Some New Fixed Ellipse Results and Discontinuity at Fixed Point in Sp-metric Space

Kaushik Lahkar and Nilakshi Goswami

***Abstract*—This paper provides some new solutions to Rhoades' open problem regarding discontinuity at fixed point in the context of fixed ellipse in S_p-metric space. Our results extend some works of Pant et al. and Ozgur et al. in the framework of S_p-metric space. Suitable examples are provided in support of our results. Applications are demonstrated for discontinuous activation function and Volterra integral equation.**

***Index Terms*—S_p-metric space, fixed point, fixed ellipse, activation function.**

I. Introduction

THE fixed point theory is an essential area in nonlinear analysis. Initially the metric fixed point condition was provided by Banach in 1922 under which a contraction mapping on a complete metric space has a unique fixed point [3]. Rhoades' work [18] explored around two hundred and fifty contractive definitions and concluded that, a large class of these definitions do not require the mapping to be continuous throughout the entire domain, but they do maintain continuity at the fixed point. Rhoades [19] posed an open problem regarding contractive conditions that are strong enough to ensure the existence of a fixed point but do not necessarily require the mapping to be continuous at the fixed point. Pant in [16] provided an answer to Rhoades' open problem in 1999 and several research work is going on in this direction.

In [20], Sedghi et al. introduced the concept of S-metric space and established some fixed point results in such space. Later on, many other researchers (refer to [5], [8], [11], [12], [14], [21], [23]) provided different fixed point results in various metric spaces. The concept of S_b-metric space was introduced by Souayaha and Mlaiki in their work in [22]. Both these concepts were extended to a larger framework by the introduction of S_p-metric space by Mustafa et al. [9] in 2019. After that, the study of fixed point theorem in S_p-metric space opens up a new area with promising dimension. As well as the study of geometrical structures of fixed point sets is taken up by many researchers considering fixed circle or fixed ellipse results.

In this paper, we study Rhoades' open problem considering some fixed ellipse theorems in S_p-metric space. Our work determines a broader framework for analysing fixed points in spaces equipped with a generalized metric structure. The geometrical interpretation of fixed points provided by the discontinuity points on the ellipse gives an insight into the Rhoades' open problem regarding discontinuity. Our results also establish a connection between fixed ellipse and discontinuous activation function. In the study of artificial neural networks, activation functions play an important role shaping the network's behaviour and enabling complex mappings between input and output spaces. In section V, we discuss the geometric properties exhibited by the fixed point sets of certain discontinuous activation functions and the last section of our paper contains an application regarding the existence and uniqueness of the solution of a Volterra integral equation.

Manuscript received October 29, 2024; revised March 22, 2025.

Kaushik Lahkar is a Research Fellow in the Department of Mathematics, Gauhati University, Guwahati, 781014, India (Corresponding author, e-mail: kaushiklahkar1696@gmail.com).

Nilakshi Goswami is a Professor in the Department of Mathematics, Gauhati University, Guwahati, 781014, India (e-mail: nila_g2003@yahoo.co.in).

II. Preliminaries

Sedghi et al. in [20], introduced the concept of an S-metric space as follows:

Definition 2.1 [20] Let X be a non-empty set and suppose that $S : X^3 \to \mathbb{R}^+ \cup \{0\}$ be a mapping satisfying the following conditions:

(S_1) $S(x,y,z) = 0$ if and only if $x = y = z$;
(S_2) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$ for all $a,x,y,z \in X$ (rectangle inequality).

Then (X,S) is called an S-metric space.

Mustafa et al. defined S_p-metric space in [9] as follows:

Definition 2.2 [9] Let X be a non-empty set and $\Omega : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function such that $\Omega^{-1}(t) \leq t \leq \Omega(t)$ for all $t > 0$ and $\Omega(0) = 0$. Suppose that $\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ be a mapping satisfying the following conditions:

(S_p1) $\overline{S}(x,y,z) = 0$ if and only if $x = y = z$;
(S_p2) $\overline{S}(x,y,z) \leq \Omega(\overline{S}(x,x,a) + \overline{S}(y,y,a) + \overline{S}(z,z,a))$ for all $a,x,y,z \in X$ (rectangle inequality).

Then $(X,\overline{S})$ is called an S_p-metric space.

Following are some examples of S_p-metric space.

Example 2.3 Let $X = \mathbb{N} \cup \{0\}$ and define $\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by
$\overline{S}(x,y,z) = \ln(1 + S(x,y,z))$, where

$$S(x,y,z) = \begin{cases} 0, & x = y = z; \\ \max(x,y,z) & otherwise, \end{cases}$$

with $\Omega(t) = \ln(1 + t), t \in [0,\infty)$. Then $(X,\overline{S})$ is an S_p-metric space.

Example 2.4 Let (X,S) be an S-metric space and $\overline{S}(x,y,z) = \sinh(S(x,y,z))$; $x,y,z \in X$. Then $\overline{S}$ is an S_p-metric with $\Omega(t) = \sinh t, t \in [0,\infty)$.

Example 2.5 Let (X,S) be an S-metric space and

define $\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by
$\overline{S}(x,y,z) = \sec^{-1}(e^{S(x,y,z)})$ for all $x,y,z \in X$ with $\Omega(t) = \sec^{-1}(e^t), t \in [0,\infty)$.
Then it can be easily verified that $(X,\overline{S})$ is an S_p-metric space.

Definition 2.6 [9] Let $(X,\overline{S})$ be an S_p-metric space. A sequence $\{x_n\}$ in X is said to be

(i) S_p-convergent to a point $p \in X$ if and only if for each $\epsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$, $\overline{S}(x_n,x_n,p) < \epsilon$.
(ii) S_p-Cauchy if and only if for each $\epsilon > 0$, there exists a positive integer n_0 such that for all $m,n \geq n_0$, $\overline{S}(x_m,x_n,x_n) < \epsilon$.

X is called S_p-complete if and only if every S_p-Cauchy sequence is S_p-convergent in X.

In [6] Joshi et al. introduced the concept of fixed ellipse in an S-metric space. In a similar way an ellipse in an S_p-metric space can be defined.

Definition 2.7 [6] An ellipse having foci at c_1 and c_2 in an S_p-metric space $(X,\overline{S})$ is defined as

$$E(c_1,c_2,a) = \{x \in X : \overline{S}(c_1,c_1,x) + \overline{S}(c_2,c_2,x) = 2a\}, \quad c_1,c_2 \in X, a \in [0,\infty).$$

Clearly, for an ellipse, $\overline{S}(c_1,c_1,c_2) < 2a$.

Following Caristi [4], the Caristi map in S_p-metric space can be defined as follows.

Definition 2.8 [4] A self-mapping T on an S_p-metric space $(X,\overline{S})$ is a Caristi map on X if there is a lower semi continuous function $\mu : X \to \mathbb{R}^+ \cup \{0\}$ such that

$$\overline{S}(x,x,Tx) \leq \mu(x) - \mu(Tx) \quad \text{for all} \quad x \in X. \tag{1}$$

III. Main Results

In this section, we derive some fixed ellipse results to study the geometry of non unique fixed points in the framework of S_p-metric space with some examples. In the next section, some discontinuity results at fixed point and fixed ellipse are established.

Let $(X,\overline{S})$ be a complete S_p-metric space. For $x,y,z \in X$, we take

$$M(x,y,z) = \max\Big\{\alpha(\overline{S}(x,y,z) + \overline{S}(Tx,Ty,Tz)), \frac{\beta}{3}(\overline{S}(x,x,Tx) + \overline{S}(y,y,Ty) + \overline{S}(z,z,Tz)), \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x,Ty,z)) + \Omega^{-1}(\overline{S}(z,Tx,Ty)) + \Omega^{-1}(\overline{S}(Ty,Tz,Tx)))\Big\}, \tag{2}$$

where $\alpha,\gamma \in [0,\frac{1}{2})$ and $\beta \in [0,1)$.
Also for $x \in X$, define a mapping $\mu : X \to \mathbb{R}^+ \cup \{0\}$ by

$$\mu(x) = \overline{S}(c_1,c_1,x) + \overline{S}(c_2,c_2,x), \quad x \in X. \tag{3}$$

Theorem 3.1 In an S_p-metric space $(X,\overline{S})$, let $E(c_1,c_2,a)$ be an ellipse for $c_1,c_2 \in X, a \in [0,\infty)$. Let T be a self-mapping on X satisfying:

(i) $\overline{S}(x,x,Tx) \leq \mu(x) - \mu(Tx)$, for all $x \in E(c_1,c_2,a)$;
(ii) $\overline{S}(c_1,c_1,Tx) + \overline{S}(c_2,c_2,Tx) \geq 2a$, for all $x \in E(c_1,c_2,a)$.

Then $E(c_1,c_2,a)$ is a fixed ellipse of T in X. Moreover, if for some $\lambda \in [0,1]$, T satisfies the following additional condition:

(iii) $\overline{S}(Tx,Ty,Tz) \leq \lambda M(x,y,z)$, for all $x,y \in E(c_1,c_2,a)$ and $z \in X \setminus E(c_1,c_2,a)$,

then $E(c_1,c_2,a)$ is the unique fixed ellipse of T.

Proof: We consider an arbitrary point x in $E(c_1,c_2,a)$. Using (1) and (3), we get,

$$\begin{aligned}\overline{S}(x,x,Tx) &\leq \overline{S}(c_1,c_1,x) + \overline{S}(c_2,c_2,x) - \overline{S}(c_1,c_1,Tx) - \overline{S}(c_2,c_2,Tx)\\ &\leq 2a - 2a\\ &= 0.\end{aligned}$$

Hence $\overline{S}(x,x,Tx) = 0$ and thus, $Tx = x$.
This shows that for all $x \in E(c_1,c_2,a)$, x is a fixed point of T, i.e., T fixes the ellipse $E(c_1,c_2,a)$.
To show the uniqueness, suppose there exist two fixed ellipses $E(c_1,c_2,a)$ and $E(c_1',c_2',a')$ of T.
Let $x \in E(c_1,c_2,a)$ and $y \in E(c_1',c_2',a')$.
Using condition (iii),

$$\overline{S}(x,x,y) = \overline{S}(Tx,Tx,Ty) \leq \lambda M(x,x,y) \leq M(x,x,y) \tag{4}$$

$$\begin{aligned}&= \max\Big\{2\alpha\overline{S}(x,x,y), \frac{\beta}{3}(2\overline{S}(x,x,Tx) + \overline{S}(y,y,Ty)), \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x,Tx,y)) + \Omega^{-1}(\overline{S}(y,Tx,Tx)) + \Omega^{-1}(\overline{S}(Tx,Ty,Tx)))\Big\}\\ &= \max\Big\{2\alpha\overline{S}(x,x,y), 0, \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x,x,y)) + \Omega^{-1}(\overline{S}(y,x,x)) + \Omega^{-1}(\overline{S}(x,y,x)))\Big\}\end{aligned} \tag{5}$$

If $M(x,x,y) = 2\alpha\overline{S}(x,x,y)$, then from (5), we get

$$\overline{S}(x,x,y) \leq 2\alpha\overline{S}(x,x,y) < \overline{S}(x,x,y),$$

which is a contradiction.
If $M(x,x,y) = \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x,x,y)) + \Omega^{-1}(\overline{S}(y,x,x)) + \Omega^{-1}(\overline{S}(x,y,x)))$, then

$$\begin{aligned}\overline{S}(x,x,y) &\leq \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x,x,y)) + \Omega^{-1}(\Omega(2\overline{S}(x,x,y))) + \Omega^{-1}(\Omega(2\overline{S}(x,x,y))))\\ &\leq \frac{\gamma}{3}(\overline{S}(x,x,y) + 2\overline{S}(x,x,y) + 2\overline{S}(x,x,y))\\ &= \frac{5\gamma}{3}\overline{S}(x,x,y)\\ &< \overline{S}(x,x,y), \quad \text{a contradiction.}\end{aligned}$$

Thus, $M(x,x,y) = 0$, i.e., $\overline{S}(x,x,y) = 0$ i.e., $x = y$.
Hence $E(c_1,c_2,a)$ is the unique fixed ellipse of T. ∎

The following examples exhibit Theorem 3.1.

Example 3.2 Let $X = \mathbb{R}$ and $\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ be defined by

$$\overline{S}(x,y,z) = \max\{e^{|x-y|}, e^{|y-z|}, e^{|z-x|}\} - 1, x,y,z \in X.$$

Then $(X,\overline{S})$ is an S_p-metric space with $\Omega(t) = e^t - 1$, $t \in [0,\infty)$.
Now,

$$\begin{aligned} E(0, \ln 12, \frac{23}{3}) &= \{x \in X : \overline{S}(0,0,x) \\ &\quad + \overline{S}(\ln 12, \ln 12, x) = \frac{46}{3}\} \\ &= \{x \in X : \max\{1, e^{|0-x|}, e^{|x-0|}\} \\ &\quad + \max\{1, e^{|\ln 12 - x|}, e^{|x - \ln 12|}\} = \frac{52}{3}\} \\ &= \{\ln \frac{3}{4}, \ln 16\} \end{aligned}$$

is an ellipse having foci at $c_1 = 0$ and $c_2 = \ln 12$.

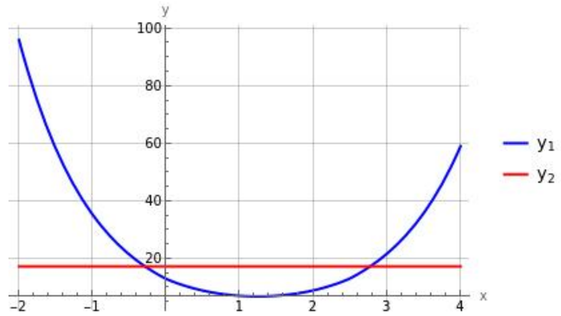

Figure 1

As given in the above figure (Figure 1), the points of intersection of the red coloured curve (i.e., $y_1 = \max\{1, e^{|0-x|}, e^{|x-0|}\} + \max\{1, e^{|\ln 12 - x|}, e^{|x-\ln 12|}\}$) and the blue coloured line (representing $y_2 = \frac{52}{3}$) give the ellipse $E(0, \ln 12, \frac{23}{3})$.
Define $T : X \to X$ by

$$T(x) = \begin{cases} x, & x \in E(0, \ln 12, \frac{23}{3}); \\ 0, & otherwise. \end{cases}$$

For $x = \ln 16$, $\overline{S}(\ln 16, \ln 16, \ln 16) = 0$ and

$$\begin{aligned} \mu(x) - \mu(Tx) &= \overline{S}(0,0, \ln 16) + \overline{S}(\ln 12, \ln 12, \ln 16) \\ &\quad - \overline{S}(0,0, \ln 16) \\ &\quad - \overline{S}(\ln 12, \ln 12, \ln 16) \\ &= 0. \end{aligned}$$

So, $\overline{S}(x,x,Tx) = \mu(x) - \mu(Tx)$ for $x = \ln 16$.
Thus condition (i) of the Theorem 3.1 is satisfied for $x = \ln 16$.
Again,

$$\begin{aligned} &\overline{S}(0,0, \ln 16) + \overline{S}(\ln 12, \ln 12, \ln 16) \\ &= \max\{1,, e^{|0-\ln 16|}, e^{|\ln 16 - 0|}\} - 1 \\ &+ \max\{1, e^{|\ln 12 - \ln 16|}, e^{|\ln 16 - \ln 12|}\} - 1 \\ &= 16 - 1 + \frac{4}{3} - 1 = \frac{46}{3}. \end{aligned}$$

So, $\overline{S}(c_1,c_1,Tx) + \overline{S}(c_2,c_2,Tx) = 2a$, for $x = \ln 16$, i.e., satisfies the condition (ii) of the Theorem 3.1.

Similarly, the point $x = \ln \frac{3}{4}$ also satisfy both the condition of the Theorem 3.1.
Hence T satisfies the conditions (i) and (ii) of Theorem 3.1 and clearly,
$E(0, \ln 12, \frac{23}{3})$ is a fixed ellipse of T.

Example 3.3 Let $X = M^2$ with $M = \{1,2,3\}$ and define an S_p-metric
$\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by

$$\overline{S}(x,y,z) = \sinh(S(x_1,y_1,z_1)) + \sinh(S(x_2,y_2,z_2)),$$

$x = (x_1,x_2), y = (y_1,y_2), z = (z_1,z_2) \in X$ with $\Omega(t) = \sinh(t), t \in [0,\infty)$, where (M,S) is an S-metric space, where $S : M^3 \to \mathbb{R}^+ \cup \{0\}$ is defined by

$$\begin{aligned} &S(1,1,2) = S(2,2,1) = \ln 5, \\ &S(2,2,3) = S(3,3,2) = S(1,1,3) = S(3,3,1) = \ln 3, \\ &S(x,y,z) = 0, \text{ if } x = y = z, \\ &S(x,y,z) = \ln 2, \text{ otherwise.} \end{aligned}$$

Now, for $c_1 = (2,2), c_2 = (3,3)$ and $a = \frac{38}{15}$, the equation of the ellipse in $(X,\overline{S})$ with foci at (c_1,c_2) is

$$\begin{aligned} E(c_1,c_2,\frac{38}{15}) &= \{x \in X : \overline{S}(c_1,c_1,x) + \overline{S}(c_2,c_2,x) = \frac{76}{15}\} \\ &= \{x \in X : \sinh(S(2,2,x_1)) + \sinh(S(2,2,x_2)) \\ &\quad + \sinh(S(3,3,x_1)) + \sinh(S(3,3,x_2)) = \frac{76}{15}\}. \end{aligned}$$

The points of the ellipse are $(1,3), (3,1), (2,1)$ and $(1,2)$.
Define $T : X \to X$ by

$$T(x,y) = \begin{cases} (1,y), & x < 2; \\ (x,1), & x \geq 2. \end{cases}$$

For $x = (1,3)$,
$T(1,3) = (1,3)$, $\overline{S}(x,x,Tx) = 0$ and

$$\begin{aligned} \mu(x) - \mu(Tx) &= \overline{S}((2,2),(2,2),(1,3)) + \overline{S}((3,3),(3,3),(1,3)) \\ &\quad - \overline{S}((2,2),(2,2),(1,3)) - \overline{S}((3,3),(3,3),(1,3)) \\ &= 0. \end{aligned}$$

So, $\overline{S}(x,x,Tx) = \mu(x) - \mu(Tx)$ for $x = (1,3)$.
Thus condition (i) of the Theorem 3.1 is satisfied for $x = (1,3)$.
Again,

$$\begin{aligned} &\overline{S}((2,2),(2,2),(1,3)) + \overline{S}((3,3),(3,3),(1,3)) \\ &= \sinh(S(2,2,1)) + \sinh(S(2,2,3)) + \sinh(S(3,3,1)) \\ &+ \sinh(S(3,3,3)) \\ &= \sinh(\ln 5) + \sinh(\ln 3) + \sinh(\ln 3) \\ &= \frac{76}{15}. \end{aligned}$$

So, $\overline{S}(c_1,c_1,Tx) + \overline{S}(c_2,c_2,Tx) = 2a$, i.e., for $x = (1,3)$, T satisfies the condition (ii) of the Theorem 3.1.
Similarly, at the points $(3,1), (2,1), (1,2)$ both the conditions of the Theorem 3.1 are satisfied.
Hence by Theorem 3.1, $E(c_1,c_2,\frac{38}{15})$ is a fixed ellipse of T.

However, if we define $T : X \to X$ as

$$T(x,y) = \begin{cases} (1,1), & x < 2; \\ (x,1), & x \geq 2, \end{cases}$$

Then for $x = (1,3)$, $T(1,3) = (1,1)$ and

$$\begin{aligned}\overline{S}(x,x,Tx) &= \overline{S}((1,3),(1,3),(1,1))\\ &= \sinh(S(1,1,1)) + \sinh(S(3,3,1))\\ &= \frac{4}{3}.\end{aligned}$$

Also,

$$\begin{aligned}&\mu(x) - \mu(Tx)\\ &= \overline{S}((2,2),(2,2),(1,3)) + \overline{S}((3,3),(3,3),(1,3))\\ &- \overline{S}((2,2),(2,2),(1,1)) - \overline{S}((3,3),(3,3),(1,1))\\ &= \sinh(S(2,2,1)) + \sinh(S(2,2,3)) + \sinh(S(3,3,1))\\ &- 2\sinh(S(2,2,1)) - 2\sinh(S(3,3,1))\\ &= -\sinh(\ln 5)\\ &= -2.4.\end{aligned}$$

So, $\overline{S}(x,x,Tx) > \mu(x) - \mu(Tx)$ for $x = (1,3)$.
Thus condition (i) of Theorem 3.1 is not satisfied for $x = (1,3)$.
But it is seen that T satisfies condition (ii) for all $x \in E(c_1,c_2,\frac{38}{15})$. Clearly $E(c_1,c_2,\frac{38}{15})$ is not a fixed ellipse here, although T fixes the points (3,1) and (2,1) of the ellipse.

Example 3.4 Let $X = \mathbb{R}^2$ and define an S_p-metric $\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by

$$\overline{S}(x,y,z) = e^{|x_1-y_1|+|y_1-z_1|+|x_2-y_2|+|y_2-z_2|} - 1,$$

$x = (x_1,x_2)$, $y = (y_1,y_2), z = (z_1,z_2) \in X$ with $\Omega(t) = e^t - 1, t \in [0,\infty)$. Now, for $c_1 = (0,0)$, $c_2 = (\ln 6, \ln 6)$, $a = 19.45$, the equation of the ellipse with foci at (c_1,c_2) is

$$\begin{aligned}E(c_1,c_2,19.45) &= \{x \in X : \overline{S}(c_1,c_1,x)\\ &\quad + \overline{S}(c_2,c_2,x) = 38.9\}\\ &= \{x \in X : e^{|x_1|+|x_2|}\\ &\quad + e^{|\ln 6 - x_1| + |\ln 6 - x_2|} = 40.9\}.\end{aligned}$$

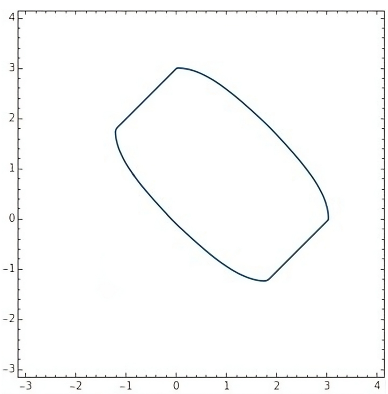

Figure 2

The figure (Figure 2) depicts the ellipse $E(c_1,c_2,19.45)$.
Define $T : X \to X$ by

$$T(x,y) = \begin{cases}(x,y), & (x,y) \in E(c_1,c_2,19.45);\\ (1,1), & otherwise.\end{cases}$$

For any arbitrary point $(x,y) \in E(c_1,c_2,19.45)$, $T(x,y) = (x,y)$.
So, $\overline{S}(x,x,Tx) = \mu(x) - \mu(Tx)$.
Thus condition (i) of the Theorem 3.1 is satisfied for all $(x,y) \in E(c_1,c_2,19.45)$.
Again,

$$\begin{aligned}&\overline{S}(c_1,c_1,Tx) + \overline{S}(c_2,c_2,Tx)\\ &= \overline{S}((0,0),(0,0),T(x,y)) + \overline{S}((\ln 6, \ln 6),(\ln 6, \ln 6),T(x,y))\\ &= \overline{S}((0,0),(0,0),(x,y)) + \overline{S}((\ln 6, \ln 6),(\ln 6, \ln 6),(x,y))\\ &= 38.9\end{aligned}$$

So, condition (ii) of the Theorem 3.1 is satisfied for all the points of the ellipse. Therefore, by Theorem 3.1, $E(c_1,c_2,19.45)$ is a fixed ellipse of T.

Example 3.5 Let $X = \{1,2,3,4\}$ and define an S_p-metric $\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by $\overline{S}(x,y,z) = e^{S(x,y,z)} - 1$ with $\Omega(t) = e^t - 1$, $t \in [0,\infty)$, where (X,S) is an S-metric space with

$$\begin{aligned}&S(1,1,4) = S(4,4,1) = S(1,1,2) = S(2,2,1) = 4,\\ &S(2,2,3) = S(3,3,2) = S(4,4,3) = S(3,3,4) = 2,\\ &S(x,y,z) = 0, \text{ if } x = y = z,\\ &S(x,y,z) = 1, \text{ otherwise.}\end{aligned}$$

We consider the ellipse

$$\begin{aligned}&E(1,3,\frac{e^4 + e^2 - 2}{2})\\ &= \{x \in X : \overline{S}(1,1,x) + \overline{S}(3,3,x) = e^4 + e^2 - 2\}\\ &= \{2,4\}.\end{aligned}$$

Let $T : X \to X$ be defined by

$$T(x) = \begin{cases}1, & x = 1;\\ 2, & x \neq 1.\end{cases}$$

For $x = 4$, $Tx = 2$ and $\overline{S}(x,x,Tx) = \overline{S}(4,4,2) = e - 1$.
Also,

$$\begin{aligned}&\mu(x) - \mu(Tx)\\ &= \overline{S}(1,1,4) + \overline{S}(3,3,4) - \overline{S}(1,1,2) - \overline{S}(3,3,2)\\ &= e^4 - 1 + e^2 - 1 - e^4 + 1 - e^2 + 1\\ &= 0.\end{aligned}$$

So, $\overline{S}(x,x,Tx) > \mu(x) - \mu(Tx)$ for $x = 4$.
Thus, condition (i) of the Theorem 3.1 is not satisfied for $x = 4$.
It can be verified that condition (ii) is satisfied for $x = 2$ and $x = 4$.
Hence T does not satisfy the condition (i) but satisfies the condition (ii) of the Theorem 3.1. Here, T fixes the point $x = 2$ of the ellipse.

In the next result, we use the following mapping $\psi_a : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ defined by

$$\psi_a(p) = \begin{cases}p - \frac{a}{8}, & p > 0;\\ 0, & p = 0,\end{cases}$$

where $a \in [0,\infty)$.

Theorem 3.6 Let $(X,\overline{S})$ be an S_p-metric space and for $c_1,c_2 \in X$, $a \in [0,\infty)$, $E(c_1,c_2,a)$ be an ellipse on X. Let T be a self-mapping on X satisfying

(i) $\overline{S}(c_1,c_1,Tx) + \overline{S}(c_2,c_2,Tx) = 2a$ for all $x \in E(c_1,c_2,a)$,

(ii) $\Omega^{-1}(\overline{S}(Tx,Ty,Tz)) > \frac{a}{8}$ for all $x,y,z \in E(c_1,c_2,a)$ with $x \neq y \neq z$,

(iii) $\Omega^{-1}(\overline{S}(Tx,Ty,Tz)) \leq \Omega^{-1}(\overline{S}(x,y,z)) - \psi_a(\overline{S}(Tx,Tx,x) + \overline{S}(T^2x,T^2x,x))$ for all $x,y,z \in E(c_1,c_2,a)$.

Then $E(c_1,c_2,a)$ is a fixed ellipse of T.

Proof: Let $x \in E(c_1,c_2,a)$. By (i), $Tx \in E(c_1,c_2,a)$ and $T^2x \in E(c_1,c_2,a)$.
We assume that $x \neq Tx$. Using (ii), for $y = Tx, z = T^2x$, we have,

$$\Omega^{-1}(\overline{S}(Tx,T^2x,T^3x)) > \frac{a}{8}. \tag{6}$$

Now, using (iii), for $y = Tx,\ \ z = T^2x$, we get,

$$\begin{aligned}
&\Omega^{-1}(\overline{S}(Tx,T^2x,T^3x))\\
&\leq \Omega^{-1}(\overline{S}(x,Tx,T^2x)) - \psi_a(\overline{S}(Tx,Tx,x)\\
&+ \overline{S}(T^2x,T^2x,x))\\
&= \Omega^{-1}(\overline{S}(x,Tx,T^2x)) - \overline{S}(Tx,Tx,x) - \overline{S}(T^2x,T^2x,x)\\
&+ \frac{a}{8}\\
&\leq \Omega^{-1}(\Omega(\overline{S}(x,x,x) + \overline{S}(Tx,Tx,x) + \overline{S}(T^2x,T^2x,x))\\
&- \overline{S}(Tx,Tx,x) - \overline{S}(T^2x,T^2x,x) + \frac{a}{8}\\
&= \overline{S}(x,x,x) + \overline{S}(Tx,Tx,x) + \overline{S}(T^2x,T^2x,x)\\
&- \overline{S}(Tx,Tx,x) - \overline{S}(T^2x,T^2x,x) + \frac{a}{8}\\
&= \frac{a}{8},
\end{aligned}$$

which contradicts the inequality (6) . Hence $x = Tx$.
So, $E(c_1,c_2,a)$ is a fixed ellipse of T. ■

We present the following examples to demonstrate the above Theorem.

Example 3.7 Let $X = \mathbb{R}^+$ and define an S_p-metric $\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by
$\overline{S}(x,y,z) = |\ln x - \ln y| + |\ln y - \ln z|$ for all $x,y,z \in X$ with $\Omega(t) = 2t$, $t \in [0,\infty)$.
Then $(X,\overline{S})$ is an S_p-metric space. For $c_1 = e^2, c_2 = e^5, a = 2.5$,

$$\begin{aligned}
E(e^2,e^5,2.5) &= \{x \in X : \overline{S}(e^2,e^2,x) + \overline{S}(e^5,e^5,x) = 5\}\\
&= \{x \in X : |2 - \ln x| + |5 - \ln x| = 5\}\\
&= \{e,e^6\}.
\end{aligned}$$

We define $T : X \to X$ by

$$T(x) = \begin{cases} x, & x \in E(e^2,e^5,2.5);\\ e, & otherwise. \end{cases}$$

Then T satisfies all the conditions of Theorem 3.6 and T fixes the ellipse $E(e^2,e^5,2.5)$.

Example 3.8 Let $X = \{0,2,3,4,\ln 6\}$ and define an S_p-metric
$\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by $\overline{S}(x,y,z) = \sec^{-1}(e^{S(x,y,z)})$ with $\Omega(t) = \sec^{-1}(e^t)$, $t \in [0,\infty)$, where (X,S) is an S-metric space with

$$\begin{aligned}
&S(0,0,2) = S(2,2,0) = S(\ln 6, \ln 6,3) = S(3,3,\ln 6)\\
&= S(\ln 6, \ln 6,4) = S(4,4,\ln 6) = \ln 2;
\end{aligned}$$

and

$$\begin{aligned}
&S(0,0,3) = S(3,3,0) = S(0,0,4) = S(4,4,0) = S(\ln 6, \ln 6,2)\\
&= S(2,2,\ln 6) = \ln\sqrt{2};
\end{aligned}$$

Again,

$$\begin{aligned}
&S(x,y,z) = 0, \quad x = y = z;\\
&S(x,y,z) = \ln\frac{2}{\sqrt{3}}, \quad \text{otherwise.}
\end{aligned}$$

Then $(X,\overline{S})$ is an S_p-metric space. We consider the following ellipse:

$$\begin{aligned}
E(0,\ln 6,\frac{7\pi}{24}) &= \{x \in X : \overline{S}(0,0,x) + \overline{S}(\ln 6, \ln 6,x) = \frac{7\pi}{12}\}\\
&= \{2,3,4\}.
\end{aligned}$$

Define $T : X \to X$ by

$$T(x) = \begin{cases} x, & x \in E(0,\ln 6,\frac{7\pi}{24});\\ e, & \text{otherwise.} \end{cases}$$

Obviously, the condition (i) of the Theorem 3.6 is satisfied for all the points of the ellipse.
Now,

$$\begin{aligned}
\Omega^{-1}(\overline{S}(Tx,Ty,Tz)) &= \Omega^{-1}(\overline{S}(2,3,4))\\
&= \Omega^{-1}(\sec^{-1}(e^{\ln\frac{2}{\sqrt{3}}}))\\
&= \Omega^{-1}(\frac{\pi}{6})\\
&= \ln(\sec\frac{\pi}{6})\\
&= 0.1438 > \frac{7\pi}{192}.
\end{aligned}$$

So, $\Omega^{-1}(\overline{S}(Tx,Ty,Tz)) > \frac{a}{8}$.
Again,

$$\Omega^{-1}(\overline{S}(x,y,z)) = \Omega^{-1}(\overline{S}(2,3,4)) = \Omega^{-1}(\overline{S}(Tx,Ty,Tz))$$

and

$$\psi_a(\overline{S}(Tx,Tx,x) + \overline{S}(T^2x,T^2x,x)) = \psi_a(0) = 0.$$

So,

$$\begin{aligned}
\Omega^{-1}(\overline{S}(Tx,Ty,Tz)) = \Omega^{-1}(\overline{S}(x,y,z)) - \psi_a(\overline{S}(Tx,Tx,x)\\
+ \overline{S}(T^2x,T^2x,x))
\end{aligned}$$

for all $x,y,z \in E(c_1,c_2,a)$.
Thus, T satisfies all the conditions of the Theorem 3.6 and so, $E(0,\ln 6,\frac{7\pi}{24})$ is a fixed ellipse of T.

Remark 3.9 It is seen that the mapping T in Example 3.4 does not satisfy the condition (ii) of the Theorem 3.6 although it has a fixed ellipse. So, the conditions of the Theorem 3.6 are not necessary for existence of a fixed ellipse.

IV. Some discontinuity results at fixed point and fixed ellipse

It is noteworthy that in all the results of the above section, the self-mapping T is continuous in the fixed ellipse. In this context, it is interesting to investigate the discontinuity of the mapping at the fixed point. Our next results deal with such situations. Here we assume the S_p-metric $\overline{S}$ to be continuous. We consider $M(x,y,z)$ and α,β,γ as defined in (2).

Theorem 4.1 Let $(X,\overline{S})$ be a complete S_p-metric space and T be a self-mapping on X satisfying the conditions:

(i) there exists a function $\phi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that $\phi(t) < t$ for each $t > 0$ and
$$\overline{S}(Tx,Ty,Tz) \leq \phi(M(x,y,z))$$
for all $x,y,z \in X$,

(ii) for a given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < M(x,y,z) < \varepsilon + \delta$ implies $\overline{S}(Tx,Ty,Tz) \leq \varepsilon$ for all $x,y,z \in X$.

Then T has a unique fixed point $p \in X$. Also, T is discontinuous at p if and only if $\lim_{x\to p} M(x,x,p) \neq 0$.

Proof: First, we define a number
$$\eta = \max\left\{\frac{\alpha}{1-\alpha}, \frac{2\beta}{3-\beta}, \frac{\gamma}{3-2\gamma}\right\},$$
where α,β,γ are as in (2).
Clearly, $\eta < 1$.
By (i), there exists $\phi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that
$$\overline{S}(Tx,Ty,Tz) \leq \phi(M(x,y,z)) < M(x,y,z), \tag{7}$$
whenever $M(x,y,z) > 0$.
For $x_0 \in X$ with $Tx_0 \neq x_0$, we consider the Picard's sequence $\{x_n\}$ as
$x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$.
If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then x_n will be a fixed point of T. We assume that $x_n \neq x_{n+1}$, for each $n \in \mathbb{N}$.
Now,
$$\begin{aligned}
&\overline{S}(x_n,x_n,x_{n+1})\\
&= \overline{S}(Tx_{n-1},Tx_{n-1},Tx_n)\\
&\leq \phi(M(x_{n-1},x_{n-1},x_n))\\
&< M(x_{n-1},x_{n-1},x_n)\\
&= \max\Big\{\alpha(\overline{S}(x_{n-1},x_{n-1},x_n) + \overline{S}(x_n,x_n,x_{n+1})),\\
&\frac{\beta}{3}(2\overline{S}(x_{n-1},x_{n-1},x_n) + \overline{S}(x_n,x_n,x_{n+1})),\\
&\frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x_{n-1},x_n,x_n)) + \Omega^{-1}(\overline{S}(x_n,x_n,x_n))\\
&+ \Omega^{-1}(\overline{S}(x_n,x_{n+1},x_n)))\Big\}
\end{aligned}$$
If
$$\begin{aligned}
&M(x_{n-1},x_{n-1},x_n)\\
&= \alpha(\overline{S}(x_{n-1},x_{n-1},x_n) + \overline{S}(x_n,x_n,x_{n+1})),
\end{aligned}$$
then
$$\overline{S}(x_n,x_n,x_{n+1}) < \alpha(\overline{S}(x_{n-1},x_{n-1},x_n) + \overline{S}(x_n,x_n,x_{n+1})),$$
which implies, $\overline{S}(x_n,x_n,x_{n+1}) < \frac{\alpha}{1-\alpha}\overline{S}(x_{n-1},x_{n-1},x_n) \leq \eta\overline{S}(x_{n-1},x_{n-1},x_n)$,
$$\text{i.e., } \overline{S}(x_n,x_n,x_{n+1}) < \overline{S}(x_{n-1},x_{n-1},x_n). \tag{8}$$
If
$$\begin{aligned}
M(x_{n-1},x_{n-1},x_n) = \frac{\beta}{3}(2\overline{S}(x_{n-1},x_{n-1},x_n)\\
+ \overline{S}(x_n,x_n,x_{n+1})),
\end{aligned}$$
then
$$\overline{S}(x_n,x_n,x_{n+1}) < \frac{\beta}{3}(2\overline{S}(x_{n-1},x_{n-1},x_n)+\overline{S}(x_n,x_n,x_{n+1})),$$
which implies, $\overline{S}(x_n,x_n,x_{n+1}) < \frac{2\beta}{3-\beta}\overline{S}(x_{n-1},x_{n-1},x_n) \leq \eta\overline{S}(x_{n-1},x_{n-1},x_n)$,
$$\text{i.e., } \overline{S}(x_n,x_n,x_{n+1}) < \overline{S}(x_{n-1},x_{n-1},x_n). \tag{9}$$
If
$$\begin{aligned}
&M(x_{n-1},x_{n-1},x_n)\\
&= \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x_{n-1},x_n,x_n)) + \Omega^{-1}(\overline{S}(x_n,x_n,x_n))\\
&+ \Omega^{-1}(\overline{S}(x_n,x_{n+1},x_n))),
\end{aligned}$$
then
$$\begin{aligned}
\overline{S}(x_n,x_n,x_{n+1}) &< \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x_{n-1},x_n,x_n))\\
&+ \Omega^{-1}(\overline{S}(x_n,x_n,x_n)) + \Omega^{-1}(\overline{S}(x_n,x_{n+1},x_n)))\\
&< \frac{\gamma}{3}(\Omega^{-1}(\Omega(\overline{S}(x_{n-1},x_{n-1},x_n)))\\
&+ \Omega^{-1}(\Omega(2\overline{S}(x_n,x_n,x_{n+1}))))
\end{aligned}$$
which implies, $\overline{S}(x_n,x_n,x_{n+1}) < \frac{\gamma}{3-2\gamma}\overline{S}(x_{n-1},x_{n-1},x_n) \leq \eta\overline{S}(x_{n-1},x_{n-1},x_n)$,
$$\text{i.e., } \overline{S}(x_n,x_n,x_{n+1}) < \overline{S}(x_{n-1},x_{n-1},x_n). \tag{10}$$
If we set $l_n = \overline{S}(x_n,x_n,x_{n+1})$, then by (8), (9) and (10), we have,
$$l_n < l_{n-1}. \tag{11}$$
Thus, $\{l_n\}$ is a decreasing sequence of positive real numbers which converges to some $l \geq 0$.
Assume $l > 0$. Then for $\delta(l) > 0$, there exists a positive integer $k \in \mathbb{N}$ such that
$$n \geq k \implies l < l_n < l + \delta(l). \tag{12}$$
$$\begin{aligned}
\text{Now, } \alpha(\overline{S}(x_n,x_n,x_{n+1})+\overline{S}(x_{n+1},x_{n+1},x_{n+2}))\\
< 2\alpha\overline{S}(x_n,x_n,x_{n+1})\\
< \overline{S}(x_n,x_n,x_{n+1}).
\end{aligned}$$
$$\begin{aligned}
\text{Similarly, } \frac{\beta}{3}(2\overline{S}(x_n,x_n,x_{n+1}) + \overline{S}(x_{n+1},x_{n+1},x_{n+2}))\\
< \beta\overline{S}(x_n,x_n,x_{n+1})\\
< \overline{S}(x_n,x_n,x_{n+1}).
\end{aligned}$$
Again,
$$\begin{aligned}
&\frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x_n,x_{n+1},x_{n+1})) + \Omega^{-1}(\overline{S}(x_{n+1},x_{n+2},x_{n+1})))\\
&\leq \frac{\gamma}{3}(\Omega^{-1}(\Omega(\overline{S}(x_n,x_n,x_{n+1})))\\
&+ \Omega^{-1}(\Omega(2\overline{S}(x_{n+1},x_{n+1},x_{n+2}))))
\end{aligned}$$

$$= \frac{\gamma}{3}(\overline{S}(x_n,x_n,x_{n+1}) + 2\overline{S}(x_{n+1},x_{n+1},x_{n+2})) < \gamma\overline{S}(x_n,x_n,x_{n+1}) < \overline{S}(x_n,x_n,x_{n+1}).$$

Thus, $M(x_n,x_n,x_{n+1}) < \overline{S}(x_n,x_n,x_{n+1}) = l_n$.
Again,

$$l < l_{n+1} = \overline{S}(x_{n+1},x_{n+1},x_{n+2}) \leq \phi(M(x_n,x_n,x_{n+1})) < M(x_n,x_n,x_{n+1}).$$

Hence

$$l < M(x_n,x_n,x_{n+1}) < l + \delta(l). \tag{13}$$

Using the condition (ii) and inequality (13), we get,

$$\overline{S}(Tx_n,Tx_n,Tx_{n+1}) \leq l, \text{ i.e., } \overline{S}(x_{n+1},x_{n+1},x_{n+2}) \leq l,$$
$$\text{i.e., } l_{n+1} \leq l \text{ for } n \geq k, \text{a contradiction to}(12).$$

Therefore, $l = 0$.
Now, we prove that $\{x_n\}$ is a Cauchy sequence.
We fix $\varepsilon > 0$, and without loss of generality, we take $\delta < \varepsilon$.
Since $l_n \to 0$, there exists $k \in \mathbb{N}$ such that $\overline{S}(x_n,x_n,x_{n+1}) = l_n < \frac{\delta}{4}$ for all $n \geq k$.
For $k < n$,

$$\begin{aligned}
&\overline{S}(x_k,x_n,x_n)\\
&\leq \Omega(\overline{S}(x_k,x_k,x_{k+1}) + 2\overline{S}(x_n,x_n,x_{k+1}))\\
&\leq \Omega(\overline{S}(x_k,x_k,x_{k+1}) + 2\Omega(\overline{S}(x_{k+1},x_{k+1},x_{k+2})\\
&+ 2\overline{S}(x_n,x_n,x_{k+2})))\\
&\leq \Omega(\overline{S}(x_k,x_k,x_{k+1}) + 2\Omega(\overline{S}(x_{k+1},x_{k+1},x_{k+2})\\
&+ 2\Omega(\overline{S}(x_{k+2},x_{k+2},x_{k+3}) + 2\overline{S}(x_n,x_n,x_{k+3}))))\\
&\leq \Omega(\overline{S}(x_k,x_k,x_{k+1}) + 2\Omega(\overline{S}(x_{k+1},x_{k+1},x_{k+2})\\
&+ 2\Omega(\overline{S}(x_{k+2},x_{k+2},x_{k+3}) + 2\Omega(\overline{S}(x_{k+3},x_{k+3},x_{k+4})\\
&+ 2\overline{S}(x_n,x_n,x_{k+4})))))\\
&\leq \Omega(\overline{S}(x_k,x_k,x_{k+1}) + 2\Omega(\overline{S}(x_{k+1},x_{k+1},x_{k+2})\\
&+ 2\Omega(\overline{S}(x_{k+2},x_{k+2},x_{k+3}) + 2\Omega(\overline{S}(x_{k+3},x_{k+3},x_{k+4}) + ...\\
&+ 2\Omega(\overline{S}(x_{n-1},x_{n-1},x_n) + 2\overline{S}(x_n,x_n,x_n))))))
\end{aligned}$$

Thus, when $k \to \infty$, $\overline{S}(x_k,x_k,x_n) \to 0$.
So, $\{x_n\}$ is a Cauchy sequence in X. Since $(X,\overline{S})$ is complete, there exists $p \in X$ such that $x_n \to p$.
Now, we prove that p is a fixed point of T.
If $Tp \neq p$, then using (i) and the property of ϕ, we obtain,

$$\begin{aligned}
&\overline{S}(Tx_n,Tx_n,Tp)\\
&\leq \phi(M(x_n,x_n,p))\\
&< M(x_n,x_n,p)\\
&= \max\Big\{\alpha(\overline{S}(x_n,x_n,p) + \overline{S}(Tx_n,Tx_n,Tp)),\\
&\frac{\beta}{3}(2\overline{S}(x_n,x_n,Tx_n) + \overline{S}(p,p,Tp)), \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x_n,Tx_n,p))\\
&+ \Omega^{-1}(\overline{S}(p,Tx_n,Tx_n)) + \Omega^{-1}(\overline{S}(Tx_n,Tp,Tx_n)))\Big\}.
\end{aligned} \tag{14}$$

If $M(x_n,x_n,p) = \alpha(\overline{S}(x_n,x_n,p) + \overline{S}(Tx_n,Tx_n,Tp))$, then

$$\overline{S}(Tx_n,Tx_n,Tp) < \alpha(\overline{S}(x_n,x_n,p) + \overline{S}(Tx_n,Tx_n,Tp)).$$

Taking limit as $n \to \infty$ and using the continuity of $\overline{S}$, we get,

$$\overline{S}(p,p,Tp) < \alpha\overline{S}(p,p,Tp)$$
$$i.e., \overline{S}(p,p,Tp) < \overline{S}(p,p,Tp). \tag{15}$$

If $M(x_n,x_n,p) = \frac{\beta}{3}(2\overline{S}(x_n,x_n,Tx_n) + \overline{S}(p,p,Tp))$, then

$$\overline{S}(Tx_n,Tx_n,Tp) < \frac{\beta}{3}(2\overline{S}(x_n,x_n,Tx_n) + \overline{S}(p,p,Tp)),$$

and taking limit as $n \to \infty$, we get,

$$\overline{S}(p,p,Tp) < \frac{\beta}{3}\overline{S}(p,p,Tp)$$
$$i.e., \overline{S}(p,p,Tp) < \overline{S}(p,p,Tp). \tag{16}$$

For

$$\begin{aligned}
&M(x_n,x_n,p)\\
&= \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x_n,Tx_n,p)) + \Omega^{-1}(\overline{S}(p,Tx_n,Tx_n))\\
&+ \Omega^{-1}(\overline{S}(Tx_n,Tp,Tx_n))),
\end{aligned}$$

we have,

$$\begin{aligned}
&\overline{S}(Tx_n,Tx_n,Tp)\\
&< \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x_n,Tx_n,p)) + \Omega^{-1}(\overline{S}(p,Tx_n,Tx_n))\\
&+ \Omega^{-1}(\overline{S}(Tx_n,Tp,Tx_n)))\\
&\leq \frac{\gamma}{3}(\overline{S}(x_n,Tx_n,p) + \overline{S}(p,Tx_n,Tx_n)\\
&+ \Omega^{-1}(\Omega(2\overline{S}(Tx_n,Tx_n,Tp))))\\
&= \frac{\gamma}{3}(\overline{S}(x_n,Tx_n,p) + \overline{S}(p,Tx_n,Tx_n)\\
&+ 2\overline{S}(Tx_n,Tx_n,Tp)).
\end{aligned}$$

Taking limit as $n \to \infty$, we get,

$$\overline{S}(p,p,Tp) < \frac{2\gamma}{3}\overline{S}(p,p,Tp)))$$
$$i.e., \overline{S}(p,p,Tp) < \overline{S}(p,p,Tp). \tag{17}$$

So, by (15), (16) and (17), we have a contradiction.
Therefore, $Tp = p$.
To show the uniqueness, let q be another fixed point of T such that $p \neq q$.
Now,

$$\begin{aligned}
\overline{S}(p,p,q) &= \overline{S}(Tp,Tp,Tq) \leq \phi(M(p,p,q)) < M(p,p,q)\\
&= \max\Big\{\alpha(\overline{S}(p,p,q) + \overline{S}(Tp,Tp,Tq)),\\
&\frac{\beta}{3}(2\overline{S}(p,p,Tp) + \overline{S}(q,q,Tq)), \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(p,Tp,q))\\
&+ \Omega^{-1}(\overline{S}(q,Tp,Tp)) + \Omega^{-1}(\overline{S}(Tp,Tq,Tp)))\Big\}.
\end{aligned}$$

Considering different cases for $M(p,p,q)$, it can be shown that

$$\overline{S}(p,p,q) < \overline{S}(p,p,q),$$

a contradiction. Therefore, $p = q$, i.e., T has a unique fixed point $p \in X$.
Now, we show that T is continuous at p if and only if $\lim_{x_n \to p} M(x_n,x_n,p) = 0$.
Suppose, T is continuous at the fixed point p and $x_n \to p$.
So, $Tx_n \to Tp = p$.

Now, from (14), it is clear that $\lim_{x_n \to p} M(x_n,x_n,p) = 0$.
Conversely, we assume that $\lim_{x_n \to p} M(x_n,x_n,p) = 0$.
Then

$$\lim_{x_n \to p} \max \Big\{ \alpha(\overline{S}(x_n,x_n,p) + \overline{S}(Tx_n,Tx_n,Tp)), \frac{\beta}{3}(2\overline{S}(x_n,x_n,Tx_n) + \overline{S}(p,p,Tp)), \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x_n,Tx_n,p)) + \Omega^{-1}(\overline{S}(p,Tx_n,Tx_n)) + \Omega^{-1}(\overline{S}(Tx_n,Tp,Tx_n))) \Big\} = 0,$$

and so, $\overline{S}(Tx_n,Tx_n,p) \to 0$ as $x_n \to p$, i.e., $Tx_n \to p = Tp$.
Hence T is continuous at p. ■

Considering $\alpha = 0 = \gamma$ and $\beta = \frac{1}{3}$ with $\phi(t) = \frac{3t}{4}$, $t \geq 0$ in the above theorem, we get the following.

Corollary 4.2 Let $(X,\overline{S})$ be a complete S_p-metric space and T be a self-mapping on X satisfying the conditions:

(i) there exists a function $\phi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that $\phi(t) < t$ for each $t > 0$ and

$$\overline{S}(Tx,Ty,Tz) \leq \frac{1}{12}(\overline{S}(x,x,Tx)+\overline{S}(y,y,Ty)+\overline{S}(z,z,Tz))$$

for all $x,y,z \in X$,

(ii) for a given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < \frac{1}{9}(\overline{S}(x,x,Tx) + \overline{S}(y,y,Ty) + \overline{S}(z,z,Tz)) < \varepsilon + \delta$ implies $\overline{S}(Tx,Ty,Tz) \leq \varepsilon$ for all $x,y,z \in X$.

Then T has a unique fixed point $p \in X$. Also, T is discontinuous at p if and only if $\lim_{x \to p} M(x,x,p) \neq 0$.

Remark 4.3 Theorem 4.1 can be taken as an extension of Theorem 2.1 of [17] and Theorem 1 of [11] in the setting of S_p-metric space.

Example 4.4 Let $X = [0,2]$ and $(X,\overline{S})$ be a S_p-metric space defined as
$\overline{S}(x,y,z) = e^{|x-y|+|x+y-2z|} - 1$ for all $x,y,z \in X$ with $\Omega(t) = e^t - 1$.
Define $T : X \to X$ by

$$T(x) = \begin{cases} 1, & x \leq 1; \\ 0, & x > 1. \end{cases}$$

Then T has a unique fixed point at $x = 1$ which is also the point of discontinuity of T.
We take $\alpha = \gamma = \frac{1}{8}$ and $\beta = \frac{3}{4}$.
Now

$$\overline{S}(Tx,Ty,Tz) = 0 \ \ and \ \ 0 < M(x,y,z) < \frac{3}{4}(e^2 - 1), \quad (18)$$

when $x,y,z \leq 1$.

$$\overline{S}(Tx,Ty,Tz) = e^2 - 1$$

$$\text{and} \ \ \frac{1}{4}(e^4 - 1) < M(x,y,z) < \frac{1}{4}(e^4 + 2e^2 - 3), \quad (19)$$

when $x > 1, y,z \leq 1$ or $z > 1, x,y \leq 1$ or $y > 1,x,z \leq 1$.

$$\overline{S}(Tx,Ty,Tz) = e^2 - 1$$

$$\text{and} \ \ \frac{1}{2}(e^4 - 1) < M(x,y,z) < \frac{1}{4}(2e^4 + e^2 - 3), \quad (20)$$

when $x \leq 1,y,z > 1$ or $y \leq 1,x,z > 1$ or $z \leq 1,x,y > 1$.

$$\overline{S}(Tx,Ty,Tz) = 0 \ \ and \ \ \frac{3}{4}(e^2-1) < M(x,y,z) < \frac{3}{4}(e^4-1), \quad (21)$$

when $x,y,z > 1$.
Then T satisfies the condition (i) of Theorem 4.1 with

$$\phi(t) = \begin{cases} \frac{t}{2}, & t > 1; \\ \frac{t}{3}, & t \leq 1. \end{cases}$$

Again, T satisfies the condition (ii) of Theorem 4.1 with

$$\delta(\varepsilon) = \begin{cases} \frac{3}{4}(1-\varepsilon)(e^2 - 1), & \varepsilon < 1; \\ \frac{3}{4}e^2(e^2 - 1), & \varepsilon \geq 1. \end{cases}$$

It can be easily seen that $\lim_{x \to 1} M(x,x,1) \neq 0$.

In the same line as [11], we observe that the following analogous results also hold in complete S_p-metric space.

Theorem 4.5 Let $(X,\overline{S})$ be a complete S_p-metric space and T be a self-mapping on X satisfying the following conditions:

(i) $\overline{S}(Tx,Ty,Tz) < M(x,y,z)$ for any $x,y,z \in X$ with $M(x,y,z) > 0$;

(ii) There exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < M(x,y,z) < \varepsilon + \delta$ implies $\overline{S}((Tx,Ty,Tz) < \varepsilon$ for a given $\varepsilon > 0$.

Then T has a unique fixed point $p \in X$. Also, T is discontinuous at p if and only if $\lim_{x \to p} M(x,x,p) \neq 0$.

Theorem 4.6 Let $(X,\overline{S})$ be a complete S_p-metric space and T be a self-mapping on X satisfying the conditions:

(i) There exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ and $\overline{S}(T^m x,T^m y,T^m z) \leq \phi(M^*(x,y,z))$ where

$$M^*(x,y,z) = \max \Big\{ \alpha(\overline{S}(x,y,z) + \overline{S}(T^m x,T^m y,T^m z)), \frac{\beta}{3}(\overline{S}(x,x,T^m x) + \overline{S}(y,y,T^m y) + \overline{S}(z,z,T^m z)), \frac{\gamma}{3}(\Omega^{-1}(\overline{S}(x,T^m y,z)) + \Omega^{-1}(\overline{S}(z,T^m x,T^m y)) + \Omega^{-1}(\overline{S}(T^m y,T^m z,T^m x))) \Big\},$$

for all $x,y,z \in X$.

(ii) There exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < M^*(x,y,z) < \delta + \varepsilon$ implies $\overline{S}(T^m x,T^m y,T^m z) \leq \varepsilon$ for a given $\varepsilon > 0$.

Then, T has a unique fixed point $p \in X$. Also, T is discontinuous at p if and only if $\lim_{x \to p} M^*(x,x,p) \neq 0$.

Proof: By Theorem 4.1, T^m has a unique fixed point p. So, $T^m p = p$.
Hence we have, $Tp = TT^m p = T^m Tp$. So, Tp is another fixed point of T^m.
From the uniqueness of the fixed point, we have $Tp = p$. Consequently, T has a fixed point p. The next part follows as in Theorem 4.1. ■

Theorem 4.7 Let $E(c_1,c_2,a)$; $c_1,c_2 \in X$; $a \in [0,\infty)$ be a fixed ellipse of a self-mapping T in a complete S_p-metric space $(X,\overline{S})$. Then T is discontinuous at $E(c_1,c_2,a)$ if and only if $\lim_{x \to p} M(x,x,p) \neq 0$.

The proof follows as in the last part of Theorem 4.1.

In [15], Pant et al. defined k-continuity of a mapping as a weaker form of continuity. Accordingly, a self-mapping T on a metric space X is called k-continuous for $k \geq 1$ if $f^k x_n \to ft$ whenever $\{x_n\}$ is a sequence in X such that $f^{k-1}x_n \to t$. It is noteworthy that 1-continuity is equivalent to continuity and continuity $\implies$ 2-continuity $\implies$ 3-continuity $\implies$ .., but not conversely (refer to [15]). Here we prove an analogous result as in Theorem 4.1 in case of a k-continuous mapping.

Theorem 4.8 Let $(X,\overline{S})$ be a complete S_p-metric space and T be a k-continuous self-mapping on X for some $k \geq 1$, satisfying the conditions:

(i) there exists a function $\phi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that $\phi(t) < t$ for each $t > 0$ and
$$\overline{S}(Tx,Ty,Tz) \leq \phi(M(x,y,z))$$
for all $x,y,z \in X$,

(ii) for a given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < M(x,y,z) < \varepsilon + \delta$ implies $\overline{S}(Tx,Ty,Tz) \leq \varepsilon$ for all $x,y,z \in X$,

Then T has a unique fixed point $p \in X$. Also, T is discontinuous at p if and only if $\lim_{x \to p} M(x,x,p) \neq 0$.

Proof: For $x_0 \in X$, as in Theorem 4.1, we can show that the Picard's sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists p in X such that $x_n \to p$. Moreover, for each integer $m \geq 1$, we have $T^m x_n \to p$.

Since $T^{k-1}x_n \to p$, k-continuity of T implies that $T^k x_n \to Tp$. Hence $p = Tp$ and so, p is fixed point of T.
Uniqueness and discontinuity at fixed point follows as in Theorem 4.1. ■

Remark 4.9 Theorem 4.1, Theorem 4.5, Theorem 4.6 and Theorem 4.8 give a new solution to Rhoades' open problem for existence of some new types of contractive mappings which are discontinuous at fixed point and fixed ellipse in S_p-metric space.

V. Application to discontinuous activation function

In recent times, neural networks have experienced remarkable advancement in many areas such as associative memory, pattern recognition, image processing etc. Due to the practical relevance, neural networks utilizing discontinuous activation function have gained much attention in research (refer to [7], [10]). In [24], Wang et al. investigated the neural networks with a class of Maxican-hat-type non-monotonic discontinuous activation function defined as

$$T_i(x) = \begin{cases} r_i, & -\infty < x < a_i \\ m_{i,1}x + n_{i,1}, & a_i \leq x \leq b_i \\ m_{i,2}x + n_{i,2}, & b_i < x \leq c_i \\ s_i, & c_i < x < +\infty. \end{cases} \quad (22)$$

where a_i, b_i, c_i, r_i, $m_{i,1}$, $m_{i,2}$, $n_{i,1}$, $m_{i,2}$ are constants satisfying

$$-\infty < a_i < b_i < c_i < +\infty, m_{i,1} > 0, m_{i,2} < 0,$$

$$r_i = m_{i,1}a_i + n_{i,1} = m_{i,2}c_i + n_{i,2},$$

$$m_{i,1}b_i + n_{i,1} = m_{i,2}b_i + n_{i,2},$$

$$s_i > Tb_i, i = 1,2,...,n.$$

It is found that employing discontinuous activation functions can significantly enhance the storage capacity of neural networks.

In this section, we give an application of our results obtained in Section III to discontinuous activation function.

Example 5.1 Let $X = \{1,3,4,8\}$ and define an S_p-metric $\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by $\overline{S}(x,y,z) = e^{S(x,y,z)} - 1$ with $\Omega(t) = e^t - 1, t \in [0,\infty)$, where (X,S) is an S-metric space with
$S(1,1,4) = S(4,4,1) = S(1,1,8) = S(8,8,1) = 5$,
$S(8,8,3) = S(3,3,8) = S(4,4,3) = S(3,3,4) = 2$,
$S(x,y,z) = 0$, if $x = y = z$,
$S(x,y,z) = \frac{3}{2}$, otherwise.
We consider the ellipse

$$E(1,3,\frac{e^5+e^2-2}{2}) = \{x \in X : \overline{S}(1,1,x) + \overline{S}(3,3,x) = e^5 + e^2 - 2\} = \{4,8\}.$$

Taking $m_{i,1} = 2$, $m_{i,2} = -1$, $n_{i,1} = 2$, $n_{i,2} = 8$, $r_i = 2$, $s_i = 8$, $a_i = 0$, $b_i = 2$, $c_i = 6$, we get,

$$T(x) = \begin{cases} 2, & -\infty < x < 0; \\ 2x + 2, & 0 \leq x \leq 2; \\ -x + 8, & 2 < x \leq 6; \\ 8, & 6 < x < +\infty. \end{cases} \quad (23)$$

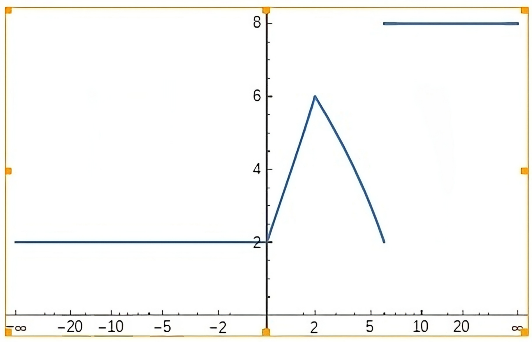

Figure 3: The discontinuous activation function T represented by (23).

T satisfies the conditions (i) and (ii) of the Theorem 3.1 for the ellipse $E(1,3,\frac{e^5+e^2-2}{2})$, where $c_1 = 1$, $c_2 = 3$ and $a = \frac{e^5+e^2-2}{2}$.
Hence, T fixes the ellipse $E(1,3,\frac{e^5+e^2-2}{2})$.
We have,
$\lim_{x \to 4} M(x,x,4) = 0$ and $\lim_{x \to 8} M(x,x,8) = 0$.
Clearly, T is continuous at $x = 4$ and $x = 8$.

Example 5.2 Let $X = \mathbb{N} \cup \{0\}$ and define an S_p-metric $\overline{S} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by $\overline{S}(x,y,z) = e^{S(x,y,z)} - 1$ with

$\Omega(t) = e^t - 1, t \in [0,\infty)$, where (X,S) is an S-metric space as defined in Example 2.3 of [1] as

$$S(x,y,z) = \begin{cases} 0, & x = y = z; \\ x + y + z, & otherwise. \end{cases} \qquad (24)$$

We consider the ellipse

$$\begin{aligned} E(0,3,\frac{e^7+e-2}{2}) &= \{x \in X : \overline{S}(0,0,x) \\ &\quad + \overline{S}(3,3,x) = e^7 + e - 2\} \\ &= \{1\}. \end{aligned}$$

Taking $m_{i,1} = 6,\ \ m_{i,2} = -4,\ \ n_{i,1} = 25,\ \ n_{i,2} = 5,\ \ r_i = 1,\ \ s_i = 15,\ \ a_i = -4,\ b_i = -2,\ \ c_i = 2$, we get,

$$T(x) = \begin{cases} 1, & -\infty < x < -4; \\ 6x + 25, & -4 \le x \le -2; \\ -4x + 5, & -2 < x \le 1; \\ 15, & 1 < x < +\infty. \end{cases} \qquad (25)$$

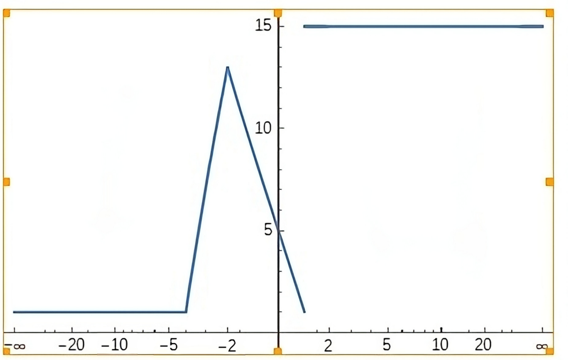

Figure 4: The discontinuous activation function T represented by (25).

T satisfies the conditions (i) and (ii) of the Theorem 3.1 for the ellipse $E(0,3,\frac{e^7+e-2}{2})$, where $c_1 = 0,\ \ c_2 = 3$ and $a = \frac{e^7+e-2}{2}$.
Hence, T fixes the ellipse $E(0,3,\frac{e^7+e-2}{2})$.
We have,
$\lim_{x \to 1} M(x,x,1)$ does not exist. Clearly, T is discontinuous at $x = 1$.

VI. An Application to Existence of Solution of an Integral Equation

In this section, we demonstrate the relevance of Theorem 4.1 to investigate the existence and uniqueness of solution of the following Volterra integral equation:

$$x(t) = p(t) + \int_0^T \lambda(t,r) f(r,x(r)) dr, \qquad (26)$$

$t \in [0,T]$, where $T > 0$. Here, $f,p : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\lambda : [0,T] \times \mathbb{R} \to [0,\infty)$ is also continuous function.
Define a metric $\overline{S}$ on $X = C(I,\mathbb{R})$ (the set of continuous function defined on $I = [0,T]$) by

$$\overline{S}(x,y,z) = e^{\sup_{t\in I}(|x(t)-y(t)| + |y(t)-z(t)| + |z(t)-x(t)|)} - 1,$$

for all $x,y,z \in X$. Then $(X,\overline{S})$ is S_p-metric space with $\Omega(t) = e^t - 1$.
We consider the following conditions:

(i) $\sup_{r\in I} |\lambda(t,r)| \le \frac{1}{T}$.
(ii) $|(f(r,x(r)) - f(r,y(r)))| < |x(t) - y(t)| + \ln\sqrt{a}$, where $a \in [0,\frac{1}{2})$.

Theorem 6.1 Under the assumptions (i) and (ii) equation (25) has a solution in X.
We define $T : X \to X$ by
$T(x(t)) = p(t) + \int_0^T \lambda(t,r) f(r,x(r)) dr$.
Now,

$$\begin{aligned} &\overline{S}(Tx,Tx,Ty) \\ &= e^{2\sup_{t\in I}|Tx(t) - Ty(t)|} - 1 \\ &\le e^{2\sup_{t\in I}\int_0^T \sup_{r\in I}|\lambda(t,r)||(f(r,x(r)) - f(r,y(r)))|dr} - 1 \\ &\le e^{2\sup_{t\in I}\frac{1}{T}\int_0^T |(f(r,x(r)) - f(r,y(r)))|dr} - 1 \\ &\le e^{\frac{2}{T}\sup_{t\in I}\int_0^T (|x(t)-y(t)| + \ln\sqrt{a})dr} - 1 \\ &\le e^{\frac{2}{T}\sup_{t\in I}\int_0^T (|x(t)-y(t)+\ln\sqrt{a}|)dr} - 1 \\ &\le e^{\frac{2}{T}\sup_{t\in I}|x(t)-y(t)+\ln\sqrt{a}|\int_0^T dr} - 1 \\ &= e^{2\sup_{t\in I}|x(t)-y(t)| + \ln\sqrt{a}} - 1 \\ &= ae^{2\sup_{t\in I}|x(t)-y(t)|} - 1 \\ &\le ae^{2\sup_{t\in I}|x(t)-y(t)|} - a \\ &= a(e^{2\sup_{t\in I}|x(t)-y(t)|} - 1) \\ &= a\overline{S}(x,x,y) \\ &= \frac{1}{2}2a\overline{S}(x,x,y) \\ &\le \frac{1}{2}(a\overline{S}(x,x,y) + a\overline{S}(Tx,Tx,Ty)) \\ &\le \frac{1}{2}M(x,x,y) \\ &= \phi(M(x,x,y)) \end{aligned}$$

where $\phi(t) = \frac{t}{2}$ for all $t \in \mathbb{R}^+ \cup \{0\}$.
Consider

$$\delta(\varepsilon) = \begin{cases} 3a\overline{S}(x,x,y) - \varepsilon, & \varepsilon \le 2a\overline{S}(x,x,y); \\ 4a\overline{S}(x,x,y), & \varepsilon > 2a\overline{S}(x,x,y). \end{cases}$$

For, $a\overline{S}(x,x,y) < M(x,x,y) < 3a\overline{S}(x,x,y)$
we have, $\overline{S}(Tx,Tx,Ty) \le a\overline{S}(x,x,y)$.
So, all the conditions of the Theorem 4.1 is satisfied and hence T has a unique fixed point which is the unique solution of the integral equation (25).

VII. Conclusion

In this paper, we derive some fixed ellipse results with analysis of discontinuity at fixed point and fixed ellipse in S_p-metric space. An application is given for discontinuous activation function arising in neural networks. The paper concludes with an application to integral equation. The derived results have several motivations considering future perspective. In 2020, Adewale et al. [2] defined the notion of A_p-metric space and derived some fixed point results with an application to nonlinear integral equation. Analogous study can be done in A_p-metric space using our defined type of contractive conditions. In 2022, Joshi et al. [6] introduced an $\mathcal{M}$-class function in S-metric space which is very useful for finding the existence of a fixed circle and fixed points. In a similar manner, the contractive conditions derived in this paper can be modified using $\mathcal{M}$-class functions, and discontinuity results can be investigated at the fixed point and fixed ellipse.

References

[1] O.K. Adewale and C. Iluno, "Fixed point theorems on rectangular S-metric spaces," Scientific African, vol. 16, pp. e01202, 2022.

[2] O. K. Adewale, J. C. Umudu and A. A. Mogbademu, "Fixed point theorems on A_p-metric spaces," International Journal of Mathematical Sciences and Optimization: Theory and Applications, vol. 2020, no. 1, pp. 657-668, 2020.

[3] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," Fundamenta Mathematicae, vol. 3, no. 1, pp. 133-181, 1922.

[4] J. Caristi, "Fixed point theorems for mappings satisfying inwardness conditions," Transactions of the American Mathematical Society, vol. 215, pp. 241-251, 1976.

[5] M. Dhanraj and A. J. Gnanaprakasam, "On Orthogonal (phi, digamma)-Contraction Type Mappings and Relevant Fixed Point Theorems with Applications," IAENG International Journal of Applied Mathematics, vol. 54, no. 9, pp. 1788-1796, 2024.

[6] M. Joshi, A. Tomar and T. Abdeljawad, "On fixed points, their geometry and application to satellite web coupling problem in S-metric spaces," AIMS Mathematics, vol. 8, no. 2, pp. 4407-4441, 2023.

[7] P. Liu, Z. Zeng and J. Wang, "Multistability of delayed recurrent neural networks with Mexican hat activation functions," Neural Computation, vol. 29, no. 2, pp. 423-457, 2017.

[8] M. Madhuri and M. V. R. Kameswari, "Fixed Point Theorems in Bicomplex Partial S Metric Spaces and Applications to Boundary Value Problems," IAENG International Journal of Applied Mathematics, vol. 54, no. 12, pp. 2824-2831, 2024.

[9] Z. Mustafa, R. J. Shahkoohi, V. Parvaneh, Z. Kadelburg and M. M. M. Jaradat, "Ordered S_p-metric spaces and some fixed point theorems for contractive mappings with application to periodic boundary value problems," Fixed Point Theory and Applications, vol. 2019, pp. 1-20, 2019.

[10] X. Nie and W. X. Zheng, "Dynamical behaviors of multiple equilibria in competitive neural networks with discontinuous nonmonotonic piecewise linear activation functions," IEEE Transactions on Cybernetics, vol. 46, no. 3, pp. 679-693, 2015.

[11] N. Y. Özgür and N. Taş, "A new solution to the Rhoades' open problem with an application," Acta Universitatis Sapientiae, Mathematica, vol. 13, no. 2, pp. 427-441, 2019.

[12] N. Y. Özgür and N. Taş, "Fixed-circle problem on S-metric spaces with a geometric viewpoint," arXiv preprint arXiv:1704.08838, 2017.

[13] N. Y. Özgür and N. Taş, "Some fixed-circle theorems and discontinuity at fixed circle," American Institute of Physics, pp. 020048, 2018.

[14] N. Y. Özgür, N. Taş and U. Çelik, "New fixed-circle results on S-metric spaces," Bull. Math. Anal. Appl., vol. 9, pp. 10-23, 2017.

[15] A. Pant and R. P. Pant, "Fixed points and continuity of contractive maps," Filomat, vol. 31, no. 11, pp. 3501-3506, 2017.

[16] R. P. Pant, "Discontinuity and fixed points," Journal of mathematical analysis and applications, vol. 240, no. 1, pp. 284-289, 1999.

[17] R. P. Pant, N. Y. Özgür and N. Taş, "On discontinuity problem at fixed point," Bulletin of the Malaysian Mathematical Sciences Society, vol. 43, pp. 499-517, 2020.

[18] B. E. Rhoades, "A comparison of various definitions of contractive mappings," Transactions of the American Mathematical Society, vol. 226, pp. 257-290, 1977.

[19] B. E. Rhoades, "Contractive definitions and continuity," Contemporary Math., vol. 72, pp. 233–245, 1988.

[20] S. Sedghi, N. Shobe and A. Aliouche, "A generalization of fixed point theorems in S-metric spaces," Matematički Vesnik., vol. 64, no. 249, pp. 258-266, 2012.

[21] T. C. Singh, "Some Theorems on Fixed Points in N-Cone Metric Spaces with Certain Contractive Conditions," Engineering Letters, vol. 32, no. 9, pp. 1833-1839, 2024.

[22] N. Souayah and N. Mlaiki, "A fixed point theorem in S_b-metric spaces," J. Math. Computer Sci., vol. 16, no. 315, pp. 131-139, 2016.

[23] R. A. Thirumalai and S. Thalapathiraj, "A novel approach for some fixed point results in complete G-metric space by using asymptotically regular mapping in digital image compression and decompression applications," International Journal of Applied Mathematics, vol. 37, no. 3, pp. 311-332, 2024.

[24] L. Wang and T. Chen, "Multistability of neural networks with Mexican-hat-type activation functions," IEEE Transactions on Neural Networks and Learning Systems, vol. 23, no. 11, pp. 1816-1826, 2012.