \mathbf{e}_i$, where $\mathbf{e}_i$ are the standard basis vectors in $\mathbb{R}^n$, then $\mathbf{B}$ is the leading principal $m \times m$ submatrix of $\mathbf{A}$. The matrix $\mathbf{B}_m$ has the form

$$\mathbf{B}_m = \begin{bmatrix} \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_2 \rangle & \cdots & \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_m \rangle \\ \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2 \rangle & \cdots & \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_m \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{A}\mathbf{v}_m, \mathbf{v}_1 \rangle & \langle \mathbf{A}\mathbf{v}_m, \mathbf{v}_2 \rangle & \cdots & \langle \mathbf{A}\mathbf{v}_m, \mathbf{v}_m \rangle \end{bmatrix} \tag{19}$$

We shall denote the eigenvalues of $\mathbf{B}_m$ by β_i, $i = 1, 2, \cdots, m$, where

$$\beta_1 \le \beta_2 \le \cdots \le \beta_m. \tag{20}$$

Theorem 3: The extremal eigenvalues of $\mathbf{A}$ and $\mathbf{B}_m$ satisfy

$$\lambda_1 \le \beta_1 \le \beta_m \le \lambda_n \tag{21}$$

Proof: We shall prove only the right hand side of inequality (21).

$$\begin{aligned} \beta_m &= \max_{\|\mathbf{x}\|_2=1} \langle \mathbf{B}_m\mathbf{x}, \mathbf{x} \rangle \\ &= \max_{\|\mathbf{x}\|_2=1} \langle \mathbf{AVx}, \mathbf{Vx} \rangle \\ &= \max_{\|\mathbf{x}\|_2=1} \left\langle \frac{\mathbf{AVx}}{\|\mathbf{Vx}\|_2}, \frac{\mathbf{Vx}}{\|\mathbf{Vx}\|_2} \right\rangle \|\mathbf{Vx}\|_2^2 \\ &= \max_{\mathbf{y}\in \mathrm{span}(S_m)} \langle \mathbf{Ay}, \mathbf{y} \rangle \qquad \mathbf{y} = \frac{\mathbf{Vx}}{\|\mathbf{Vx}\|_{\mathbf{2}}} \\ &\le \max_{\substack{\|\mathbf{z}\|_2=1 \\ \mathbf{z}\in\mathbb{R}^n}} \langle \mathbf{Az}, \mathbf{z} \rangle \\ &= \lambda_n \end{aligned} \tag{22}$$

The left hand side of (21) is proved similarly by considering $\min_{\|\mathbf{x}\|_2=1}\langle \mathbf{B}_m\mathbf{x}, \mathbf{x} \rangle$. ■

It follows from Theorem 3 that β_1 and β_m are inner bounds for the extremal eigenvalues of $\mathbf{A}$. Usually fairly good approximations to the eigenvectors of $\mathbf{A}$ are available from numerical techniques, like say the power method with deflation. The following theorem shows how the eigenvalues of $\mathbf{A}$ may be approximated in this sense.

Theorem 4: Let $\{\mathbf{v}_i\}_{i=1}^m$ be good approximations to the eigenvectors $\{\mathbf{u}_i\}_{i=1}^m$ of $\mathbf{A}$. We assume that $\mathbf{v}_i \perp \mathbf{v}_j$, $i \ne j$, and that $\|\mathbf{v}_i\|_2 = 1$, then the eigenvalues of the matrix (19) satisfy $|\beta_i - \lambda_i| = \mathcal{O}(\varepsilon^2)$,where ε is a small parameter described in the proof below.

Proof: It is sufficient to write

$$\mathbf{v}_i = \frac{\mathbf{u}_i + \epsilon_i \mathbf{u}_i^{\perp}}{\sqrt{1 + \epsilon_i^2}}, \tag{23}$$

where we have assumed that $\|\mathbf{u}_i\|_2 = \|\mathbf{u}_i^{\perp}\|_2 = 1$ and $|\epsilon_i| < 1$, thus $\mathbf{v}_i$ is normalized. From

$$\mathbf{A}\mathbf{v}_i = \frac{\lambda_i \mathbf{u}_i + \epsilon_i \mathbf{A}\mathbf{u}_i^{\perp}}{\sqrt{1 + \epsilon_i^2}}, \tag{24}$$

we have

$$\begin{aligned} &\sqrt{1 + \epsilon_i^2}\sqrt{1 + \epsilon_j^2}\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j \rangle \\ &= \lambda_i \langle \mathbf{u}_i, \mathbf{u}_j \rangle + \lambda_i \epsilon_j \langle \mathbf{u}_i, \mathbf{u}_j^{\perp} \rangle + \lambda_j \epsilon_i \langle \mathbf{u}_i^{\perp}, \mathbf{u}_j \rangle \\ &+ \epsilon_i \epsilon_j \langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp} \rangle. \end{aligned} \tag{25}$$

■

For $j = i$, we have from (21)

$$\begin{aligned} \langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_i \rangle &= \frac{\lambda_i + \epsilon_i^2 \langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_i \perp \rangle}{1 + \epsilon_i^2} \\ &= \lambda_i + (\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_i \perp \rangle - \lambda_i)\epsilon_i^2 \end{aligned} \tag{26}$$

If $\varepsilon = \max_i |\epsilon_i|$, we conclude that

$$|\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_i \rangle - \lambda_i| = \mathcal{O}(\varepsilon^2). \tag{27}$$

For $j \neq i$, we have from (23)

$$\sqrt{1+\epsilon_i^2}\sqrt{1+\epsilon_j^2}\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \epsilon_i\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j\rangle + \epsilon_j\langle \mathbf{u}_i, \mathbf{u}_j^{\perp}\rangle + \epsilon_i\epsilon_j\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle = 0. \quad (28)$$

Let $M = \max\{\lambda_i, \lambda_j\}$ and $m = \min\{\lambda_i, \lambda_j\}$, then from (25) and (28), it follows that

$$\begin{aligned}&\sqrt{1+\epsilon_i^2}\sqrt{1+\epsilon_j^2}\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j\rangle \\ &\leq M(\epsilon_j\langle \mathbf{u}_i, \mathbf{u}_j^{\perp}\rangle + \epsilon_i\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j\rangle) + \epsilon_i\epsilon_j\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle \\ &= \epsilon_i\epsilon_j(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - M\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle. \end{aligned} \quad (29)$$

Similarly

$$\begin{aligned}&\sqrt{1+\epsilon_i^2}\sqrt{1+\epsilon_j^2}\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j\rangle \\ &\geq m(\epsilon_j\langle \mathbf{u}_i, \mathbf{u}_j^{\perp}\rangle + \epsilon_i\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j\rangle) + \epsilon_i\epsilon_j\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle \\ &= \epsilon_i\epsilon_j(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - m\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle. \end{aligned} \quad (30)$$

From (29), we have

$$\begin{aligned}&\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j\rangle \\ &\leq \epsilon_i\epsilon_j(1+\epsilon_i^2)^{-\frac{1}{2}}(1+\epsilon_j^2)^{-\frac{1}{2}}(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - M\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle) \\ &\approx \epsilon_i\epsilon_j(1 - \tfrac{1}{2}\epsilon_i^2 - \tfrac{1}{2}\epsilon_j^2)(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - M\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle) \\ &\approx \epsilon_i\epsilon_j(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - M\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle) \end{aligned} \quad (31)$$

to second order in ϵ_i, ϵ_j. Similarly from (30) one can show that

$$\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j\rangle \geq \epsilon_i\epsilon_j(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - m\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle) \quad (32)$$

Let $\epsilon = \max\{|\epsilon_i|, |\epsilon_j|\}$, then (31) implies that

$$\begin{aligned}\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j\rangle &\leq \epsilon_i\epsilon_j(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - M\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle) \\ &\leq |\epsilon_i||\epsilon_j|(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - M\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle) \\ &\leq \epsilon^2(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - M\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle) \end{aligned} \quad (33)$$

Similarly (32) implies that

$$\begin{aligned}\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j\rangle &\geq -|\epsilon_i||\epsilon_j|(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - m\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle \\ &\geq -\epsilon^2(\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - m\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle) \end{aligned} \quad (34)$$

Let

$$\begin{aligned}K = \max\{&|\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - M\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle|, \\ &|\langle \mathbf{A}\mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle - m\langle \mathbf{u}_i^{\perp}, \mathbf{u}_j^{\perp}\rangle|\},\end{aligned}$$

then $-K\epsilon^2 \leq \langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j\rangle \leq K\epsilon^2$ or $|\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j\rangle| \leq K\epsilon^2$. Recall that $\varepsilon = \max_i|\epsilon_i|$, thus we may conclude that all off diagonal elements of the matrix $\mathbf{B}_m$ are $\mathcal{O}(\varepsilon^2)$. It then follows from Gerschgorin's theorem , for ε small enough, that the Gerschogrin disks are mutually disjoint and that the radii are $\mathcal{O}(\varepsilon^2)$. Hence we may conclude that each circle contains at least one eigenvalue (two or more circles may coincide for identical eigenvalues). Thus we may conclude from (27) that

$$\begin{aligned}|\beta_i - \lambda_i| &\leq |\beta_i - \langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_i\rangle| + |\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_i\rangle - \lambda_i| \\ &= \mathcal{O}(\varepsilon^2). \end{aligned} \quad (35)$$

Theorem 4 provides a useful means of approximating few of the eigenvalues of $\mathbf{A}$ by evaluating the corresponding eigenvalues of the smaller matrix $\mathbf{B}$. In addition it shows that the approximation is $\mathcal{O}(\varepsilon^2)$, thus maintaining good accuracy.

Theorem 5: Let $\mathbf{x} \in \operatorname{span}(S_2)$, $\|\mathbf{x}\|_2 = 1$ where, $S_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$, with $\|\mathbf{v}_1\|_2 = \|\mathbf{v}_2\|_2 = 1$, then

$$\begin{aligned}&\max_{\mathbf{x}\in S_2 \|\mathbf{x}\|_2=1} \langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle \\ &= \tfrac{1}{2}\big[\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle + \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle \\ &+\sqrt{(\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle - \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle)^2 + 4\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_2\rangle^2}\big] \end{aligned} \quad (36)$$

and

$$\begin{aligned}&\min_{\substack{\mathbf{x}\in S_2 \\ \|\mathbf{x}\|_2=1}} \langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle \\ &= \tfrac{1}{2}\big[\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle + \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle \\ &-\sqrt{(\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle - \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle)^2 + 4\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_2\rangle^2}\big] \end{aligned} \quad (37)$$

Proof: Let

$$\mathbf{x} = \frac{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2}{\sqrt{\alpha_1^2 + \alpha_2^2}}, \quad (38)$$

where $\alpha_1, \alpha_2 \neq 0$, then

$$\mathbf{x} = \frac{\frac{\alpha_1}{|\alpha_1|}\mathbf{v}_1 + \frac{\alpha_2}{|\alpha_1|}\mathbf{v}_2}{\sqrt{1 + (\frac{\alpha_2}{\alpha_1})^2}}. \quad (39)$$

We shall assume that $\alpha_1 > 0$ as the proof is similar if $\alpha_1 < 0$, thus (39) may be written as

$$\mathbf{x} = \frac{\mathbf{v}_1 + \alpha\mathbf{v}_2}{\sqrt{1+\alpha^2}}, \quad (40)$$

where $\alpha = \frac{\alpha_2}{\alpha_1}$ is a parameter to be determined. Let $f(\alpha) = \langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle$, then

$$f(\alpha) = \frac{\langle \mathbf{A}\mathbf{v}_1 + \alpha\mathbf{A}\mathbf{v}_2, \mathbf{v}_1 + \alpha\mathbf{v}_2\rangle}{1+\alpha^2} \quad (41)$$

Thus

$$\begin{aligned}&f(\alpha)(1+\alpha^2) \\ &= \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle + 2\alpha\langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_1\rangle + \alpha^2\langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle \end{aligned} \quad (42)$$

Differentiate (42) with respect to α and set $f'(\alpha) = 0$, to get

$$\alpha f(\alpha) = \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_1\rangle + \alpha\langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle \quad (43)$$

Using (42) in (43), we get

$$\begin{aligned}&\alpha[\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle + 2\alpha\langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_1\rangle + \alpha^2\langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle] \\ &=\langle \mathbf{A}\mathbf{v}_2,\mathbf{v}_1\rangle + \alpha\langle \mathbf{A}\mathbf{v}_2,\mathbf{v}_2\rangle + \alpha^2\langle \mathbf{A}\mathbf{v}_2,\mathbf{v}_1\rangle + \alpha^3\langle \mathbf{A}\mathbf{v}_2,\mathbf{v}_2\rangle \end{aligned} \quad (44)$$

Simplifying (44) yields the quadratic

$$\alpha^2 a + \alpha b - a = 0, \quad (45)$$

where $a = \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_1\rangle$ and $b = \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle - \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle$. Thus

$$\alpha = \frac{-b \pm \sqrt{b^2 + 4a^2}}{2a}. \quad (46)$$

From (42) and (46) it may shown that

$$\begin{aligned}f(\alpha) &= \frac{b + \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle}{1+\alpha^2} + \frac{2\alpha a}{1+\alpha^2} + \frac{\alpha^2\langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle}{1+\alpha^2} \\ &= \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle + \frac{b + 2\alpha a}{1+\alpha^2} \\ &= \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle \pm \frac{\sqrt{b^2 + 4a^2}}{1+\alpha^2} \end{aligned} \quad (47)$$

From (46) and rationalization, it may be shown that

$$\frac{1}{1+\alpha^2} = \frac{2a^2}{\sqrt{b^2+4a^2}(\sqrt{b^2+4a^2} \mp b)} \tag{48}$$

Using (48), equation (47) simplifies to

$$\begin{aligned} f(\alpha) &= \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle \pm \frac{2a^2}{\sqrt{b^2+4a^2} \mp b)} \\ &= \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle \pm \frac{2a^2(\sqrt{b^2+4a^2} \pm b)}{4a^2} \\ &= \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle + \tfrac{1}{2}b \pm \tfrac{1}{2}\sqrt{b^2+4a^2} \\ &= \tfrac{1}{2}\Big[\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle + \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle \\ &\quad \pm\sqrt{(\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle - \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle)^2 + 4\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_2\rangle^2}\Big] \end{aligned} \tag{49}$$

Thus both (36) and (37) are proved. ■

The calculus approach to proving theorem 5 is instinctive, though if one chooses $\mathbf{x} \in S_m$, $m > 2$, then this leads to a non linear system of $m-1$ unknowns which is difficult to solve. The following algebraic approach however holds for all $0 < m \le n$. We shall consider only the case $m = 2$ as an illustration. Since $\mathbf{x} \in S_2$, we have that $\mathbf{x} = \mathbf{V}\mathbf{y}$ for some $\mathbf{y} \in \mathbb{R}^2$. Also $\|\mathbf{x}\|_2 = 1 \implies \|\mathbf{y}\|_2 = 1$. Thus

$$\begin{aligned} \max_{\substack{\mathbf{x}\in S_2\\ \|\mathbf{x}\|_2=1}} \langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle &= \max_{\substack{\mathbf{y}\in\mathbb{R}^2\\ \|\mathbf{y}\|_2=1}} \langle \mathbf{A}\mathbf{V}\mathbf{y}, \mathbf{V}\mathbf{y}\rangle \\ &= \max_{\substack{\mathbf{y}\in\mathbb{R}^2\\ \|\mathbf{y}\|_2=1}} \langle \mathbf{V}^{\mathbf{t}}\mathbf{A}\mathbf{V}\mathbf{y}, \mathbf{y}\rangle \\ &= \max_{\substack{\mathbf{y}\in\mathbb{R}^2\\ \|\mathbf{y}\|_2=1}} \langle \mathbf{B}_2\mathbf{y}, \mathbf{y}\rangle \\ &= \beta_2 \end{aligned} \tag{50}$$

Note that $\mathbf{B}_2$ is a 2×2 matrix and that β_2 is its largest eigenvalues. Replacing **max** by **min** in the above argument yields β_1, the smallest eigenvalue of $\mathbf{B}_2$. Since $\mathbf{B}_2$ has the simple form

$$\mathbf{B}_2 = \begin{bmatrix} \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_1\rangle & \langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_2\rangle \\ \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_1\rangle & \langle \mathbf{A}\mathbf{v}_2, \mathbf{v}_2\rangle \end{bmatrix} \tag{51}$$

the calculation of its eigenvalues is elementary and yields both (36) and (37).

Theorem 6: Let $(\tilde{\lambda}_k, \mathbf{v}_k)$ be a good approximation to $(\lambda_k, \mathbf{u}_k)$, that is $\mathbf{v}_k = \frac{\mathbf{u}_k+\epsilon\mathbf{u}_k^\perp}{\sqrt{1+\epsilon^2}}$ and $\tilde{\lambda}_k = \langle \mathbf{A}\mathbf{v}_k, \mathbf{v}_k\rangle$. Let $\mathbf{w} = \frac{\boldsymbol{\delta}\mathbf{v}_k}{\|\boldsymbol{\delta}\mathbf{v}_k\|_2}$, where the residual $\boldsymbol{\delta}\mathbf{v}_k = \mathbf{A}\mathbf{v}_k - \tilde{\lambda}_k\mathbf{v}_k$, then $\mathbf{v}_k \perp \mathbf{w}$. If further $\|\mathbf{u}_k^\perp - \mathbf{u}_p\| << 1$ for some eigenvector $\mathbf{u}_p$ of $\mathbf{A}$, then $\langle \mathbf{A}\mathbf{w}, \mathbf{w}\rangle \approx \lambda_p$.

Proof: Since $\langle \boldsymbol{\delta}\mathbf{v}_k, \mathbf{v}_k\rangle = \langle \mathbf{A}\mathbf{v}_k, \mathbf{v}_k\rangle - \tilde{\lambda}_k = 0$, we have $\mathbf{w} \perp \mathbf{v}_k$. For the rest of the proof we shall work to $\mathcal{O}(\epsilon^2)$. Now

$$\begin{aligned} \langle \mathbf{A}\mathbf{w}, \mathbf{w}\rangle &= \frac{\langle \mathbf{A}^2\mathbf{v}_k - \tilde{\lambda}_k\mathbf{A}\mathbf{v}_k, \mathbf{A}\mathbf{v}_k - \tilde{\lambda}_k\mathbf{v}_k\rangle}{\|\mathbf{A}\mathbf{v}_k - \tilde{\lambda}_k\mathbf{v}_k\|_2^2} \\ &= \frac{\langle \mathbf{A}^3\mathbf{v}_k, \mathbf{v}_k\rangle - 2\tilde{\lambda}_k\langle \mathbf{A}^2\mathbf{v}_k, \mathbf{v}_k\rangle + \tilde{\lambda}_k^3}{\langle \mathbf{A}^2\mathbf{v}_k, \mathbf{v}_k\rangle - \tilde{\lambda}_k^2} \\ &= \frac{N}{D}, \end{aligned} \tag{52}$$

where N denotes the numerator and D denotes the denominator. It is trivial to show that

$$\begin{aligned} \langle \mathbf{A}^r\mathbf{v}_k, \mathbf{v}_k\rangle &= \frac{\langle \lambda_k^r\mathbf{u}_k + \epsilon\mathbf{A}^r\mathbf{u}_k^\perp, \mathbf{u}_k + \epsilon\mathbf{u}_k^\perp\rangle}{1+\epsilon^2} \\ &= \frac{\lambda_k^r + \epsilon^2\langle \mathbf{A}^r\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle}{1+\epsilon^2} \\ &= \lambda_k^r + (\langle \mathbf{A}^r\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k^r)\epsilon^2. \end{aligned} \tag{53}$$

Thus using (53) N and D simplify to

$$\begin{aligned} N &= \lambda_k^3 + (\langle \mathbf{A}^3\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k^3)\epsilon^2 \\ &\quad - 2[\lambda_k + (\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k)\epsilon^2] \\ &\quad = [\lambda_k^2 + (\langle \mathbf{A}^2\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k^2)\epsilon^2] \\ &\quad + [\lambda_k + (\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k)\epsilon^2]^3 \\ &\quad = [\langle \mathbf{A}^3\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k^3 - 2\lambda_k(\langle \mathbf{A}^2\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k^2) \\ &\quad - 2\lambda_k^2(\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k) \\ &\quad + 3\lambda_k^2(\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k)]\epsilon^2 \\ &\quad = [\langle \mathbf{A}^3\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - 2\lambda_k\langle \mathbf{A}^2\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle + \lambda_k^2\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle]\epsilon^2 \end{aligned} \tag{54}$$

$$\begin{aligned} D &= \lambda_k^2 + (\langle \mathbf{A}^2\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k^2)\epsilon^2 \\ &\quad - [\lambda_k + (\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k)\epsilon^2]^2 \\ &\quad = [\langle \mathbf{A}^2\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k^2 - 2\lambda_k(\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda_k)]\epsilon^2 \\ &\quad = [\langle \mathbf{A}^2\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - 2\lambda_k\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle + \lambda_k^2]\epsilon^2. \end{aligned} \tag{55}$$

Thus (52) simplifies to

$$\begin{aligned} \langle \mathbf{A}\mathbf{w}, \mathbf{w}\rangle &= \frac{\langle \mathbf{A}^3\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - 2\lambda_k\langle \mathbf{A}^2\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle + \lambda_k^2\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle}{\langle \mathbf{A}^2\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - 2\lambda_k\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle + \lambda_k^2} \\ &= \frac{\lambda_p^3 - 2\lambda_k\lambda_p^2 + \lambda_k^2\lambda_p}{\lambda_p^2 - 2\lambda_k\lambda_p + \lambda_k^2} \\ &= \lambda_p \end{aligned} \tag{56}$$

■

Theorem 7: Let $(\tilde{\lambda}_k, \mathbf{v}_k)$ and $\mathbf{w}$ be as defined in theorem 6, then $\langle \mathbf{A}\mathbf{v}_k, \mathbf{w}\rangle$ is $\mathcal{O}(\epsilon)$.

Proof: We shall work to $\mathcal{O}(\epsilon^2)$.

$$\mathbf{v}_k = \tfrac{\mathbf{u}_k+\epsilon\mathbf{u}_k^\perp}{\sqrt{1+\epsilon^2}} \tag{57}$$

$$= \mathbf{u}_k + \epsilon\mathbf{u}_k^\perp - \tfrac{1}{2}\epsilon^2\mathbf{u}_k. \tag{58}$$

$$\begin{aligned} \tilde{\lambda} &= \langle \mathbf{A}\mathbf{v}_k, \mathbf{v}_k\rangle \\ &= \lambda + (\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \lambda)\epsilon^2 \end{aligned} \tag{59}$$

$$\tilde{\lambda}\mathbf{v}_k = \lambda\mathbf{u}_k + \epsilon\lambda\mathbf{u}_k^\perp + \epsilon^2(\langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle - \tfrac{3}{2})\mathbf{u}_k \tag{60}$$

$$\begin{aligned} \mathbf{A}\mathbf{v}_k &= \frac{\lambda\mathbf{u}_k + \epsilon\mathbf{A}\mathbf{u}_k^\perp}{\sqrt{1+\epsilon^2}} \\ &= \lambda\mathbf{u}_k + \epsilon\mathbf{A}\mathbf{u}_k^\perp - \tfrac{1}{2}\epsilon^2\lambda\mathbf{u}_k \end{aligned} \tag{61}$$

$$\begin{aligned} \boldsymbol{\delta}\mathbf{v}_k &= \mathbf{A}\mathbf{v}_k - \tilde{\lambda}_k\mathbf{v}_k \\ &= \epsilon(\mathbf{A}\mathbf{u}_k^\perp - \lambda\mathbf{u}_k^\perp) + \epsilon^2(\lambda - \langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle)\mathbf{u}_k \end{aligned} \tag{62}$$

It follows that

$$\|\boldsymbol{\delta}\mathbf{v}_k\|_2 = |\epsilon|\|\mathbf{A}\mathbf{u}_k^\perp - \lambda\mathbf{u}_k^\perp\|_2 \tag{63}$$

Thus from (62) and (63) it follows that

$$\begin{aligned} \mathbf{w} &= \frac{\boldsymbol{\delta}\mathbf{v}_k}{\|\boldsymbol{\delta}\mathbf{v}_k\|_2} \\ &= \frac{\pm(\mathbf{A}\mathbf{u}_k^\perp - \lambda\mathbf{u}_k^\perp) + |\epsilon|(\lambda - \langle \mathbf{A}\mathbf{u}_k^\perp, \mathbf{u}_k^\perp\rangle)\mathbf{u}_k}{\|\mathbf{A}\mathbf{u}_k^\perp - \lambda\mathbf{u}_k^\perp\|_2}, \end{aligned} \tag{64}$$

where we use the $+$ sign for $\epsilon > 0$ and $-$ sign for $\epsilon < 0$. Using (61) results in

$$\begin{aligned}\langle \mathbf{Av}, \mathbf{w}\rangle &= \frac{|\epsilon|\lambda^2 - \lambda\langle \mathbf{Au}_k^\perp, \mathbf{u}_k^\perp\rangle(|\epsilon| \pm \epsilon) \pm \epsilon\langle \mathbf{Au}_k^\perp, \mathbf{Au}_k^\perp\rangle}{\|\mathbf{Au}_k^\perp - \lambda\mathbf{u}_k^\perp\|_2} \\ &= \pm\epsilon\frac{\|\mathbf{Au}_k^\perp - \lambda\mathbf{u}_k^\perp\|_2^2}{\|\mathbf{Au}_k^\perp - \lambda\mathbf{u}_k^\perp\|_2} \\ &= \pm\epsilon\|\mathbf{Au}_k^\perp - \lambda\mathbf{u}_k^\perp\|_2 \end{aligned} \tag{65}$$

■

Theorem 8: Let $\widetilde{\mathbf{w}} \perp \mathbf{v}_k$ be any normalized vector, then

$$|\langle \mathbf{Av}_k, \mathbf{w}\rangle| \leq |\epsilon|\|\mathbf{Au}_k^\perp - \lambda\mathbf{u}_k^\perp\|_2 \tag{66}$$

Proof: If $\mathbf{x} \perp \mathbf{v}$, then let $\widetilde{\mathbf{w}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ and from (57) it follows that

$$\langle \mathbf{u}_k + \epsilon\mathbf{u}_k^\perp, \mathbf{x}\rangle = 0 \tag{67}$$

$$\langle \mathbf{u}_k, \mathbf{x}\rangle = -\epsilon\langle \mathbf{u}_k^\perp, \mathbf{x}\rangle \tag{68}$$

Thus using (58), (68) and retaining at most terms of $\mathcal{O}(\epsilon^2)$ yields

$$\langle \mathbf{Av}_k, \widetilde{\mathbf{w}}\rangle = \frac{\langle \lambda\mathbf{u}_k + \epsilon\mathbf{Au}_k^\perp - \frac{1}{2}\epsilon^2\lambda\mathbf{u}_k, \mathbf{x}\rangle}{\|\mathbf{x}\|_2} \tag{69}$$

$$\begin{aligned} &= \frac{\lambda\langle \mathbf{u}_k, \mathbf{x}\rangle + \epsilon\langle \mathbf{Au}_k^\perp, \mathbf{x}\rangle - \frac{1}{2}\epsilon^2\lambda\langle \mathbf{u}_k, \mathbf{x}\rangle}{\|\mathbf{x}\|_2} \\ &= \frac{\epsilon(\langle \mathbf{Au}_k^\perp, \mathbf{x}\rangle - \lambda\langle \mathbf{u}_k^\perp, \mathbf{x}\rangle)}{\|\mathbf{x}\|_2} \\ &= \frac{\epsilon\langle \mathbf{Au}_k^\perp - \lambda\mathbf{u}_k^\perp, \mathbf{x}\rangle}{\|\mathbf{x}\|_2} \end{aligned} \tag{70}$$

The result then follows by taking the modulus and applying the Cauchy-Schwarz inequality. ■

There are many possibilities for choosing $\tilde{\mathbf{w}}$. Suppose that $n = 2m$ and $\mathbf{v}_k^t = [\mathbf{v}_1^t\ \mathbf{v}_2^t]$, or $n = 2m+1$ and $\mathbf{v}_k^t = [\mathbf{v}_1^t\ \mu\ \mathbf{v}_2^t]$ where $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors of length m, then $\tilde{\mathbf{w}}_k^t = [\mathbf{v}_2^t\ -\mathbf{v}_1^t]$ and $\tilde{\mathbf{w}}_k^t = [\mathbf{v}_2^t\ 0\ -\mathbf{v}_1^t]$ will suffice.

III. Bounds

From (53) it is simple to show that for $r = 1$

$$\begin{aligned}|\lambda_k - \langle \mathbf{Av}_k, \mathbf{v}_k\rangle| &= |\lambda_k - \langle \mathbf{Au}_k^\perp, \mathbf{u}_k^\perp\rangle|\epsilon^2 \\ &\leq (|\lambda_k| + \rho(\mathbf{A}))\epsilon^2\end{aligned} \tag{71}$$

Equation (71), although $\mathcal{O}(\epsilon^2)$, is of not much use bounding λ_k due to the presence of unknowns on the right hand side. However if we use the results of theorem 6,8 and 9, then we may bound λ_k

Theorem 9:

1) If $\langle \mathbf{Av}, \mathbf{v}\rangle > \langle \mathbf{Aw}, \mathbf{w}\rangle$, the β^+ is closest to $\langle \mathbf{Av}, \mathbf{v}\rangle$ with $\langle \mathbf{Av}, \mathbf{v}\rangle < \beta^+$
2) If $\langle \mathbf{Av}, \mathbf{v}\rangle < \langle \mathbf{Aw}, \mathbf{w}\rangle$, the β^- is closest to $\langle \mathbf{Av}, \mathbf{v}\rangle$ with $\beta^- < \langle \mathbf{Av}, \mathbf{v}\rangle$

Proof: We shall prove (1), as (2) is proved in a similar manner.

$$\begin{aligned}&|\beta^\pm - \langle \mathbf{Av}, \mathbf{v}\rangle| \\ &= \tfrac{1}{2}|\langle \mathbf{Aw}, \mathbf{w}\rangle - \langle \mathbf{Av}, \mathbf{v}\rangle \\ &\pm\sqrt{(\langle \mathbf{Av}, \mathbf{v}\rangle - \langle \mathbf{Aw}, \mathbf{w}\rangle)^2 + 4\langle \mathbf{Av}, \mathbf{w}\rangle^2}|\end{aligned} \tag{72}$$

Clearly if $\langle \mathbf{Aw}, \mathbf{w}\rangle - \langle \mathbf{Av}, \mathbf{v} < 0$, then (72) is minimized by using the positive sign prepending the radical, so that β^+ is closest to $\langle \mathbf{Av}, \mathbf{v}\rangle$ and

$$\begin{aligned}\beta^+ - \langle \mathbf{Av}, \mathbf{v}\rangle = \tfrac{1}{2}\Big(&\sqrt{(\langle \mathbf{Av}, \mathbf{v}\rangle - \langle \mathbf{Aw}, \mathbf{w}\rangle)^2 + 4\langle \mathbf{Av}, \mathbf{w}\rangle^2} \\ &-(\langle \mathbf{Av}, \mathbf{v}\rangle - \langle \mathbf{Aw}, \mathbf{w}\rangle)\Big) \\ &> 0.\end{aligned} \tag{73}$$

■

From Gerschgorin's theorem it follows that whether $\beta = \beta^+$ or $\beta = \beta^-$ is closest to that$\langle \mathbf{Av}, \mathbf{v}\rangle$, that

$$|\beta - \langle \mathbf{Av}, \mathbf{v}\rangle| \leq |\langle \mathbf{Av}, \mathbf{w}\rangle| \tag{74}$$

Now

$$|\lambda - \beta| \leq |\lambda - \langle \mathbf{Av}, \mathbf{v}\rangle| + |\beta - \langle \mathbf{Av}, \mathbf{v}\rangle| \tag{75}$$

We note from (59) that the first term is $\mathcal{O}(\epsilon^2)$. We shall show that the second term is also $\mathcal{O}(\epsilon^2)$, specifically for β^+. Let $\gamma = \langle \mathbf{Av}, \mathbf{v}\rangle - \langle \mathbf{Aw}, \mathbf{w}\rangle > 0$ in (73), then

$$\begin{aligned}\beta^+ - \langle \mathbf{Av}, \mathbf{v}\rangle &= \tfrac{1}{2}\left(\sqrt{\gamma^2 + 4\langle \mathbf{Av}, \mathbf{w}\rangle^2} - \gamma\right) \\ &= \tfrac{1}{2}\gamma\left[\left(1 + \frac{4\langle \mathbf{Av}, \mathbf{w}\rangle^2}{\gamma}\right)^{\frac{1}{2}} - 1\right] \\ &\approx \tfrac{1}{2}\gamma\left(\frac{2\langle \mathbf{Av}, \mathbf{w}\rangle^2}{\gamma^2}\right) \\ &= \frac{\langle \mathbf{Av}, \mathbf{w}\rangle^2}{\gamma} \\ &= \mathcal{O}(\epsilon^2).\end{aligned} \tag{76}$$

Since a similar proof may be applied for β^-, we conclude that $|\lambda - \beta| = \mathcal{O}(\epsilon^2)$. Thus from (74) we have

$$\langle \mathbf{Av}, \mathbf{v}\rangle - |\langle \mathbf{Av}, \mathbf{w}\rangle| \leq \beta \leq \langle \mathbf{Av}, \mathbf{v}\rangle + |\langle \mathbf{Av}, \mathbf{w}\rangle| \tag{77}$$

Thus for $\epsilon << 1$ we may replace β by λ to get

$$\langle \mathbf{Av}, \mathbf{v}\rangle - |\langle \mathbf{Av}, \mathbf{w}\rangle| \leq \lambda \leq \langle \mathbf{Av}, \mathbf{v}\rangle + |\langle \mathbf{Av}, \mathbf{w}\rangle| \tag{78}$$

In view of theorem 7 and 8 , we conclude that sharper bounds should be obtained for $\mathbf{w} \neq \widetilde{\mathbf{w}}$.

IV. Results

We shall consider two examples regarding matrices taken from taken from [4]. The exact eigenvalues and eigenvectors are computed using Julia 1.10.5. Each eigenvector $\mathbf{u}_i$ is perturbed by $0.01 \times$ **rand**(n), where **rand**(n) denotes a random vector of order n. The resulting vector is then normalized to give $\mathbf{v}_i$. By taking the inner product with $\mathbf{u}_i$ in (23) it is simple to show that

$$\epsilon_i = \sqrt{\frac{1}{\langle \mathbf{v}_i, \mathbf{u}_i\rangle} - 1} \tag{79}$$

Example 1: Consider the matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & -2 & 0 & -2 & 5 \\ 1 & 6 & -3 & 2 & 0 & 6 \\ -2 & -3 & 8 & -5 & -6 & 0 \\ 0 & 2 & -5 & 5 & 1 & -2 \\ -2 & 0 & -6 & 1 & 6 & -3 \\ 5 & 6 & 0 & -2 & -3 & 8 \end{bmatrix}$$

The matrix has three distinct eigenvalues each of algebraic multiplicity two. The eigenvalues together with the corresponding perturbation parameter ϵ_i, of the exact eigenvectors is summarized in table I. In table II the bounding intervals using $\tilde{\mathbf{w}}$ and $\mathbf{w}$ are shown.

TABLE I
λ AND ϵ, EXAMPLE 1

Eigenvalue	Epsilon
-1.598734	0.016015
4.455990	0.016196
16.142745	0.012149

TABLE II
BOUNDS, EXAMPLE 1

Bounds(w)	Bounds($\widetilde{\text{w}}$)
$[-1.676944, -1.518775]$	$[-1.796466, -1.400970]$
$[\ 4.349058,\ \ 4.561257]$	$[\ 4.306216,\ \ 4.601440]$
$[\ 16.102869, 16.178121]$	$[\ 15.939010, 16.346463]$

Example 2: Consider the matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The matrix has distinct eigenvalues. The eigenvalues together with the corresponding perturbation parameter ϵ_i, of the exact eigenvectors is summarized in table III. In table IV the bounding intervals using $\tilde{\mathbf{w}}$ and $\mathbf{w}$ are shown. An exact expression for the eigenvalues is known [4] and given below.

$$\lambda_i = \frac{1}{2}\left[1 - \cos\left(\frac{(2i-1)\pi}{15}\right)\right]^{-1}, \quad i = 1, 2, \cdots, 7.$$

TABLE III
λ AND ϵ, EXAMPLE 2

Eigenvalue	Epsilon
0.261295	0.017217
0.299557	0.017225
0.381966	0.017225
0.558365	0.016614
1.000000	0.016623
2.618034	0.017204
22.880783	0.006299

Example 3: Consider the positive definite heptadiagonal

TABLE IV
BOUNDS, EXAMPLE 2

Bounds(w)	Bounds($\widetilde{\text{w}}$)
$[\ 0.239662,\ 0.288647]$	$[-0.102376,\ 0.625006]$
$[\ 0.252151,\ 0.348060]$	$[-0.063277,\ 0.662429]$
$[\ 0.368628,\ 0.404184]$	$[\ 0.020379,\ 0.743585]$
$[\ 0.541704,\ 0.585187]$	$[\ 0.201288,\ 0.915461]$
$[\ 0.789894,\ 1.209886]$	$[\ 0.649961,\ 1.350025]$
$[\ 2.374878,\ 2.860885]$	$[\ 2.291871,\ 2.944032]$
$[22.813824, 22.946412]$	$[\ 22.741855, 23.019710]$

matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 1 & 1 & & & & & & & \\ 2 & 6 & 3 & 1 & 1 & & & & & & \\ 1 & 3 & 6 & 3 & 1 & 1 & & & & & \\ 1 & 1 & 3 & 6 & 3 & 1 & 1 & & & & \\ & 1 & 1 & 3 & 6 & 3 & 1 & 1 & & & \\ & & 1 & 1 & 3 & 6 & 3 & 1 & 1 & & \\ & & & 1 & 1 & 3 & 6 & 3 & 1 & 1 & \\ & & & & 1 & 1 & 3 & 6 & 3 & 1 & 1 \\ & & & & & 1 & 1 & 3 & 6 & 3 & 1 \\ & & & & & & 1 & 1 & 3 & 6 & 2 \\ & & & & & & & 1 & 1 & 2 & 5 \end{bmatrix}$$

The eigenvalues together with the corresponding perturbation parameter ϵ_i, of the exact eigenvectors is summarized in table V. In table VI the bounding intervals using $\tilde{\mathbf{w}}$ and $\mathbf{w}$ are shown.

TABLE V
λ AND ϵ, EXAMPLE 3

Eigenvalue	Epsilon
0.522282	0.021184
1.803848	0.020927
3.171573	0.021179
4.000000	0.021085
4.0000000	0.020553
4.129248	0.021181
4.406650	0.020875
6.000000	0.021119
8.828427	0.020571
12.196152	0.021044
14.941819	0.009671

Example 4: Our final example is the famous Lehmer matrix $\mathbf{A}$, where

$$a_{ij} = \frac{\min(i,j)}{\max(i,j)}.$$

This is a real symmetric positive definite matrix. Here we use order $n = 10$. The eigenvalues together with the corresponding perturbation parameter ϵ_i, of the exact eigenvectors is summarized in table VII. In table VIII the bounding intervals using $\tilde{\mathbf{w}}$ and $\mathbf{w}$ are shown.

TABLE VI
BOUNDS, EXAMPLE 3

Bounds(w)	Bounds($\widetilde{w}$)
$[\ 0.526975,\ 0.528967]$	$[\ 0.242471,\ 0.802380]$
$[\ 1.797944,\ 1.820540]$	$[\ 1.550785,\ 2.057141]$
$[\ 3.175254,\ 3.176891]$	$[\ 2.944870,\ 3.398416]$
$[\ 3.992056,\ 4.014621]$	$[\ 3.790134,\ 4.209923]$
$[\ 4.000111,\ 4.068231]$	$[\ 3.790780,\ 4.209276]$
$[\ 4.130921,\ 4.137039]$	$[\ 3.921557,\ 4.336982]$
$[\ 4.406327,\ 4.414866]$	$[\ 4.203549,\ 4.609756]$
$[\ 5.955749,\ 6.047857]$	$[\ 5.828303,\ 6.171400]$
$[\ 8.826198,\ 8.834651]$	$[\ 8.704952,\ 8.949913]$
$[12.155654, 12.233509]$	$[12.116111, 12.280658]$
$[14.937480, 14.944540]$	$[14.852639, 15.030891]$

TABLE VII
λ AND ϵ, EXAMPLE 4

Eigenvalue	Epsilon
0.066657	0.019956
0.090525	0.01984
0.122013	0.019869
0.166994	0.020008
0.235515	0.019869
0.346963	0.019566
0.539329	0.019987
0.896943	0.019589
1.776579	0.019957
5.758482	0.007304

TABLE VIII
BOUNDS, EXAMPLE 4

Bounds(w)	Bounds($\widetilde{w}$)
$[0.067642, 0.069662]$	$[-0.039554, 0.172898]$
$[0.069131, 0.110762]$	$[-0.015671, 0.196748]$
$[0.086807, 0.151175]$	$[\ 0.016935, 0.227116]$
$[0.137391, 0.193049]$	$[\ 0.062540, 0.271469]$
$[0.228430, 0.245543]$	$[\ 0.132070, 0.338974]$
$[0.322741, 0.372631]$	$[\ 0.246271, 0.447660]$
$[0.489283, 0.569278]$	$[\ 0.441890, 0.636752]$
$[0.854561, 0.971002]$	$[\ 0.806377, 0.987447]$
$[1.705401, 1.846259]$	$[\ 1.701522, 1.851381]$
$[5.719552, 5.797409]$	$[\ 5.101889, 6.717488]$

V. CONCLUSION

It is evident from Table II, Table IV, Table VI and Table VIII, that sharper bounds are obtained by using $\mathbf{w}$ than $\tilde{\mathbf{w}}$. This validates Theorems 7 and 8. We have thus provided a relatively simple means for bounding eigenvalues of real symmetric matrices, provided that approximations to their eigenvectors are known.

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