Brunn-Minkowski Inequalities of Dual Harmonic Quermassintegrals

Chao Li and Dandan Ren

***Abstract*—The notion of harmonic quermassintegrals was introduced by Hadwiger. Later, Yuan, Yuan and Leng proposed the concept of dual harmonic quermassintegrals for star bodies. We derive several Brunn-Minkowski type inequalities for dual harmonic quermassintegrals associated with Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms.**

***Index Terms*—dual harmonic quermassintegral, Brunn-Minkowski type inequality, Blaschke-Minkowski homomorphism, radial Blaschke-Minkowski homomorphism**

I. Introduction

LET $\mathcal{K}^n$ denote a set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbf{R}^n$. Let $\mathcal{K}_o^n$ denote the set of convex bodies in $\mathbf{R}^n$ that contain the origin in their interiors. The set of star bodies in $\mathbf{R}^n$ is denoted by $\mathcal{S}^n$. Let S^{n-1} denote the unit sphere, $\mathrm{vol}_i(K \mid \zeta)$ denote the i-dimensional volume of the orthogonal projection of K onto an i-dimensional subspace $\zeta \in \mathbf{R}^n$, and k_n denote the volume of the unit ball B_n in $\mathbf{R}^n$.

The concept of quermassintegrals plays an important role in convex geometry analysis. It is defined as follows: For $K \in \mathcal{K}^n$ and $0 \le i \le n$, the quermassintegrals, $W_{n-i}(K)$, of K is defined by

$$W_{n-i}(K) = k_n \int_{G(n,i)} \frac{\mathrm{vol}_i(K \mid \zeta)}{k_i} d\mu_i(\zeta). \tag{1}$$

Here, the Grassmann manifold $G(n,i)$ is endowed with the normalized Haar measure. The quermassintegrals are generalizations of the surface area and the volume. Moreover, if $i = n-1$ or $i = n$ in (1), then $nW_1(K)$ is the surface area of K, and $W_0(K)$ is the volume of K.

The dual quermassintegrals for star bodies were introduced by Lutwak [12]. For $K \in \mathcal{S}^n$ and $0 \le i \le n$, the dual quermassintegrals, $\widetilde{W}_{n-i}(K)$, of K is defined by

$$\widetilde{W}_{n-i}(K) = k_n \int_{G(n,i)} \frac{\mathrm{vol}_i(K \cap \zeta)}{k_i} d\mu_i(\zeta). \tag{2}$$

Here, $\mathrm{vol}_i(K \cap \zeta)$ denotes the i-dimensional volume of slice of K by an i-dimensional subspace $\zeta \in \mathbf{R}^n$. If $i = n$ or $i = 0$ in (2), then $\widetilde{W}_0(K) = \mathrm{vol}(K)$ denotes the n-dimensional volume of the body K, $\widetilde{W}_n(K) = k_n$.

The study of quermassintegrals of convex bodies and dual quermassintegrals of star bodies has attracted extensive attention in convex geometry. The book [20, Chapters 5] by Schneider provides a comprehensive account of the classical Brunn-Minkowski theory and its recent developments.In addition, for further discussions on quermassintegrals and dual quermassintegrals, as well as their latest research results, see [10], [15], [16], [21], [27], [28], [29].

The concept of harmonic quermassintegrals, introduced by Hadwiger [9, page 267], can be defined as follows: Let $K \in \mathcal{S}^n$ and $0 \le i \le n$, the harmonic quermassintegrals, $\widehat{W}_{n-i}(K)$, of K is defined by

$$\widehat{W}_{n-i}(K) = k_n \left[\int_{G(n,i)} \left(\frac{\mathrm{vol}_i(K \mid \zeta)}{k_i} \right)^{-1} d\mu_i(\zeta) \right]^{-1}. \tag{3}$$

If $i = n$ or $i = 0$ in (3), then $\widehat{W}_0(K) = \mathrm{vol}(K)$, $\widehat{W}_n(K) = k_n$.

At the same time, Hadwiger obtained the Brunn-Minkowski inequality for harmonic quermassintegrals. Later, Lutwak [13] established the Blaschke-Santaió inequality and the affine inequality for harmonic quermassintegrals. Recently, Ji and Zeng [18] introduced the notion of Orlicz mixed harmonic quermassintegrals, proved the variational formula with respect to the Orlicz combination, and established the Minkowski-type inequality and the Brunn-Minkowski-type inequality for Orlicz mixed harmonic quermassintegrals.

The concept of dual harmonic quermassintegrals is the dual of harmonic quermassintegrals. The dual quermassintegrals were first introduced by Yuan, Yuan and Leng [22] as follows: For $K \in \mathcal{S}^n$ and $0 \le i \le n$, the dual harmonic quermassintegrals, $\breve{W}_{n-i}(K)$, of K is defined by

$$\breve{W}_{n-i}(K) = k_n \left[\int_{G(n,i)} \left(\frac{\mathrm{vol}_i(K \cap \zeta)}{k_i} \right)^{-1} d\mu_i(\zeta) \right]^{-1}. \tag{4}$$

Specially, if $i = n$ or $i = 0$ in (4), then $\breve{W}_0(K) = \mathrm{vol}(K)$, $\breve{W}_n(K) = k_n$. In addition, from the Schwarz or Hölder inequality, Yuan, Yuan and Leng [22] obtained: For $K \in \mathcal{S}^n$ and $0 \le i \le n$, then

$$\breve{W}_i(K) \le \widetilde{W}_i(K), \tag{5}$$

with equality if and only if L is of constant $(n-i)$-section.

Meanwhile, Yuan, Yuan and Leng [22] are also introduced the concept of mixed p-dual harmonic quermassintegrals as follows: Let $K, L \in \mathcal{K}_o^n$, $\zeta \in G(n,i)$, and $0 \le p \le i$, the mixed p-dual harmonic quermassintegrals $\breve{W}_{p,n-i}(K,L)$, of K and L is defined by

$$\breve{W}_{p,n-i}(K,L) = k_n \left[\int_{G(n,i)} \left(\frac{V_{p,i}(K,L;\zeta)}{k_i} \right)^{-1} d\mu_i(\zeta) \right]^{-1}, \tag{6}$$

where

$$V_{p,i}(K,L;\zeta) = \mathrm{vol}(K \cap \zeta, i-p; L \cap \zeta, p). \tag{7}$$

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If $p=1$ in (6), then $\breve{W}_{p,n-i}(K,L)=\breve{W}_i(K,L)$. If $L=K$ and $0\le p\le n-i$ in (6), then $\breve{W}_{p,i}(K,K)=\breve{W}_i(K)$ and $\breve{W}_{n-i,i}(K,L)=\breve{W}_i(L)$.

Recently, Ma and Wang [17] extended the notion of dual harmonic mixed quermassintegrals from the classical Brunn-Minkowski theory to Orlicz dual harmonic mixed quermassintegrals in the Orlicz-Brunn-Minkowski theory. They established the dual Minkowski isoperimetric inequality and the dual Brunn-Minkowski inequality for dual Orlicz harmonic mixed quermassintegrals (see also [30]).

The projection bodies were introduced by Minkowski at the turn of the previous century. For $K\in\mathcal{K}^n$, the projection body, ΠK of $K$is an origin-symmetric convex body whose support function is defined by

$$h(\Pi K,u)=\frac{1}{2}\int_{S^{n-1}}|u\cdot v|dS(K,v),$$

for all $u\in S^{n-1}$. Here, $S(K,\cdot)$ denotes the surface area measure of K.

The projection body is a central object of study in the Brunn-Minkowski theory. A great deal of results are collected in two excellent books (see [8], [20]). In 2006, based on the properties of projection bodies, Schuster [19] introduced the notion of Blaschke-Minkowski homomorphisms as follows:

Definition 1.A. *A map $\Phi:\mathcal{K}^n\to\mathcal{K}^n$ is called a Blaschke-Minkowski homomorphisms if it satisfies the following conditions:*

(a) Φ is continuous.

(b) For all $K,L\in\mathcal{K}^n$,

$$\Phi(K\#L)=\Phi K+\Phi L. \tag{8}$$

(c) For all $K\in\mathcal{K}^n$ and every $\vartheta\in SO(n)$, $\Phi(\vartheta K)=\vartheta\Phi K$.

Here, $SO(n)$ is the group of rotations in n dimensions. $\Phi K+\Phi L$ denotes the Minkowski sum of ΦK and ΦL, $K\#L$ denotes the Blaschke addition of convex bodies K and L. Additionally, we denote the polar of ΦK for the polar of Φ^*K.

The intersection body is the dual form of the projection body. The concept of intersection bodies was first introduced by Lutwak [14], and its definition is given by: for $K\in\mathcal{S}^n$, the intersection body, IK, of K is a star body whose radial function in the direction $u\in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^\perp$, the hyperplane orthogonal to u, i.e., for all $u\in S^{n-1}$,

$$\rho(IK,u)=\mathrm{vol}_{n-1}(K\cap u^\perp),$$

where vol_{n-1} denotes $(n-1)$-dimensional volume.

In recent years, intersection bodies and their generalizations in the Brunn-Minkowski theory have attracted increased attention (see [8], [20]). Later, based on the properties of the well-known intersection operators, Schuster [19] introduced a special class of valuations: radial Blaschke-Minkowski homomorphisms, which are defined as follows.

Definition 1.B. *A map $\Psi:\mathcal{S}^n\to\mathcal{S}^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:*

(d) Ψ is continuous.

(e) For all $K,L\in\mathcal{S}^n$,

$$\Psi(K\widehat{+}L)=\Psi K\widetilde{+}\Psi L. \tag{9}$$

(f) For all $K\in\mathcal{S}^n$ and every $\vartheta\in SO(n)$, $\Psi(\vartheta K)=\vartheta\Psi K$.

Here, $\Psi K\widetilde{+}\Psi L$ denotes the radial addition of ΨL and ΨL, $K\widehat{+}L$ denotes the radial Blaschke addition of convex bodies K and L.

In addition, Schuster investigated Busemann-Petty type problems for Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms. Later, Wang [23], [24] extended the Blaschke-Minkowski homomorphism and the radial Blaschke-Minkowski homomorphism to the L_p space and studied their Busemann-Petty type problems. In 2020, Wang [25] provided a lower bound for the dual quermassintegrals of mixed radial Blaschke-Minkowski homomorphisms. Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms have attracted considerable interest; see, for example, [2], [3], [4], [5], [6], [11], [26].

The purpose of this paper is to establish Brunn-Minkowski type inequalities for dual harmonic quermassintegrals of Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms, respectively. First, we establish the following Brunn-Minkowski type inequality for Blaschke-Minkowski homomorphisms.

Theorem 1.1. *Let $\Phi:\mathcal{K}^n\to\mathcal{K}^n$ be a Blaschke-Minkowski homomorphism. If $K,L\in\mathcal{K}^n_o$, and $0\le i<n$, then*

$$\breve{W}_i(\Phi(K\#L))^{\frac{1}{n-i}}\ge\breve{W}_i(\Phi K)^{\frac{1}{n-i}}+\breve{W}_i(\Phi L)^{\frac{1}{n-i}}, \tag{10}$$

with equality if and only if ΦK and ΦL are dilates.

Note that the special case $\Phi=\Pi$ of Theorem 1.1 provide a new Brunn-Minkowski inequality for the dual harmonic quermassintegrals of projection bodies.

Corollary 1.1. *If $K,L\in\mathcal{K}^n_o$, and $0\le i<n$, then*

$$\breve{W}_i(\Pi(K\#L))^{\frac{1}{n-i}}\ge\breve{W}_i(\Pi K)^{\frac{1}{n-i}}+\breve{W}_i(\Pi L)^{\frac{1}{n-i}},$$

with equality if and only if ΠK and ΠL are dilates.

Specially, if $i=0$ in Corollary 1.1, we obtain the following result.

Corollary 1.2. *Let $K,L\in\mathcal{K}^n_o$, then*

$$\mathrm{vol}(\Pi(K\#L))^{\frac{1}{n}}\ge\mathrm{vol}(\Pi K)^{\frac{1}{n}}+\mathrm{vol}(\Pi L)^{\frac{1}{n}},$$

with equality if and only if ΠK and ΠL are dilates.

In particular, if ΠK have constant $(n-i)$-section, then, we have that $\breve{W}_i(\Pi K)=\widetilde{W}_i(\Pi K)$, by the equality condition of (5), then Corollary 1.1 yields:

Corollary 1.3. *Let $K,L\in\mathcal{K}^n_o$, ΠK and ΠL have constant $(n-i)$-section, if $0\le i<n$, then*

$$\widetilde{W}_i(\Pi(K\#L))^{\frac{1}{n-i}}\ge\widetilde{W}_i(\Pi K)^{\frac{1}{n-i}}+\widetilde{W}_i(\Pi L)^{\frac{1}{n-i}},$$

with equality if and only if ΠK and ΠL are dilates.

The next theorem shows that polar of Blaschke-Minkowski homomorphisms also satisfy a Brunn-Minkowski inequality.

Theorem 1.2. *Let $\Phi:\mathcal{K}^n\to\mathcal{K}^n$ be a Blaschke-Minkowski homomorphism. If $K,L\in\mathcal{K}^n_o$ and $0\le i<n$, then*

$$\breve{W}_i(\Phi^*(K\#L))^{-\frac{1}{n-i}}\ge\breve{W}_i(\Phi^*K)^{-\frac{1}{n-i}}+\breve{W}_i(\Phi^*L)^{-\frac{1}{n-i}}, \tag{11}$$

with equality if and only if Φ^*K and Φ^*L are dilates.

Since the projection body is a special example of a Blaschke-Minkowski homomorphism, by Theorem 1.2, we obtain the following Brunn-Minkowski inequality for polar projection bodies.

Corollary 1.4. *If $K, L \in \mathcal{K}_o^n$ and $0 \le i < n$, then*

$$\breve{W}_i(\Pi^*(K\#L))^{-\frac{1}{n-i}} \ge \breve{W}_i(\Pi^* K)^{-\frac{1}{n-i}} + \breve{W}_i(\Pi^* L)^{-\frac{1}{n-i}},$$

with equality if and only if $\Pi^* K$ and $\Pi^* L$ are dilates.

In addition, if $M = \Pi K$ and $N = \Pi L$, then $(M+N)^* = \Pi^*(K\#L)$, since $\Pi K + \Pi L = \Pi(K\#L)$, then we can obtain the following Brunn-Minkowski inequality for polar of the Minkowski combination.

Corollary 1.5. *If $K, L \in \mathcal{K}_o^n$ and $0 \le i < n$, then*

$$\breve{W}_i((M+N)^*)^{-\frac{1}{n-i}} \ge \breve{W}_i(M^*)^{-\frac{1}{n-i}} + \breve{W}_i(N^*)^{-\frac{1}{n-i}},$$

with equality if and only if M and N are dilates.

Let $i = 0$ in Corollary 1.5, and in relation to $\breve{W}_i(K) = \mathrm{vol}(K)$. Hence, Corollary 1.4 can also be obtained directly by the following classical result of Firey [7].

Corollary 1.6. *If $M, N \in \mathcal{K}_o^n$, then*

$$\mathrm{vol}((M+N)^*)^{-\frac{1}{n}} \ge \mathrm{vol}(M^*)^{-\frac{1}{n}} + \mathrm{vol}(N^*)^{-\frac{1}{n}},$$

with equality if and only if M and N are dilates.

Finally, we establish the following Brunn-Minkowski type inequality of dual harmonic quermassintegrals for the radial Blaschke-Minkowski homomorphisms

Theorem 1.3. *Let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. If $K, L \in \mathcal{S}^n$, for $n-1 < i < n$, then*

$$\breve{W}_i(\Psi(K \hat{+} L))^{\frac{1}{n-i}} \ge \breve{W}_i(\Psi K)^{\frac{1}{n-i}} + \breve{W}_i(\Psi L)^{\frac{1}{n-i}}, \quad (12)$$

with equality if and only if ΨK and ΨL are dilates.

Since the intersection operator I is an example of a radial Blaschke-Minkowski homomorphism, Theorem 1.3 provides the following new Brunn-Minkowski inequality for the dual harmonic quermassintegrals of intersection bodies.

Corollary 1.7. *If $K, L \in \mathcal{S}^n$, for $n-1 < i < n$, then*

$$\breve{W}_i(I(K \hat{+} L))^{\frac{1}{n-i}} \ge \breve{W}_i(IK)^{\frac{1}{n-i}} + \breve{W}_i(IL)^{\frac{1}{n-i}},$$

with equality if and only if IK and IL are dilates.

II. Notations and Background Materials

In this section, some notations and basic facts about convex bodies are presented. For general reference, readers may consult the books by Gardner [8] and Schneider [20].

A. Support function, radial function and polar body

For $K \in \mathcal{K}^n$, its support function, $h_K = h(K, \cdot) : \mathbf{R}^n \to \mathbf{R}$, is defined by

$$h(K, x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbf{R}^n, \quad (13)$$

where $x \cdot y$ denotes the standard inner product of x and y. If K is origin symmetric, then $h(K, \cdot) = h(-K, \cdot)$.

For $K, L \in \mathcal{K}_o^n$ and $\lambda, \mu \ge 0$ (not both zero), the Minkowski combination, $\lambda K + \mu L$, of K and L is defined by

$$h(\lambda K + \mu L, \cdot) = \lambda h(K, \cdot) + \mu h(L, \cdot), \quad (14)$$

where " $+$ " denotes the Minkowski addition and $\lambda K = \{\lambda x : x \in K\}$. Let K be a compact star-shaped set (about the origin) in $\mathbf{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbf{R}^n \setminus \{0\} \to [0, \infty)$, is defined by

$$\rho(K, x) = \max\{\lambda \ge 0 : \lambda x \in K\}, \quad x \in \mathbf{R}^n \setminus \{0\}. \quad (15)$$

If ρ_K is positive and continuous, K will be called a star body (with respect to the origin).

If $K, L \in \mathcal{S}_o^n$, and $\lambda, \mu \ge 0$ (not both zero), the radial linear combination, $\lambda \circ K \tilde{+} \mu \circ L$, of K and L is defined by

$$\rho(\lambda \circ K \tilde{+} \mu \circ L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot). \quad (16)$$

If $K \in \mathcal{K}_o^n$, the polar body, K^*, of K is defined by

$$K^* = \{x \in \mathbf{R}^n : x \cdot y \le 1, y \in K\}. \quad (17)$$

From (13), (15) and (17), it follows that if $K \in \mathcal{K}_o^n$, then

$$h(K^*, \cdot) = \frac{1}{\rho(K, \cdot)}, \quad \rho(K^*, \cdot) = \frac{1}{h(K, \cdot)}. \quad (18)$$

B. Mixed volumes

For $K_1, K_2, \cdots, K_n \in \mathcal{K}^n$, the mixed volumes $\mathrm{vol}(K_1, K_2, \cdots, K_n)$, of $K_1, K_2, \cdots, K_n$ is defined by

$$\begin{aligned} & \mathrm{vol}(K_1, K_2, \cdots, K_n) \\ = \; & \frac{1}{n!} \sum_{j=i}^{n} (-1)^{n+j} \sum_{i_1 < \cdots < i_j} \mathrm{vol}(K_{i_1} + K_{i_2} + \cdots + V_{i_j}). \end{aligned}$$

Next, we give some elementary properties of mixed volumes.

(a) If $K_1, K_2, \ldots, K_{n-1}, K, L \in \mathcal{K}^n$, then

$$\begin{aligned} & \mathrm{vol}(K_1, K_2, \ldots, K_{n-1}, K+L) \\ = \; & \mathrm{vol}(K_1, K_2, \ldots, K_{n-1}, K) + \mathrm{vol}(K_1, K_2, \ldots, K_{n-1}, L). \end{aligned}$$

(b) If $K_1, K_2, \ldots, K_{n-1}, K, L \in \mathcal{K}^n$ and $K \subseteq L$, then

$$\mathrm{vol}(K_1, K_2, \ldots, K_{n-1}, K) \le \mathrm{vol}(K_1, K_2, \ldots, K_{n-1}, L), \quad (19)$$

with equality if and only if $K = L$.

(c) If $K_1, K_2, \ldots, K_{n-1}, K_n, K \in \mathcal{K}^n$, and $K_1 = K_2 = \cdots = K_{n-1} = K_n = K$, then

$$\mathrm{vol}(K_1, K_2, \ldots, K_{n-1}, K_n) = \mathrm{vol}(K, \ldots, K) = \mathrm{vol}(K). \quad (20)$$

If $K_1 = K_2 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$ in (20), we write

$$\mathrm{vol}(K_1, K_2, \cdots, K_n) = \mathrm{vol}(K, n-i; L, i) = \mathrm{vol}_i(K, L). \quad (21)$$

Let $K \in \mathcal{K}^n$, the volume $\mathrm{vol}(K)$ of K is given by (see [8])

$$\mathrm{vol}(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n du. \quad (22)$$

III. Proofs of the Theorems

This section provides the proofs of Theorems 1.1-1.3. To begin, we prove Theorem 1.1 using the following Minkowski inequality for p-dual harmonic quermassintegrals.

Lemma 3.1([22]). *Let $K, L \in \mathcal{K}_o^n$ and $0 \le i < n$, if $0 \le p \le n-i$, then*

$$\breve{W}_{p,i}(K, L)^{n-i} \ge \breve{W}_i(K)^{n-i-p} \breve{W}_i(K)^p, \quad (23)$$

with equality if and only if K and L are dilates.

Proof of Theorem 1.1. Since $K, L \in \mathcal{K}_o^n$, $0 \le i < n$, and $0 \le p \le n-i$, in fact, for $u \in S^{n-1} \cap \zeta$ and $\zeta \in G(n, i)$, by (14), we have

$$\begin{aligned} & h[(K+L) \cap \zeta, u] = h(K+L, u) \\ = \; & h(K, u) + h(L, u) \\ = \; & h(K \cap \zeta, u) + h(L \cap \zeta, u) \\ = \; & h(K \cap \zeta + L \cap \zeta, u), \end{aligned}$$

thus, for all $\zeta \in G(n,i)$, we have

$$(K+L)\cap\zeta=(K\cap\zeta)+(L\cap\zeta). \tag{24}$$

Let $Q\in\mathcal{K}_o^n$ and $p=1$, by (7), (8), (24) and (19), we obtain

$$\begin{aligned}&V_{1,i}(Q,\Phi(K\#L);\zeta)\\=&\mathrm{vol}(Q\cap\zeta,i-1;(\Phi K+\Phi L)\cap\zeta)\\=&\mathrm{vol}(Q\cap\zeta,i-1;(\Phi K\cap\zeta)+(\Phi L\cap\zeta))\\=&\mathrm{vol}(Q\cap\zeta,i-1;\Phi K\cap\zeta)\\&+\mathrm{vol}(Q\cap\zeta,i-1;\Phi L\cap\zeta)\\=&V_{1,i}(Q,\Phi K;\zeta)+V_{1,i}(Q,\Phi L;\zeta),\end{aligned} \tag{25}$$

by (6), (25), Minkowski integral inequality[1] and (23), we deduce

$$\begin{aligned}&\breve{W}_i(Q,\Phi(K\#L))\\=&k_n\Big[\int_{G(n,n-i)}\Big(\frac{V_{1,n-i}(Q,\Phi(K\#L);\zeta)}{k_{n-i}}\Big)^{-1}d\mu_{n-i}(\zeta)\Big]^{-1}\\=&k_n\Big[\int_{G(n,n-i)}\Big(\frac{V_{1,n-i}(Q,\Phi K;\zeta)}{k_{n-i}}\\&+\frac{V_{1,n-i}(Q,\Phi L;\zeta)}{k_{n-i}}\Big)^{-1}d\mu_{n-i}(\zeta)\Big]^{-1}\\\geq&\breve{W}_i(Q,\Phi K)+\breve{W}_i(Q,\Phi L)\\\geq&\breve{W}_i(Q)^{\frac{n-i-1}{n-i}}(\breve{W}_i(\Phi K)^{\frac{1}{n-i}}+\breve{W}_i(\Phi L)^{\frac{1}{n-i}}).\end{aligned} \tag{26}$$

Setting $Q=\Phi(K\#L)$ in (26), by $\breve{W}_{p,i}(K,K)=\breve{W}_i(K)$, we get

$$\breve{W}_i(\Phi(K\#L))^{\frac{1}{n-i}}\geq\breve{W}_i(\Phi K)^{\frac{1}{n-i}}+\breve{W}_i(\Phi L)^{\frac{1}{n-i}},$$

this yields inequality (10).

From the equality condition of (23), we see that equality holds in (26) if and only if Q, ΦK and ΦL are dilates. Thus $\Phi(K\#L)$, ΦK and ΦL are dilates, by (8), we obtain equality in (10) holds if and only if ΦK and ΦL are dilates.

Next, we give the proof of Theorem 1.2, the following lemma is necessary.

Lemma 3.2. *Let $\Phi:\mathcal{K}^n\to\mathcal{K}^n$ be a Blaschke-Minkowski homomorphism. If $K,L\in\mathcal{K}_o^n$ and $0\leq i<n$, then*

$$\begin{aligned}&\mathrm{vol}_{n-i}(\Phi^*(K\#L)\cap\zeta)^{-\frac{1}{n-i}}\\\geq&\mathrm{vol}_{n-i}(\Phi^*K\cap\zeta)^{-\frac{1}{n-i}}+\mathrm{vol}_{n-i}(\Phi^*L\cap\zeta)^{-\frac{1}{n-i}},\end{aligned} \tag{27}$$

with equality if and only if Φ^*K and Φ^*L are dilates.

Proof. Since $K,L\in\mathcal{K}_o^n$ and $0\leq i<n$, by polar coordinate formula for volume, (22), (18) and the Minkowski integral inequality[1], we have

$$\begin{aligned}&\mathrm{vol}_{n-i}(\Phi^*(K\#L)\cap\zeta)^{-\frac{1}{n-i}}\\=&\Big(\frac{1}{n-i}\int_{S^{n-1}}\rho[\Phi^*(K\#L)\cap\zeta,u]^{n-i}du\Big)^{-\frac{1}{n-i}}\\=&\Big(\frac{1}{n-i}\int_{S^{n-1}\cap\zeta}\rho[\Phi^*(K\#L),u]^{n-i}du\Big)^{-\frac{1}{n-i}}\\=&\Big(\frac{1}{n-i}\int_{S^{n-1}\cap\zeta}h[\Phi(K\#L),u]^{-(n-i)}du\Big)^{-\frac{1}{n-i}}\\=&\Big(\frac{1}{n-i}\int_{S^{n-1}\cap\zeta}h[(\Phi K+\Phi L),u]^{-(n-i)}du\Big)^{-\frac{1}{n-i}}\\\geq&\Big(\frac{1}{n-i}\int_{S^{n-1}\cap\zeta}h[(\Phi K)\cap\zeta,u]^{-(n-i)}du\Big)^{-\frac{1}{n-i}}\\&+\Big(\frac{1}{n-i}\int_{S^{n-1}}h[(\Phi L)\cap\zeta,u]^{-(n-i)}du\Big)^{-\frac{1}{n-i}}\\=&\Big(\frac{1}{n-i}\int_{S^{n-1}}\rho[\Phi^*K\cap\zeta,u]^{i}du\Big)^{-\frac{1}{n-i}}\\&+\Big(\frac{1}{i}\int_{S^{n-1}}\rho[\Phi^*L\cap\zeta,u]^{i}du\Big)^{-\frac{1}{n-i}}\\=&\mathrm{vol}_{n-i}(\Phi^*K\cap\zeta)^{-\frac{1}{n-i}}+\mathrm{vol}_{n-i}(\Phi^*L\cap\zeta)^{-\frac{1}{n-i}}.\end{aligned}$$

According to the equality condition of Minkowski integral inequality, we see that equality holds in (27) if and only if Φ^*K and Φ^*L are dilates.

Proof of Theorem 1.2. Since $K,L\in\mathcal{K}_o^n$ and $0\leq i<n$, by (4), (27) and the Minkowski integral inequality[1], we obtain

$$\begin{aligned}&\breve{W}_i(\Phi^*(K\#L))^{-\frac{1}{n-i}}\\=&\Big[\frac{k_n}{k_{n-i}}\int_{G(n,n-i)}\Big(\mathrm{vol}_{n-i}(\Phi^*(K\#L)\cap\zeta)\Big)^{-1}d\mu_{n-i}(\zeta)\Big]^{\frac{1}{n-i}}\\=&\Big[\frac{k_n}{k_{n-i}}\int_{G(n,n-i)}\Big([\mathrm{vol}_{n-i}(\Phi^*(K\#L)\cap\zeta)]^{-\frac{1}{n-i}}\Big)^{n-i}d\mu_{n-i}(\zeta)\Big]^{\frac{1}{n-i}}\\\geq&\Big[\frac{k_n}{k_{n-i}}\int_{G(n,n-i)}\Big(\mathrm{vol}_{n-i}(\Phi^*K\cap\zeta)^{-\frac{1}{n-i}}\\&+\mathrm{vol}_{n-i}(\Phi^*L\cap\zeta)^{-\frac{1}{n-i}}\Big)^{n-i}d\mu_{n-i}(\zeta)\Big]^{\frac{1}{n-i}}\\\geq&\Big[\frac{k_n}{k_{n-i}}\int_{G(n,n-i)}\mathrm{vol}_{n-i}(\Phi^*K\cap\zeta)^{-1}d\mu_{n-i}(\zeta)\Big]^{\frac{1}{n-i}}\\&+\Big[\frac{k_n}{k_{n-i}}\int_{G(n,n-i)}\mathrm{vol}_{n-i}(\Phi^*L\cap\zeta)^{-1}d\mu_{n-i}(\zeta)\Big]^{\frac{1}{n-i}}\\=&\breve{W}_i(\Phi^*K)^{-\frac{1}{n-i}}+\breve{W}_i(\Phi^*L)^{-\frac{1}{n-i}}.\end{aligned}$$

This yields desired result.

From the equality conditions of (27) and the Minkowski integral inequality, it follows that equality holds in (11) if and only if Φ^*K and Φ^*L are dilates.

Finally, in order to prove Theorem 1.3, the following lemma is required.

Lemma 3.4. *Let $\Psi:\mathcal{S}^n\to\mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. If $K,L\in\mathcal{S}^n$, for $n-1<i<n$, then*

$$\begin{aligned}&\mathrm{vol}_{n-i}(\Psi(K\widehat{+}L)\cap\zeta)^{\frac{1}{n-i}}\\\geq&\mathrm{vol}_{n-i}((\Psi K)\cap\zeta)^{\frac{1}{n-i}}+\mathrm{vol}_{n-i}((\Psi L)\cap\zeta)^{\frac{1}{n-i}},\end{aligned} \tag{28}$$

for $0\leq i<n-1$, then

$$\begin{aligned}&\mathrm{vol}_{n-i}(\Psi(K\widehat{+}L)\cap\zeta)^{\frac{1}{n-i}}\\\leq&\mathrm{vol}_{n-i}((\Psi K)\cap\zeta)^{\frac{1}{n-i}}+\mathrm{vol}_{n-i}((\Psi L)\cap\zeta)^{\frac{1}{n-i}},\end{aligned} \tag{29}$$

in each case, equality holds if and only if ΨK and ΨL are dilates.

Proof. Since $K,L\in\mathcal{K}^n$ and $n-1<i<n$, by polar coordinate formula for volume, (22), (9), (16) and the

Minkowski integral inequality[1], we get

$$\begin{aligned}
&\mathrm{vol}_{n-i}(\Psi(K\widehat{+}L)\cap\zeta)^{\frac{1}{n-i}}\\
=&\left(\frac{1}{n-i}\int_{S^{n-1}}\rho[\Psi(K\widehat{+}L)\cap\zeta,u]^{n-i}du\right)^{\frac{1}{n-i}}\\
=&\left(\frac{1}{n-i}\int_{S^{n-1}}\rho[(\Psi K\widetilde{+}\Psi L)\cap\zeta,u]^{n-i}du\right)^{\frac{1}{n-i}}\\
=&\left(\frac{1}{n-i}\int_{S^{n-1}\cap\zeta}\rho(\Psi K\widetilde{+}\Psi L,u)^{n-i}du\right)^{\frac{1}{n-i}}\\
=&\left(\frac{1}{n-i}\int_{S^{n-1}\cap\zeta}[\rho(\Psi K,u)+\rho(\Psi L,u)]^{n-i}du\right)^{\frac{1}{n-i}}\\
\geq&\left(\frac{1}{n-i}\int_{S^{n-1}\cap\zeta}\rho(\Psi K,u)^{n-i}du\right)^{\frac{1}{n-i}}\\
&+\left(\frac{1}{n-i}\int_{S^{n-1}\cap\zeta}\rho(\Psi L,u)^{n-i}du\right)^{\frac{1}{n-i}}\\
=&\mathrm{vol}_{n-i}((\Psi K)\cap\zeta)^{\frac{1}{n-i}}+\mathrm{vol}_{n-i}((\Psi L)\cap\zeta)^{\frac{1}{n-i}}.
\end{aligned}$$

The equality condition of Minkowski integral inequality implies that equality holds in (28) if and only if ΨK and ΨL are dilates.

Similar to the above method, for $0 \leq i < n-1$, the inequality (29) follows from (22), (9), (16) and inverse of the Minkowski integral inequality.

Proof of Theorem 1.3. Since $K, L \in \mathcal{S}^n$ and $n-1 < i < n$, by (4), (28) and Minkowski integral inequality[1], we obtain

$$\begin{aligned}
&\left[\frac{k_{n-i}\breve{W}_i(\Psi(K\widehat{+}L))}{k_n}\right]^{\frac{1}{n-i}}\\
=&\left[\int_{G(n,n-i)}[\mathrm{vol}_{n-i}(\Psi(K\widehat{+}L)\cap\zeta)]^{-1}d\mu_{n-i}(\zeta)\right]^{-\frac{1}{n-i}}\\
=&\left[\int_{G(n,n-i)}[\mathrm{vol}_{n-i}(\Psi(K\widehat{+}L)\cap\zeta)]^{\frac{1}{n-i}(-(n-i))}d\mu_{n-i}(\zeta)\right]^{-\frac{1}{n-i}}\\
\geq&\left[\int_{G(n,n-i)}[\mathrm{vol}_{n-i}((\Psi K)\cap\zeta)^{\frac{1}{n-i}}\right.\\
&\left.+\mathrm{vol}_{n-i}((\Psi L)\cap\zeta)^{\frac{1}{n-i}}]^{-(n-i)}d\mu_{n-i}(\zeta)\right]^{-\frac{1}{n-i}}\\
\geq&\left[\int_{G(n,n-i)}[\mathrm{vol}_{n-i}((\Psi K)\cap\zeta)^{-1}d\mu_{n-i}(\zeta)\right]^{-(n-i)}\\
&+\left[\int_{G(n,n-i)}[\mathrm{vol}_{n-i}((\Psi L)\cap\zeta)^{-1}d\mu_{n-i}(\zeta)\right]^{-(n-i)}\\
=&\left[\frac{k_{n-i}\breve{W}_i(\Psi K)}{k_n}\right]^{\frac{1}{n-i}}+\left[\frac{k_{n-i}\breve{W}_i(\Psi L)}{k_n}\right]^{\frac{1}{n-i}}.
\end{aligned}$$

From the equality condition of (28) and the Minkowski integral inequality, it follows that equality in (12) holds if and only if ΨK and ΨL are dilates.

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