Sparse Online Principal Component for Laplace Factor Models

Siqi Liu, and Guangbao Guo

***Abstract*—This paper proposes a Sparse Online Principal Component (SOPC) approach applied to the Laplace Factor Model (LFM). The method integrates the strengths of Sparse Principal Component (SPC) and Online Principal Component (OPC) techniques, enabling real-time data updates and yielding sparse solutions. Numerical simulations validate its stability and accuracy under varying sample sizes and dimensions. Results demonstrate low error rates and robust adaptability in Laplace factor models.**

***Index Terms*—Laplace factor model; sparse online principal component; stability and sensitivity; real-time updating; numerical simulation.**

I. Introduction

THE Laplace Factor Model (LFM) is widely used for analyzing high-dimensional multivariate data by extracting latent factors. Existing methods face limitations in real-time updating and sparsity. This study introduces the Sparse Online Principal Component (SOPC) method, which supports online updates and sparse parameter estimation. Simulations assess its sensitivity to sample size and dimensionality. The findings offer a practical tool for dynamic data analysis in complex scenarios.

II. Laplace factor model

A. Laplace distribution

The Laplace distribution encompasses three primary forms, defined by their probability density functions.

Consider a Laplace-distributed random variable X, parameterized by its location parameter μ, scale parameter b, and precision parameter $\tau = \frac{1}{b}$.

1) Univariate symmetric Laplace distribution:

$$f_X(x;0,b) = \frac{1}{2b}\exp\left(-\frac{|x|}{b}\right).$$

2) Precision parameterization Laplace distribution:

$$f_X(x;\mu,\tau) = \frac{\tau}{2}\exp(-\tau|x-\mu|).$$

3) Univariate asymmetric Laplace distribution:

$$f_x(x;\mu,b_1,b_2) = \begin{cases} \frac{1}{b_1}\exp\left(-\frac{|x-\mu|}{b_1}\right), & x<\mu, \\ \frac{1}{b_2}\exp\left(-\frac{|x-\mu|}{b_2}\right), & x\geq\mu. \end{cases}$$

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B. Laplace regression

We consider a distributed Laplace regression model ($k = 1,\ldots,K_n$):

$$Y_{Ik} = X_{Ik}\gamma + \epsilon_{Ik}, \quad \epsilon_{Ik} \sim \text{Laplace}(\mu_1, b_3 I_{n_{ik}\times n_{ik}})$$

where

- X_{Ik} is a submatrix of the full design matrix N_{Ik}, with $n_{Ik} \geq p$;
- ϵ_{Ik} is a subresidual vector, and $I_{n_k\times n_k}$ denotes the identity matrix;
- $\gamma = (\gamma_1,\ldots,\gamma_p)^T$ denotes the regression coefficient vector;
- μ_1 and $b_3 > 0$ represent location and scale parameters, respectively.

We notice that

$$Y = \left({Y_{l_1}}^T, {Y_{l_2}}^T, \cdots, {Y_{l_{Kn}}}^T\right)^T,$$

$$X = \left({X_{I_1}}^T, {X_{I_2}}^T, \cdots, {X_{I_{Kn}}}^T\right)^T.$$

The model can also be expressed as:

$$Y = X\gamma + \epsilon, \quad \epsilon \sim \text{Laplace}(\mu_1, b_3),$$

with $\mathbb{E}(Y) = \mu_1$, $Var(\epsilon_{Ik}) = 2b_3^2 - \mu_1^2$ and the error variance estimator $\hat{\sigma}^2 = \frac{\epsilon^T\epsilon}{n-p}$.

C. Laplace factor model

The formulation of the LFM is given by

$$X = FA^\top + \epsilon, \quad \epsilon \sim \text{Laplace}(\mu_2, b_4 I_{p\times p}) \tag{1}$$

where

- $X \in \mathbb{R}^{n\times p}$ denotes the observed data matrix;
- $A \in \mathbb{R}^{p\times m}$ constitutes the factor loading matrix;
- $F \in \mathbb{R}^{n\times m}$ represents the matrix containing latent common factor with $m < p$;
- $\epsilon \in \mathbb{R}^{n\times p}$ denotes the error matrix;
- μ_2 and b_4 represent location and scale parameters.

The assumptions of the model are as follows.

- The expected value of f_j is zero and its variance is $I_{m\times m}$.
- The expected value of ϵ_j is zero and its variance is a diagonal matrix $D = diag(\sigma_1^2, \sigma_2^2, \cdots, \sigma_p^2)$.
- The f_j and ϵ_j are independent, $cov(f_j, \epsilon_j) = 0$.

III. Method

A. Sparse principal component analysis

To describe the covariance structure among p correlated variables, we use $m \ll p$ common factors:

$$\sum = AA^T + D$$

where A denotes the factor loading matrix, and the diagonal matrix D accounts for specific variance.

Under the Sparse Principal Component (SPC) framework, eigendecomposition of the empirical covariance matrix S yields

$$S = \sum_{i=1}^{p} \lambda_i v_i v_i^T$$

where λ_i and v_i denote eigenvalues and their corresponding orthogonal eigenvectors. Let $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)$ and $V = (v_1, v_2, \cdots, v_p)$. Through sparsification, a sparse eigenvector matrix V_S is obtained, which is then used to derive the sparse loading matrix A_S.

Let V_s be initialized as V. The sparse optimization criterion is formulated as

$$\begin{aligned} V_S = \arg\min_{V_S} (&\sum_{i=1}^{n} \left\| X_{i.} - W_S V_S^T X_{i.} \right\|_F^2 \\ &+ \rho \sum_{j=1}^{m} \|V_{S.j}\|_F^2 + \theta \sum_{j=1}^{m} \|V_{S.j}\|_1) \end{aligned}$$

where $W_S^T W_S = I_{m\times m}$ and $n > p$ when $\rho = 0$. The sparsity parameter $\theta > 0$ controls eigenvector sparsity. The matrix $\hat{A}_s$ containing sparse loading and the matrix $\hat{D}_s$ containing specific variance are calculated for $i = 1, 2, \cdots, p$:

$$\begin{aligned} &\hat{A}_S = (\sqrt{\lambda_1} v_{S1}, \sqrt{\lambda_2} v_{S2}, \cdots, \sqrt{\lambda_m} v_{Sm}) = (\hat{a}_{Sij})_{p\times m}, \\ &\hat{D}_S = \mathrm{diag}\left(\hat{\sigma}_1^2, \hat{\sigma}_2^2, \cdots, \hat{\sigma}_p^2\right), \quad \hat{\sigma}_i^2 = s_{ii} - \sum_{j=1}^{m} \hat{a}_{Sij}^2. \end{aligned}$$

B. Sparse online principal component analysis

The SOPC method introduces a sparsity parameter θ to enable online sparse eigenvector updates. For the first k observations $X^k = (X_1^T, X_2^T, \cdots, X_k^T)^T$, $(k < n)$, the sample covariance matrix is $S^k = V^k \Lambda^k (V^k)^T$, with $\Lambda^k = \mathrm{diag}(\lambda_1^k, \lambda_2^k, \cdots, \lambda_p^k)$, and $V^k = (v_1^k, v_2^k, \cdots, v_p^k)$. Upon receiving the $(k+1)$-th sample X_{k+1}, the covariance updates as

$$S^{k+1} = \frac{k}{k+1} S^k + \frac{1}{k+1} X_{k+1}^T$$

and the eigen decomposition $X_{k+1} = V^{k+1} \Lambda^{k+1} (V^{k+1})^T$ is performed. Let $W_{SO} = V^{k+1}[, 1:m]$, and initialize $V_{SO}^{k+1} = V^{k+1}$. The sparse eigenvector matrix V_{SO}^{k+1} is obtained via

$$\begin{aligned} V_{SO}^{k+1} = \arg\min_{V_S^{k+1}} (&\|X_{k+1.} - W_{SO} V_{SO}^{k+1T} X_{k+1.}\|_F^2 \\ &+ \rho \sum_{j=1}^{m} \|V_{SO,j}^{k+1}\|_F^2 + \theta \sum_{j=1}^{m} \|V_{SO,j}^{k+1}\|_1), \end{aligned}$$

where $W_{SO}^T W_{SO} = I_{m\times m}$. The optimization objective can be re-expressed as

$$\begin{aligned} L(V_{SO}^{k+1}) = &\mathrm{trace}(X_{k+1.}^T X_{k+1.}) \\ &+ \sum_{j=1}^{m} [{V_{SO,j}^{k+1}}^T \left(X_{k+1.}^T X_{k+1.} + \rho\right) V_{SO,j}^{k+1} \\ &- 2 w_{SO,j}^T X_{k+1}^T X_{k+1} V_{SO,j}^{k+1} + \theta \|V_{SO,j}\|_1]. \end{aligned}$$

The updated sparse loading matrix $\hat{A}_{SO}^{k+1}$ and specific variance matrix $\hat{D}_{SO}^{k+1}$ are given by $(i = 1, 2, \ldots, p)$:

$$\begin{aligned} \hat{A}_{SO}^{k+1} &= (\sqrt{\lambda_1^{k+1}} v_{SO1}^{k+1}, \ldots, \sqrt{\lambda_m^{k+1}} v_{SOm}^{k+1}) \\ &= (\hat{a}_{SO_{ij}}^{k+1})_{p\times m}, \\ \hat{D}_{SO}^{k+1} &= \mathrm{diag}\left(\hat{\sigma}_1^2, \hat{\sigma}_2^2, \ldots, \hat{\sigma}_p^2\right), \hat{\sigma}_i^2 = s_{ii} - \sum_{j=1}^{m} \hat{a}_{so_{ij}}^{k+1^2}. \end{aligned} \tag{2}$$

C. SOPC for LFM

For the LFM in equation (1), the data matrix X is generated with Laplace distributed errors. Using the SOPC-derived sparse loading matrix, the updated parameters $\hat{A}_{SO}^{k+1}$ and $\hat{D}_{SO}^{k+1}$ follow equation (2) where

$$\hat{\sigma}_i^2 = X_{k+1}^T X_{k+1}, \quad i = 1, 2, \ldots, p.$$

This completes the integration of SOPC with the LFM.

IV. Numerical analysis

A. Preparation work

1) Statistical index: The efficacy of the SOPC algorithm is quantified through the Frobenius-norm-based Mean Squared Error (MSE) between the true and estimated factor loading matrices. In simulation studies, the true loading matrix A is predefined, enabling direct accuracy assessment. The MSE is defined as

$$MSE_{\hat{A}} = \frac{1}{p^2} \|A - \hat{A}\|_F^2$$

where $\hat{A}$ denotes the estimated loading matrix.

2) Parameter setting: The simulated data X is generated according to the generative model in (1), with the following configurations:

- Parameter ranges: Location parameter $\mu \in [0, 1000]$, while factor loadings A are bounded within $[-1, 1]$;
- Distribution specifications: Latent factors $F \sim \mathcal{N}(0, I_{m\times m})$, noise terms $\epsilon \sim$ Laplace$(0, 1)$, and idiosyncratic variances $D \in (0, 1)$.

B. Simulation 1: Baseline performance

We set $n = 1000$, $p = 10$, $m = 5$ and generate the data matrix X. Table I summarizes the KMO and Bartlett's sphericity test results.

As evidenced in Table I, all KMO values exceed 0.60, and the sphericity hypothesis is rejected at $p < 0.05$. Both statistical criteria substantiate the appropriateness of LFM for dimensionality reduction.

TABLE I: KMO and Bartlett's test results

Laplace Distribution	KMO	χ^2	df	P-value	Suitable
Univariate Symmetric	0.69	1739.30	45	0	YES
Precision Parameterization	0.62	1649.50	45	$1.42e^{-316}$	YES
Univariate Asymmetric	0.70	2679.67	45	0	YES

C. Simulation 2: Sensitivity analysis

1) Impact of sample size (n): We set $p = 10$, $m = 5$, and varying $n \in \{2000, 3000, 4000, 5000, 6000\}$.

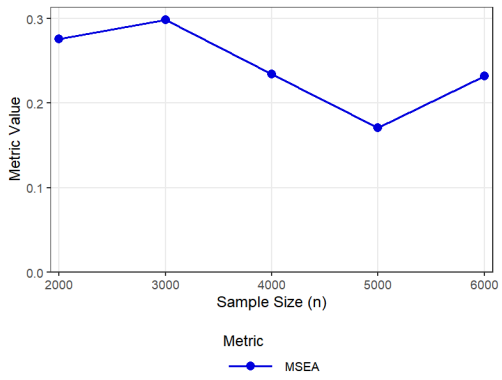

Fig. 1: $MSE_{\hat{A}}$ values of SOPC for univariate symmetric model as n varies.

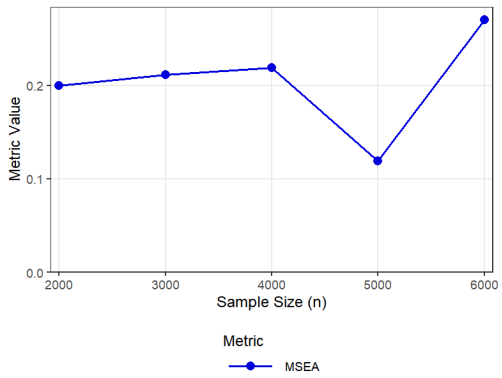

Fig. 2: $MSE_{\hat{A}}$ values of SOPC for precision parameterization model as n varies.

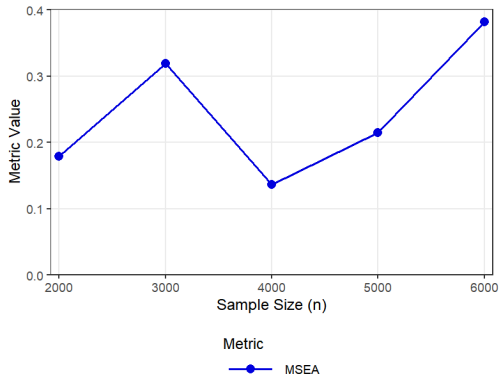

Fig. 3: $MSE_{\hat{A}}$ values of SOPC for univariate asymmetric model as n varies.

- As shown in Fig. 1, the $MSE_{\hat{A}}$ curve demonstrates strong stability, with values ranging from 0.17 to 0.29. The minimum is achieved at $n = 5000$ ($MSE_{\hat{A}} = 0.17$), followed by a slight increase at $n = 6000$ ($MSE_{\hat{A}} = 0.23$).
- Fig. 2 shows that $MSE_{\hat{A}}$ values remain consistently stable within the range of 0.12 to 0.27, with the best performance observed at $n = 5000$ ($MSE_{\hat{A}} = 0.12$).
- In Fig. 3, moderate fluctuations are observed, with values ranging from 0.14 to 0.38. Notably, there is a significant increase in error at $n = 6000$, suggesting a sensitivity to larger sample sizes.

2) Impact of sample dimension (p): We set $n = 2000$, $m = 5$, and varying $p \in \{10, 11, 12, 13, 14\}$.

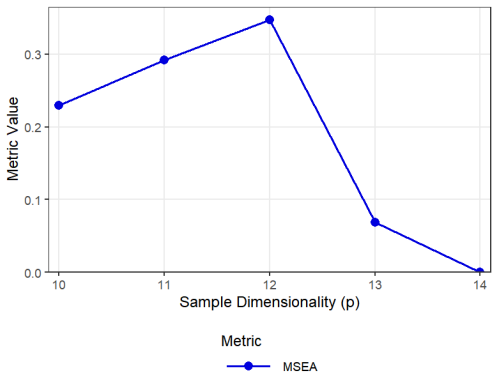

Fig. 4: $MSE_{\hat{A}}$ values of SOPC for univariate symmetric model as p varies.

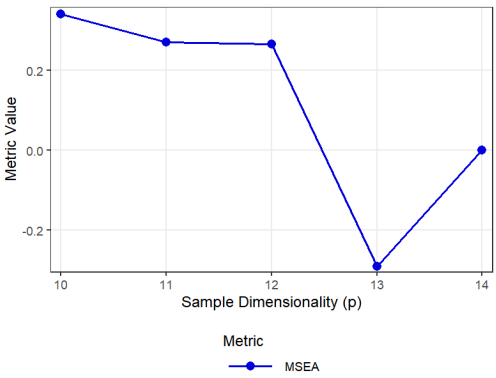

Fig. 5: $MSE_{\hat{A}}$ values of SOPC for precision parameterization model as p varies.

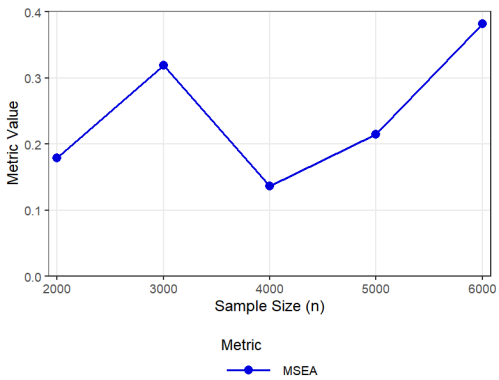

Fig. 6: $MSE_{\hat{A}}$ values of SOPC for univariate asymmetric model as p varies.

- As shown in Fig. 4, $MSE_{\hat{A}}$ peaks at $p = 12$ ($MSE_{\hat{A}} = 0.34$) before sharply declining to near-zero values at $p = 14$, suggesting a risk of overfitting with higher dimensions.
- Fig. 5 demonstrates that $MSE_{\hat{A}}$ gradually decreases from 0.26 to 0.22, but exhibits negative values at $p = 13$, indicating instability in the estimation process at this point.
- In Fig. 6, $MSE_{\hat{A}}$ fluctuates moderately within the range of 0.14 to 0.38, with the best performance observed at mid-range dimensions, specifically at $p = 12$.

D. Simulation 3: Comparative analysis

We compare the SOPC method against four alternatives: SPC, Projection Principal Component (PPC), Incremental Principal Component (IPC) and Stochastic Approximate Principal Component (SAPC). Parameters are fixed as $p = 10$, $m = 5$, $\mu \in [0, 1000]$, $A \in [-1, 1]$, $F \sim \mathcal{N}_m(0, I_{m\times m})$, $\varepsilon \sim$ Laplace$(0, 1)$, and $D \in (0, 1)$, with $n \in \{2000, 3000, 4000, 5000, 6000\}$.

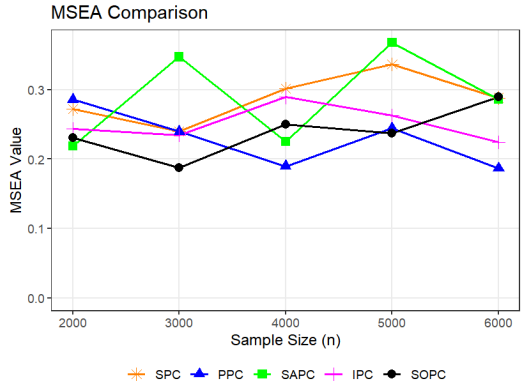

Fig. 7: Comparison of $MSE_{\hat{A}}$ performance of SOPC, PPC, IPC, SPC, and SAPC methods for univariate symmetric model with the change of n–value.

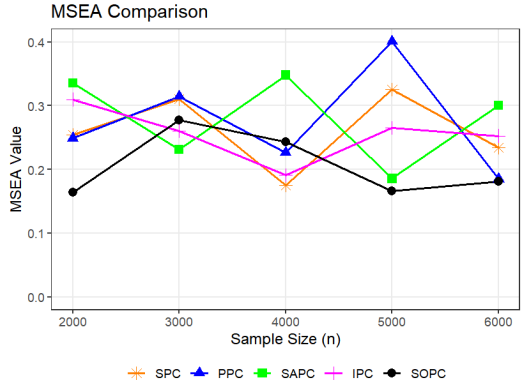

Fig. 8: Comparison of $MSE_{\hat{A}}$ performance of SOPC, PPC, IPC, SPC, and SAPC methods for precision parameterization model with the change of n–value.

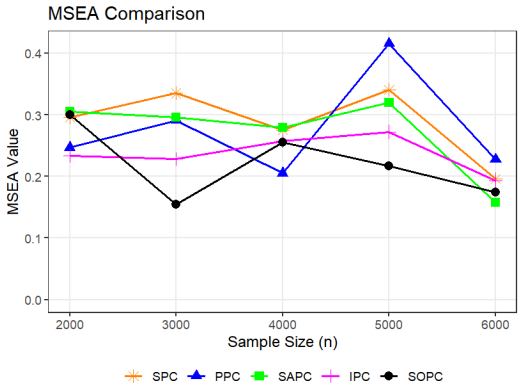

Fig. 9: Comparison of $MSE_{\hat{A}}$ performance of SOPC, PPC, IPC, SPC, and SAPC methods for univariate asymmetric model with the change of n–value.

- As shown in Fig. 7, SOPC achieves the lowest $MSE_{\hat{A}}$ values in the range of 0.19–0.24, outperforming SAPC (0.22–0.37) and all other methods.
- Fig. 8 demonstrates SOPC's superior robustness, with its $MSE_{\hat{A}}$ ranging from 0.16 to 0.28. In contrast, PPC shows considerable instability, with $MSE_{\hat{A}}$ fluctuating between 0.19 and 0.40.
- In Fig. 9, SOPC consistently achieves the lowest error rates (0.15–0.20), while PPC exhibits the largest fluctuations, with $MSE_{\hat{A}}$ spanning from 0.20 to 0.42.

These results underscore SOPC's superior adaptability to variations in sample size compared to the other methods.

V. Conclusions and Suggestions

The SOPC method exhibits high precision and stability in LFM, outperforming traditional principal component analysis. While sample size variations minimally affect performance, higher dimensions may increase estimation errors. Future work should integrate distributed computing for efficiency and extend the method to nonlinear models. This approach provides an effective solution for real-time analysis of complex datasets.

References

[1] C. Zhang, G. Guo, "The Optimal Subset Estimation of Distributed Redundant Data," *IAENG International Journal of Applied Mathematics*, vol. 55, no. 2, pp. 270-277, 2025.
[2] D. Chang, G. Guo, "LIC: An R package for optimal subset selection for distributed data," *SoftwareX*, vol. 28, p. 101909, 2024.
[3] D. Chang, G. Guo, "Research on Distributed Redundant Data Estimation Based on LIC," *IAENG International Journal of Applied Mathematics*, vol. 55, no. 1, pp.1-6, 2025.
[4] G. Guo, Y. Sun, X. Jiang, "A partitioned quasi-likelihood for distributed statistical inference," *Computational Statistics*, vol. 35, pp. 1577-1596, 2020.
[5] G. Guo, Y. Sun, G. Qian, Q. Wang, "LIC criterion for optimal subset selection in distributed interval estimation," *Journal of Applied Statistics*, vol. 50, no. 9, pp. 1900-1920, 2023.
[6] G. Guo, Q. Wang, J. Allison, G. Qian, "Accelerated distributed expectation-maximization algorithms for the parameter estimation in multivariate Gaussian mixture models," *Applied Mathematical Modelling*, vol. 137, p. 115709, 2025.
[7] G. Guo, W. You, L. Lin, G. Qian, "Covariance Matrix and Transfer Function of Dynamic Generalized Linear Models," *Journal of Computational and Applied Mathematics*, vol. 296, pp. 613-624, 2016.
[8] G. Guo, G. Qian, L. Lin, W. Shao, "Parallel inference for big data with the group bayesian method," *Metrika*, vol. 84, pp. 225-243, 2021.

[9] G. Guo, G. Qian, L. Zhu, "A scalable quasi-Newton estimation algorithm for dynamic generalized linear model," *Journal of Nonparametric Statistics*, vol. 34, no. 4, pp. 917-939, 2022.

[10] G. Guo, R. Niu, G. Qian, T. Lu, "Trimmed scores regression for k-means clustering data with high-missing ratio," *Communications in Statistics - Simulation and Computation*, vol. 53, pp. 2805-2821, 2024.

[11] G. Guo, M. Yu, G. Qian, "ORKM: Online regularized K-means clustering for online multi-view data," *Information Sciences*, vol. 680, p. 121133, 2024.

[12] G. Guo, H. Song, L. Zhu, "The COR criterion for optimal subset selection in distributed estimation," *Statistics and Computing*, vol. 34, pp. 163-176, 2024.

[13] G. Guo, "A block bootstrap for quasi-likelihood in sparse functional data," *Statistics: A Journal of Theoretical and Applied Statistics*, vol. 54, pp. 909-925, 2020.

[14] G. Guo, C. Wei, G. Qian, "Sparse online principal component analysis for parameter estimation in factor model," *Computational Statistics*, vol. 38, no. 2, pp. 1095-1116. 2022.

[15] G. Guo, G. Qian, "Optimal subset selection for distributed local principal component analysis," *Physica A: Statistical Mechanics and its Applications*, vol. 658, no. 486, p. 130308, 2025.

[16] G. Guo, W. Zhao, "Schwarz method for financial engineering," *Journal of Computational Mathematics*, vol. 39, no. 4, pp. 538-555, 2021.

[17] G. Guo, "Taylor quasi-likelihood for limited generalized linear models," *Journal of Applied Statistics*, vol. 48, pp. 669-692, 2021.

[18] G. Guo, J. Allison, L. Zhu, "Bootstrap maximum likelihood for quasi-stationary distributions," *Journal of Nonparametric Statistics*, vol. 31, no. 1, pp. 64-87, 2019.

[19] G. Guo, "Finite difference methods for the BSDEs in finance," *International Journal of Financial Studies*, vol. 6, no. 1, pp. 1-15, 2018.

[20] G. Guo, H. Song, L. Zhu, "The iterated score regression estimation algorithm for PCA-based missing data with high correlation," *Scientific Reports*, vol. 15, no. 1, p. 9067, 2025.

[21] G. Guo, G. Qian, L. Zhu, "A scalable quasi-newton estimation algorithm for dynamic generalized linear models," *Journal of Nonparametric Statistics*, vol. 34, pp. 917-939, 2022.

[22] G. Guo, W. Zhao, "Schwarz methods for quasi stationary distributions of Markov chains," *Calcolo*, vol. 49, pp. 21-39, 2012.

[23] G. Guo, S. Lin, "Schwarz method for penalized quasi-likelihood in generalized additive models," *Commun. Statist.-Theory Meth.*, vol. 39, pp. 1847-1854, 2010.

[24] G. Guo, W. You, G. Qian, "Parallel maximum likelihood estimator for multiple linear regression models," *Journal of Computational and Applied Mathematics*, vol. 273, pp. 251-263, 2015.

[25] G. Guo, W. Shao, L. Lin, "Parallel tempering for dynamic generalized linear models," *Communications in Statistics-Theory and Methods*, vol. 45, no. 21, pp. 6299-6310, 2016.

[26] G. Guo, "Parallel statistical computing for statistical inference," *Journal of Statistical Theory and Practice*, vol. 6, no. 3, pp. 536-565, 2012.

[27] G. Guo, L. Lin, "Parallel bootstrap and optimal subsample lengths in smooth function models," *Communications in Statistics-Simulation and Computation*, vol. 45, no. 6, pp. 2208-2231, 2016.

[28] G. Guo, S. Lin, "Schwarz method for penalized quasi-likelihood in generalized additive models," *Communications in Statistics—Theory and Methods*, vol. 39, no. 10, pp. 1847-1854, 2010.

[29] G. Jing, G. Guo, "TLIC: An R package for the LIC for T distribution regression analysis," *SoftwareX*, vol. 30, p. 102132, 2025.

[30] G. Jing, G. Guo, "Student LIC for distributed estimation," *IAENG International Journal of Applied Mathematics*, vol. 55, no. 3, pp. 575–581, 2025.

[31] G. Jing, G. Guo, "Student LIC for Distributed Estimation," *IAENG International Journal of Applied Mathematics*, vol. 55, no. 3, pp. 575-581, 2025.

[32] J. Li, G. Guo, "An Optimal Subset Selection Algorithm for Distributed Hypothesis Test," *IAENG International Journal of Applied Mathematics*, vol. 54, no. 12, pp. 2811-2815, 2024.

[33] L. Song, G. Guo, "Full Information Multiple Imputation for Linear Regression Model with Missing Response Variable," *IAENG International Journal of Applied Mathematics*, vol. 54, no. 1, pp. 77-81, 2024.

[34] M. Gataric, T. Wang, R. J. Samworth, "Sparse principal component analysis via axis-aligned random projections," *Journal of the Royal Statistical Society: Series B Statistical Methodology*, vol. 82, no. 2, pp. 329-359, 2020.

[35] N. T. Trendafilov, S. Fontanella, K. Adachi, "Sparse exploratory factor analysis," *Psychometrika*, vol. 82, pp. 778-794, 2017.

[36] Q. Liu, G. Guo, "Distributed Estimation of Redundant Data," *IAENG International Journal of Applied Mathematics*, vol. 55, no. 2, pp. 332-337, 2025.

[37] Q. Wang, G. Guo, G. Qian, X. Jiang, "Distributed online expectation-maximization algorithm for Poisson mixture model," *Applied Mathematical Modelling*, vol. 124, pp. 734-748, 2023.

[38] S. Kotz, T. Kozubowski, K. Podgorski, *The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering, and Finance*, Springer Science & Business Media, 2012.

[39] S. Liu, G. Guo, "LFM: An R package for Laplace factor model," *SoftwareX*, vol. 30, p. 102133, 2025.

[40] W. You, Z. Yang, G. Guo, X. F. Wan, G. Ji, "Prediction of DNA-binding proteins by interaction fusion feature representation and selective ensemble," *Knowledge-Based Systems*, vol. 163, pp. 598-610, 2018.

[41] W. Shao, G. Guo, "Multiple-Try simulated annealing algorithm for global optimization," *Mathematical Problems in Engineering*, vol. 2018, no. 1, p. 9248318, 2018.

[42] W. Shao, G. Guo, G. Zhao, "Simulated annealing for the bounds of Kendall's τ and Spearman's ρ," *Journal of Statistical Computation and Simulation*, vol. 84, no. 12, pp. 2688-2699, 2014.

[43] W. Shao, G. Guo, F. Meng, "An efficient proposal distribution for Metropolis–Hastings using a B-splines technique," *Computational Statistics & Data Analysis*, vol. 57, no. 1, pp. 465-478, 2013.

[44] Y. Zhao, M. A. Lindquist, B. S. Caffo, "Sparse Principal Component based High-Dimensional Mediation Analysis," *Computational Statistics and Data Analysis*, vol. 142, p. 106835, 2019.

[45] Y. Li, G. Guo. "Distributed Monotonic Overrelaxed Method for Random Effects Model with Missing Response," *IAENG International Journal of Applied Mathematics*, vol. 54, no. 2, pp.205-211, 2024.

[46] Y. Li, G. Guo, "General Unilateral Loading Estimation," *Engineering Letters*, vol. 32, no. 1, pp. 72-76, 2024.

[47] Z. Ma, "Sparse principal component analysis and iterative thresholding," *The Annals of Statistics*, vol. 41, no. 2, pp. 772-801, 2013.