An Introduction to 2-Domination Integrity of Graphs

Christin Sherly J and Uma Samundesvari K

***Abstract*—The vulnerability concept in communication network plays a prominent role when there is a disruption in the network. Several graph parameters exist to measure the vulnerability of a communication network. This paper introduces a new measure of graph vulnerability: 2-domination integrity of graphs. An algorithm to compute 2-domination integrity of a graph and a realization result is developed. A few bounds relating 2-domination integrity with other graph parameters are determined. Furthermore, new theorems, and results in the context of several graphs are obtained.**

***Index Terms*—dominating set, integrity, 2-dominating set, 2-domination integrity**

I. Introduction

Graph Theory offers a framework for modeling the communication networks as graph structures. Domination in graphs is a renowned branch of graph theory. 2-domination is one of the important domination parameter which was first put up by Fink and Jacobson [5]. Any communication network may be represented as a graph, with the vertices as the stations (processors), and the edges as the connections between the vertices. For network designers to rebuild a communication network once certain stations or communication links collapse, network stability is a crucial consideration. The idea of Domination Integrity was proposed by Sundareswaran and Swaminathan [13]. Connected domination integrity [7] and paired domination integrity [1] are new vulnerability parameters developed for measuring the vulnerability of communication networks. Double domination integrity was introduced in [4] and it finds an application in PMU placement problem [3]. A new vulnerability parameter 2-domination integrity of graphs is introduced in this atricle by combining the concept of 2-domination and integrity. An algorithm to find 2-domination integrity, a realization result and the 2-domination integrity of certain graphs are obtained in this article.

Lemma 1.1: [13] For any graph $G, I(G) \leq DI(G)$

II. 2-Domination Integrity of a Graph

Definition 2.1: [5] A dominating set S of a graph G is a 2-dominating set of G if every vertex of $V - S$ is adjacent to at least two vertices of S. The smallest cardinality among all the 2-dominating sets of G is the 2-domination number of G which is represented as $\gamma_{2d}(G)$. A 2-dominating set of cardinality $\gamma_{2d}(G)$ is called a γ_{2d}-set of G.

Definition 2.2: The 2-domination integrity of a connected graph G is defined by $DI_2(G) = min\{|S| + m(G - S) : S$ is a 2-dominating set of $G\}$ and $m(G - S)$ is the maximum order of the component of $G - S$.

Definition 2.3: A 2-dominating set S of G is called a 2-domination integrity set or DI_2-set of G if $|S| + m(G - S)$ is minimum.

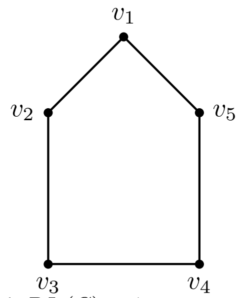

Fig. 1. A graph G with $DI_2(G) = 4$

$S = \{v_1, v_2, v_4\}$ is a DI_2-set of G. So, $|S| = 3$, $m(G-S) = 1$. Hence, $DI_2(G) = 4$

Theorem 2.4: Every pendant vertex of a graph G belongs to the DI_2-set of G.

Proof: Let v be an end vertex of a graph G. Let S represents a DI_2 set of G. To prove: v belongs to S. Suppose $v \notin S$. Since S is a DI_2-set of G and $v \in V - S$, v must be adjacent to at least two vertices of S, which implies that v cannot be an end vertex of G. This leads to a contradiction. Therefore, $v \in S$. ■

Observation 2.5: DI_2-set need not be unique. (i. e) For a graph G, there may exist more than one DI_2-set.

Result 2.6: $DI_2(H)$ need not be necessarily less than or equal to $DI_2(G)$ for any subgraph H of a graph G

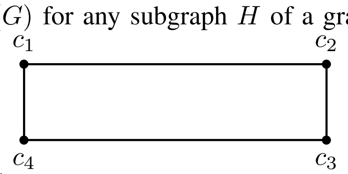

Fig. 2. Cycle C_4

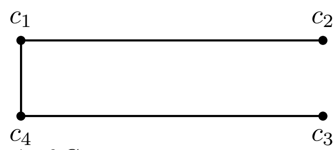

Fig. 3. A Subgraph of C_4

$DI_2(C_4) = 3$ and $DI_2(C_4 - e) = 4$ for any edge e of C_4

III. Realization Result

Theorem 3.1: For any pair a, b of integers with $2 \leq a \leq b$, there is a connected graph G of order b such that $DI_2(G) = a$.

Proof: Let $C : u_1, u_2, u_3, u_4$ be a Cycle of length 4. A graph G in Fig. 4 is got from C_4 by adding vertices $z_1, z_2, ..., z_{a-3}$ and connecting each $z_i (1 \leq i \leq a - 3)$ with

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u_2 and also adding the new vertex u_5 and adjoining u_5 with u_1 and u_3.

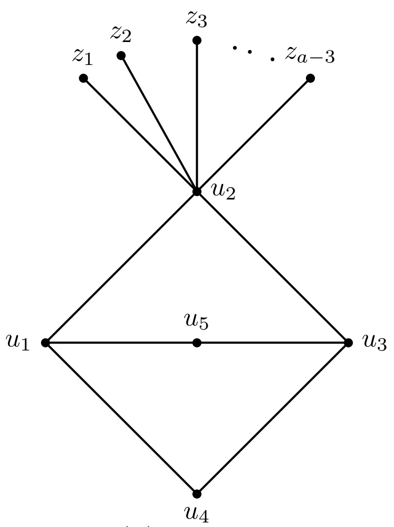

Fig. 4. Graph G with $DI_2(G) = a$

Let $S = \{z_1, z_2, ..., z_{a-3}\}$ be the set of all end vertices of G. By Theorem 2.4, S belongs to the DI_2-set of G. But S is not a 2-domination integrity set of G so that $DI_2(G) \geq a-3$. From the structure of the graph, it is obvious that $S_1 = S \cup \{u_1, u_3\}$ is a 2-dominating set of G with $|S_1| = a-3+2$ and $m(G-S_1) = 1$. Thus, $|S_1|+m(G-S_1)$ is minimum for the above mentioned set S_1. Hence, S_1 is a DI_2-set of G. Therefore, $DI_2(G) = |S_1|+m(G-S_1) = a-3+2+1 = a$. ■

IV. Algorithm to find $DI_2(G)$

Algorithm 1 Finding 2-Domination Integrity of a Simple Connected Graph

Require: A graph $G(n \geq 2)$ without isolated vertices.
Ensure: $DI_2(G)$

Step:1 Determine all the 2-dominating sets of G
Step:2 Finding $DI_2(G)$=$min\{|S|+m(G-S)$: S is a 2-dominating set of $G\}$ where $m(G-S)$ is the maximum order of the component of $G-S$
if S is a γ_{2d}-set of G and $m(G-S) = 0$ **then**
    $DI_2(G) = \gamma_{2d}(G)$
else
    $DI_2(G) = min\,\{|S| + m(G-S)\}$
end if

V. 2-Domination Integrity of Standard Graphs

Observation 5.1: For $n \geq 3$,

$$DI_2(P_n) = \begin{cases} \frac{n}{2}+2 & if \ \ n \ is \ even \\ \frac{n+1}{2}+1 & if \ \ n \ is \ odd \end{cases}$$

Observation 5.2: For $n \geq 3$, the 2-domination integrity of Cycle C_n is

$$\begin{cases} \frac{n}{2}+1 & if \ \ n \ is \ even \\ \frac{n+1}{2}+1 & if \ \ n \ is \ odd \end{cases}$$

Theorem 5.3: For $n \geq 3$, $DI_2(K_n) = n$.

Proof: Let $S = \{x, y\} \subseteq V(K_n)$ be the 2-dominating set of K_n. Then, $K_n - S$ is connected with one component and so $m(K_n - S) = n - |S|$. Hence $DI_2(K_n) = min\{|S| + m(K_n - S) : S$ is a 2-dominating set of $K_n\} = min\,\{|S| + n - |S|\} = n$. ■

Theorem 5.4: For $n \geq 2$, the 2-domination integrity of Star $K_{1,n}$ is $1+n$

Proof: Let x be the central vertex of the Star. Let $X = \{u_1, u_2, ..., u_n\}$ be the set of pendant vertices of $K_{1,n}$. Let S be the DI_2-set of $K_{1,n}$. By the Theorem 2.4, $X \subseteq S$. Obviously, $S = X$ itself is a DDI-set of $K_{1,n}$ with with $|S| = n$. $K_{1,n} - S$ contains only one isolated vertex x. Thus, $m(K_{1,n} - S) = 1$. Therefore, $DDI(K_{1,n}) = |S| + m(K_{1,n} - S) = n+1$. This completes the proof. ■

Theorem 5.5: For $m, n \geq 2$, th 2-domination integrity of Complete bipartite graph $K_{m,n}$ is $DI_2(K_{m,n}) = min\,\{m, n\} + 1$

Proof: Let $V(K_{m,n}) = V_1(K_{m,n}) \cup V_2(K_{m,n})$ and $V_1(K_{m,n}) = \{u_1, u_2, ..., u_m\}$, $V_2(K_{m,n}) = \{v_1, v_2, ..., v_n\}$; where $m, n \geq 2$.

Case: (i) $m \leq n$
$S = \{u_1, u_2, ..., u_m\}$ is a 2-dominating set of $K_{m,n}$ with minimum cardinality. Then $|S| = m = min\{m, n\}$. Removing the vertices of S from $K_{m,n}$ results in a disconnected graph containing n isolated vertices and so $m(K_{m,n} - S) = 1$. If X is any other 2-dominating set of $K_{m,n}$, then $|X| + m(K_{m,n} - S) > |S| + m(K_{m,n} - S)$. Thus, S is the DI_2-set of $K_{m,n}$. Hence, $DI_2(K_{m,n}) = |S| + m(K_{m,n} - S) = min\{m, n\} + 1$.

Case: (ii) $m \geq n$
The proof is as in Case:(i) ■

VI. Bounds on 2-Domination Integrity of Graphs

Observation 6.1: A minimum of two vertices always belong to the 2-dominating set S since every vertex in $V - S$ should be adjacent to at least two vertices of S. (i.e) $|S| \geq 2$.

Theorem 6.2: For $n \geq 2$, $I(G) \leq DI(G) \leq DI_2(G)$.

Proof: We have $\gamma(G) \leq \gamma_{2d}(G)$.
Hence, $DI(G) \leq DI_2(G)$
Using Lemma 1.1,
$I(G) \leq DI(G) \leq DI_2(G)$. ■

Theorem 6.3: $DI_2(G) \geq \gamma_{2d}(G)$

Proof: Let G be a connected graph of order $n > 2$. Let S be a DI_2 set of G. Hence $|S| \geq \gamma_{2d}(G)$ and $m(G - S) \geq 0$. Thus, $DI_2(G) = min\{|S| + m(G-S) : S$ is a 2-dominating set of $G\} \geq min\,\{\gamma_{2d}(G) + 0\} = \gamma_{2d}(G)$. Therefore, $DI_2(G) \geq \gamma_{2d}$. ■

Theorem 6.4: For $n \geq 2$, $DI_2(G) \geq \chi(G)$

Proof: Let G be a connected graph with $n \geq 2$.
Since $I(G) \geq \chi(G)$ and using Theorem 6.3,
$DI_2(G) \geq \chi(G)$. ■

Remark 6.5: The bound obtained for $DI_2(G)$ in the above Theorem is reachable. For example, the double domination integrity of Complete graph K_n is $DI_2(K_n) = \chi(K_n)$. By the given Theorem 5.3, $D_2I(K_n) = n$. Since all the vertices of K_n are adjacent to each other, $\chi(K_n) = n$.

VII. 2-Domination Integrity of Some Special Graphs

Definition 7.1: [2] A collection of t-triangles with a vertex in common is called the friendship graph $C_3^{(t)}$.

Theorem 7.2: For $n \geq 2$, $DI_2(C_3^{(n)}) = n + 2$.

Proof: Let $C_3^{(n)}$ be a friendship graph consisting of n triangles. Denote the central vertex of the graph as x and for each $i = 1, 2, ..., n$, let u_i and v_i be the other two vertices of the i^{th} triangle for $1 \leq i \leq n$. Let S denotes a 2-dominating set of $C_3^{(n)}$. Inorder for the set S to be minimal, it needs to contain the central vertex x, and S must contain one of the vertices u_i or v_i from each triangle. Therefore S contains x and one vertex among each pair $\{u_i, v_i\}$ for $i = 1, 2, 3, ...$ As a result, $|S| = n+1$ and in this case $m\left(C_3^{(n)} - S\right) = 1$ and so $|S|+m\left(C_3^{(n)} - S\right)$ is minimum for the above set S. Thus, S is the DI_2-set of $C_3^{(n)}$. Therefore, $DI_2\left(C_3^{(n)}\right) = |S| + m\left(C_3^{(n)} - S\right) = n + 2$. ∎

Definition 7.3: [5] Bistar graph $B_{m,n}$ is obtained by adjoining m pendent edges to one end and n pendent edges to other end of Path P_2. $B_{m,n}$ contains $m + n + 2$ vertices.

Theorem 7.4: $DI_2(B_{m,n}) = m + n + 2$.

Proof: Let $c_1, c_2, ..., c_m$ and $d_1, d_2, ..., d_n$ represent the m and n pendant vertices of $B_{m,n}$ respectively. Let x and y be the two vertices to which $c_1, c_2, ..., c_m$ and $d_1, d_2, ..., d_n$ are attached respectively. Let S be the DI_2 set of $B_{m,n}$. So, $|S| \geq m + n$. Clearly, one vertex among x and y belong to S. Hence $S = \{c_1, c_2, ..., c_m, d_1, d_2, ..., d_n, x\}$ or $S = \{c_1, c_2, ..., c_m, d_1, d_2, ..., d_n, y\}$. These two sets are DI_2 sets of $B_{m,n}$ since in both the cases, $m(B_{m,n} - S) = 1$. Thus, $|S| = m + n + 1$ and $m(B_{m,n} - S) = 1$. Therefore, $DI_2(B_{m,n}) = |S| + m(B_{m,n} - S) = m + n + 2$. ∎

Definition 7.5: [2] The flower graph Fl_n is acquired by adjoining every pendant vertex of the helm graph H_n to the central vertex.

Theorem 7.6: $DI_2(Fl_n) = n + 2$.

Proof: Let x be the central vertex of Fl_n which is adjacent to all the remaining $2n$ vertices of Fl_n. Let $x_1, x_2, ..., x_n$ represent the inner rim vertices and $y_1, y_2, ..., y_n$ represent the outer vertices. Let S be the γ_{2d} set of Fl_n. Since x is adjacent to all the remaining vertices in Fl_n, choose x in S ensuring that every vertex in $V - Fl_n$ is adjacent to x. Also, all the $y_i's(i = 1, 2, 3, ..., n)$ are adjacent to the corresponding $x_i's$. By the structure of the graph, there are four possible ways to choose S.

Case:(i) Let $S = \{x, x_i/1 \leq i \leq n\}$. In this case, $V(Fl_n - S) = \{y_i/1 \leq i \leq n\}$ and so $m(Fl_n - S) = 1$. Hence, $|S| + m(Fl_n - S) = n + 2$.

Case:(ii) Suppose $S = \{x, y_i/1 \leq i \leq n\}$. Then $|S| = n+1$ and $m(Fl_n - S) = n$ which implies that $|S|+m(Fl_n - S) = 2n + 1$.

Case:(iii) S can be chosen as $\{x, x_{i-1}, y_i\}$ for $i = 1, 2, 3, ..., n$. Then $|S| = n + 1$ and $m(Fl_n - S) = 1$. Hence $|S| + m(Fl_n - S) = n + 2$.

Among the above three cases, we get $|S| + m(Fl_n - S) = n+2$ in case: (i) and (iii) and $|S|+m(Fl_n - S) = 2n+1$ in case:(ii). Hence $DI_2(Fl_n) = |S| + m(Fl_n - S) = n + 2$. ∎

Definition 7.7: [12] Coconut tree $CT(m, n)$ is acquired by adjoining m pendant edges with an end vertex of Path P_n.

Theorem 7.8: For $m, n \geq 2$,

$$DI_2(CT(m, n)) = \begin{cases} \frac{n}{2} + m + 1 & if \quad n \; is \; even \\ \frac{n+1}{2} + m + 1 & if \quad n \; is \; odd \end{cases}$$

Proof: Let S be the 2-dominating set of $CT(m, n)$. $CT(m, n)$ consists of m pendant vertices each connected to an end vertex of path P_n. By Theorem 2.4, m pendant vertices and the other end vertex of Path P_n must belong to S.

Case:(i) n is odd.

Choose every alternate vertex starting from the end vertex of the path P_n, continuing until all alternative vertices are selected. This selection results in $\frac{n+1}{2}$ vertices. Hence S contains m pendant vertices and the selected $\frac{n+1}{2}$ vertices. In this case, $|S| = \frac{n+1}{2} + m + 1$ and $m((CT(m, n)) - S) = 1$. Thus, S is the DDI-set of G. Therefore, $DI_2(CT(m, n)) = \frac{n+1}{2} + m + 1$.

Case:(ii) n is even.

As in Case:(i), select alternate vertices, but now we obtain $\frac{n}{2} + m + 1$ vertices. Thus, $|S| = \frac{n}{2} + m + 1$ and $m((CT(m, n)) - S) = 1$. Therefore, the result is as desired. ∎

Definition 7.9: [10] The n-Sunlet graph is a graph which is obtained by attaching n-pendant edges to the Cycle C_n and it is denoted by S_n. S_n contains $2n$ vertices.

Theorem 7.10: For n-Sunlet graph S_n,

$$DI_2(S_n) = \begin{cases} 7 & if \quad n = 4 \\ 4\left(\frac{n}{3}\right) + 2 & if \quad n \equiv 0(mod3) \\ 4\left(\frac{n-1}{3}\right) + 4 & if \quad n \equiv 1(mod3) \\ 4\left(\frac{n-2}{3}\right) + 5 & if \quad n \equiv 2(mod3) \end{cases}$$

Proof: Let S be the DI_2-set of S_n. Since S_n contains $2n$ vertices of which n vertices are pendant vertices, $|S| \geq n$. Let $u_1, u_2, ..., u_n$ be the n vertices of the Cycle and $v_1, v_2, ..., v_n$ denote the n pendant vertices corresponding to $u_1, u_2, ..., u_n$ respectively.

Case:(i) $n = 4$

$S = \{v_1, v_2, v_3, v_4, u_2, u_4\}$. Then $|S| = 4 + 2 = 6$ and so $m(S_n - S) = 1$. Hence, $DI_2(S_n) = 7$.

Case:(ii) $n \equiv 0(mod3)$

S contains the n pendant vertices and one middle vertex for every three vertices of the Cycle (i.e) u_{3i-1} for $i = 1, 2, 3, ...$ Thus, $S = \{v_1, v_2, ..., v_n, u_2, u_5, ..., u_{n-1}\}$. Then $|S| = n + \dfrac{n}{3} = \dfrac{4n}{3}$. Here, $m(S_n - S) = 2$. Thus, $DI_2(S_n) = \dfrac{4n}{3} + 2$.

Case: (iii) $n \equiv 1(mod3)$ and $n > 4$

S contains the n pendant vertices. In addition to that, one middle vertex from every three vertices of Cycle and the n^{th} vertex belong to S; (i. e, u_{3i-1} for $i = 1, 2, 3, ...$ and u_n) which implies $S = \{v_1, v_2, ..., v_n, u_2, u_5, u_8, u_{11}, ..., u_{n-2}, u_n\}$. Then $|S| = n + \dfrac{n-1}{3} + 1 = 4\left(\dfrac{n-1}{3}\right) + 2$. Thus, $DI_2(S_n) = 4\left(\dfrac{n-1}{3}\right) + 4$.

Case: (iv) $n \equiv 2(mod3)$

$S = \{v_1, v_2, ..., v_n, u_2, u_5, ..., u_n\}$. Then $|S| = n+\dfrac{n+1}{3}+2$ and $m(S_n - S) = 2$. Hence, $DI_2(S_n) = 4\left(\dfrac{n-2}{3}\right) + 5$. ∎

VIII. 2-Domination Integrity of Graphs obtained by Vertex Switching of Some graphs

Definition 8.1: [8] For a finite undirected graph $G(V, E)$ and $v \in V$, the vertex switching of G by v is the graph G^v which is obtained from G by removing all edges incident to v and adding edges which are not adjacent to v.

Theorem 8.2: Let C_n be a Cycle of order $n \geq 3$ and let v be an arbitrary vertex of C_n. Then the value of $DI_2(C_n^v)$ is given by

$$DI_2(C_n^v) = \begin{cases} 5 & if \quad n = 4 \\ n & if \quad n = 5, 6 \\ \left\lfloor \dfrac{n}{3} \right\rfloor + 4 & if \quad n \geq 7 \end{cases}$$

Proof: Let $v, u_1, u_2, ..., u_{n-1}$ be the vertices of the Cycle C_n. Consider the graph C_n^v, which results from switching the vertex v.

Case:(i) $n = 4$

Let $S = \{u_1, v, u_3\}$ be a 2-dominating set of C_4^v. The graph $C_4^v - S$ consists of a single isolated vertex, so $m(C_4^v - S) = 1$. Therefore, S is the DI_2-set of C_4^v, and we have: $DI_2(C_4^v) = 4$.

Case:(ii) $n = 5$

Consider the sets $S_1 = \{v, u_1, u_4\}$ and $S_2 = \{u_1, u_2, u_3, u_4\}$, both of which are DI_2-sets of C_5^v since $|S_1| + m(C_5^v - S_1) = 3 + 2 = 5$ and $|S_2| + m(C_5^v - S_2) = 4 + 1 = 5$. Thus, $DI_2(C_5^v) = 5$

Case:(iii) $n = 6$

$S = \{v, u_1, u_3, u_5\}$ is the only DI_2-set of C_6^v since $|S| + m(C_6^v - S) = 5 + 1 = 6$ which is minimum. Hence $DI_2(C_6^v) = 6$

Case:(iv) $n > 6$

By the Theorem 2.4, $\{u_1, u_{n-1}\}$ is part of any DI_2-set of C_6^v. Additionally, since v is adjacent to $u_2, u_3, ..., u_{n-1}$, we can include v in the DI_2-set of C_n^v as well. For $7 \leq n \leq 10$ and $n = 12$, the specific DI_2-sets are provided in the table below.

TABLE I
DI_2-SETS S_1 OF C_n^v

n	S_1	$\|S_1\|$	$m(C_n^v - S_1)$	$DI_2(C_n^v)$
7	$\{v, u_1, u_3, u_6\}$	4	2	6
8	$\{v, u_1, u_4, u_7\}$	4	2	6
9	$\{v, u_1, u_4, u_6, u_8\}$	5	2	7
10	$\{v, u_1, u_4, u_7, u_9\}$	5	2	7
12	$\{v, u_1, u_4, u_7, u_9, u_{11}\}$	6	2	8

TABLE II
DI_2-SETS S_2 OF C_n^v

n	S_2	$\|S_2\|$	$m(C_n^v - S_2)$	$DI_2(C_n^v)$
7	$\{v, u_1, u_3, u_5, u_6\}$	5	1	6
8	$\{v, u_1, u_3, u_5, u_7\}$	5	1	6
9	$\{v, u_1, u_3, u_5, u_7, u_8\}$	6	1	7
10	$\{v, u_1, u_3, u_5, u_7, u_9\}$	6	1	7
12	$\{v, u_1, u_3, u_5, u_7, u_9, u_{11}\}$	7	1	8

For choosing the DI_2-sets of C_n^v for $n = 11$ and $n > 12$, we consider the following cases.

Sub Case:(a) When $n \equiv 0(mod3)$ and $n \geq 12$, let $S = \{v, u_1, u_{n-1}, u_4, u_7, u_{10}, ..., u_{n-3}\}$ or $S = \{v, u_1, u_{n-1}, u_4, u_7, u_{10}, ..., u_{n-2}\}$

Sub Case:(b) When $n \equiv 1(mod3)$, let $S = \{v, u_1, u_{n-1}, u_4, u_7, u_{10}, ..., u_{n-3}\}$ or $S = \{v, u_1, u_{n-1}, u_4, u_7, u_{10}, ..., u_{n-2}\}$

Sub Case:(c) When $n \equiv 2(mod3)$, let $S = \{v, u_1, u_{n-1}, u_4, u_7, u_{10}, ..., u_{n-3}\}$

In all the three subcases, $|S| = \left\lfloor \dfrac{n}{3} \right\rfloor + 2$ and $m(C_n^v - S) = 2$.

Therefore, for all $n \geq 7$, $DI_2(C_n^v) = \left\lfloor \dfrac{n}{3} \right\rfloor + 4$ ∎

Definition 8.3: [6] The Bull graph is a graph with 5 vertices and 5 edges consisting of a triangle with two disjoint pendant edges.

Theorem 8.4: Let G be the graph obtained by switching an arbitrary vertex of the Bull graph. Then $DI_2(G) = 4$.

Proof: Let u_1 and u_2 be the pendant vertices of the Bull graph. The structure of the Bull graph is shown in the Fig.5

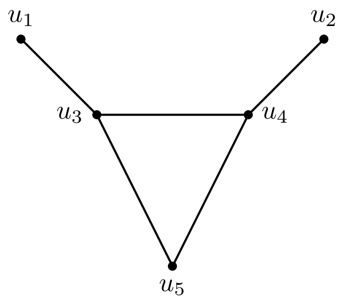

Fig. 5. Bull graph

Case:(i) Switch vertex u_1

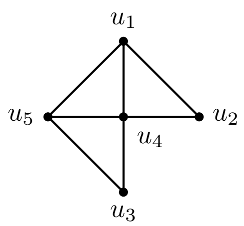

Fig. 6. Graph obtained by switching the vertex u_1 of Bull graph

In this case, the set $S = \{u_1, u_3, u_4\}$ forms the DI_2-set of G as it minimizes $|S| + m(G - S)$. Therefore, $DI_2(G) = 3 + 1 = 4$.

Case: (ii) Switch vertex u_2

Switching u_2 results in a graph with a structure similar to the one obtained in Case: (i). Here, the DI_2-set of G is $S = \{u_1, u_3, u_5\}$ with $|S| = 3$ and $m(G - S) = 1$. Therefore $DI_2(G) = 4$

Case: (iii) Switch vertices u_3 or u_4

Switching either u_3 or u_4 in the Bull graph leads to a disconnected graph.

Case:(iv) Switch vertex u_5

Switching u_5 transforms the Bull graph into a Cycle C_5. Hence, by the corresponding theorem, $DI_2(G) = 4$.

In all the above four cases, we find that $DI_2(G) = 4$. Therefore, the theorem is proven. ∎

Definition 8.5: [11] The n-Pan graph is obtained by connecting a Cycle C_n with a singleton graph by an edge.

Theorem 8.6: Let G be the graph obtained by switching the pendant vertex of the n-Pan graph. Then

$$DI_2(G) = \begin{cases} n & if \quad n = 3, 4 \\ \frac{n}{3} + 3 & if \quad n \equiv 0(mod\ 3); n \neq 6 \\ \frac{n-1}{3} + 4 & if \quad n \equiv 1(mod\ 3); n \neq 7 \\ \frac{n-2}{3} + 4 & if \quad n \equiv 2(mod\ 3); n \neq 8 \end{cases}$$

Proof: Let $u_1, u_2, u_3, ..., u_n$ be the vertices of the Cycle C_n and let x be the pendant vertex of the n-Pan graph. Let G be the graph that results from switching the vertex x.

Case:(i) $n = 3$.
Clearly, $S = \{u_2, u_3\}$ is a 2-dominating set of G with minimum cardinality. In this case, $G - S$ contains two isolated vertices. Thus, $|S| = 2$ and $m(G - S) = 1$. Hence $|S| + m(G - S)$ is minimized. Therefore, $DI_2(G) = |S| + m(G - S) = 3$

Case:(ii) $n = 4$. Obviously, $S = \{u_2, u_3, u_4\}$ is a DI_2-set of G since $|S| + m(G - S)$ is minimized. Therefore, $DI_2(G) = 4$

Case:(iii) $n \equiv 0(mod\ 3); n > 3$

Subcase:(a) For $n = 6$, there are two possible DI_2-sets: $S_1 = \{x, u_2, u_4, u_6\}$ with $|S_1| = 4$ and $m(G - S_1) = 1$. $S_2 = \{x, u_1, u_4\}$ with $|S_2| = 3$ and $m(G - S_2) = 2$. Both sets give $|S_1| + m(G - S_1) = |S_2| + m(G - S_2) = 5$. Hence, $DI_2(G) = 5$

Subcase:(b) $n > 6$. For $n > 6$, choose $S = \{x, u_1, u_4, u_7, u_10, ..., u_{n-2}\}$ as a 2-dominating set of G. The order of S is $|S| = \frac{n}{3} + 1$ and $G - S$ consists of $\frac{n}{3}$ components, each of order 2 (i.e., a Path P_2). Therefore, $m(G - S) = 2$ and the minimized value is $|S| + m(G - S) = \frac{n}{3} + 3$. Thus, S becomes the DI_2-set of G. Hence $DI_2(G) = |S| + m(G - S) = \frac{n}{3} + 3$

Case:(iv) $n \equiv 1(mod\ 3); n > 4$. There are two ways of choosing the DI_2-sets of G. $S_1 = \{x, u_1, u_4, u_7, u_{10}, ..., u_{n-2}\}$ or $S_1 = \{x, u_1, u_4, u_7, u_{10}, ..., u_{n-1}\}$. Then $|S_1| = \frac{n-1}{3} + 2$ and $G - S_1$ contains components of order 1 and 2, so $m(G - S_1) = 2$. Therefore, $|S_1| + m(G - S_1) = \frac{n-1}{3} + 2 + 2 = \frac{n-1}{3} + 4$................(1). Now $S_2 = \{x, u_n, u_2, u_5, u_8, u_{11}, ..., u_{n-2}\}$. So, $|S_2| = \frac{n-1}{3} + 2$ and $m(G - S_2) = 2$. Thus, $|S_2| + m(G - S_2) = \frac{n-1}{3} + 4$ .... (2). From (1) and (2), $|S_1| + m(G - S_1) = |S_2| + m(G - S_2)$. Thus, both S_1 and S_2 are DI_2-sets of G. Hence $DI_2(G) = \frac{n-1}{3} + 4$.

Case:(v) $n \equiv 2(mod\ 3); n \geq 5$

Subcase:(a) $n = 5$. $S_1 = \{x, u_2, u_5\}$ and $S_2 = \{x, u_1, u_3\}$ are 2-dominating sets of G with minimum cardinality. The largest component in $G - S_1$ and $G - S_2$ are a Path P_2. Both sets give $m(G - S_1) = m(G - S_2) = 2$ and $|S_1| + m(G - S_1) = |S_2| + m(G - S_2) = 5$. Now, $S_3 = \{u_2, u_3, u_4, u_5\}$ is a 2-dominating set of G with $|S_3| = 4$; which is greater than S_1 and S_2. But $G - S_3$ contains two isolated vertices and so $m(G - S_3) = 1$. Hence $|S_3| + m(G - S_3) = 5$. Thus S_1, S_2 and S_3 are DDI-sets of G. Therefore, $DI_2(G) = 5$

Subcase:(b) $n > 5$. For $n > 5$, there are two ways of choosing the DI_2-set of G. $S_1 = \{x, u_1, u_4, u_7, u_{10}, .., u_{n-1}\}$ or $S_1 = \{x, u_1, u_4, u_7, u_{10}, .., u_{n-2}\}$ with $|S_1| = \frac{n-2}{3} + 2$ and $m(G - S_1) = 2$. Also, $S_2 = \{x, u_2, u_5, u_8, u_{11}, u_{14}, ..., u_n\}$ with $|S_2| = \frac{n-2}{3} + 2$ and $m(G - S_2) = 2$. o other 2-dominating set S_3 of G satisfies $|S_3| + m(G - S_3) < |S_1| + m(G - S_1)$ and $|S_3| + m(G - S_3) < |S_2| + m(G - S_2)$. Thus, both S_1 and S_2 are DI_2-sets of G. Hence $DI_2(G) = |S_1| + m(G - S_1) = |S_2| + m(G - S_2) = \frac{n-2}{3} + 4$. ■

Definition 8.7: [9] The Double Fan graph Df_n is obtained by $P_n + 2K_1$

Theorem 8.8: $DI_2(Df_n^u) =$

$$\begin{cases} 4 & for \quad n = 3 \\ \left\lfloor \frac{n}{3} \right\rfloor + 4 & for \quad 4 \leq n \leq 6 \\ \left\lceil \frac{n}{3} \right\rceil + 4 & forn > 6 \end{cases}$$

where u is an apex vertex of Double Fan graph.

Proof: Let $v_1, v_2, ..., v_n$ represent the vertices of the Path P_n and let x and u be the apex vertices of Df_n. Let the vertex u be switched to form the graph Df_n^u.

Case:(i)$n = 3$.
Clearly, $S = \{x, u, v_2\}$ is a 2-dominating set of Df_3^u with the minimum cardinality. In this case, $Df_3^u - S$ contains two isolated vertices. Therefore, $m(Df_n^u - S) = 1$. For any other 2-dominating set X of Df_3^u, we have $|X| + m(Df_3^u - X) > |S| + m(Df_3^u - S)$. Thus, S is the DI_2-set of Df_3^u. Hence $DI_2(Df_3^u) = |S| + m(G - S) = 3 + 1 = 4$.

Case: (ii) $4 \leq n \leq 6$.
We consider three subcases for $n = 4,\ 5,\ 6$ separately.

Subcase:(a) $n = 4$.
The 2-dominating sets of Df_4^u with cardinality 4 are $S_1 = \{x, u, v_1, v_3\}$, $S_2 = \{x, u, v_2, v_4\}$ and $S_3 = \{x, u, v_2, v_3\}$. For each of these sets, $V(Df_4^u - S_1) = \{v_2, v_4\}$, $V(Df_4^u - S_2) = \{v_1, v_3\}$ and $V(Df_4^u - S_3) = \{v_1, v_4\}$. Thus, $m(Df_4^u - S_1) = m(Df_4^u - S_2) = m(Df_4^u - S_3) = 1$. Hence $|S_1| + m(Df_4^u - S_1) = |S_2| + m(Df_4^u - S_2) = |S_3| + m(Df_4^u - S_3)$ is minimized. Thus, S_1, S_2 and S_3 are DI_2-sets of Df_4^u. Therefore, $DI_2(Df_4^u) = 5$

Subcase:(b) $n = 5$.
The unique DI_2-set of Df_5^u is $S = \{x, u, v_2, v_4\}$ since there does not exist any other DI_2-set S_1 of Df_5^u that satisfies the condition $|S_1| + m(Df_5^u - S_1) < |S| + m(Df_5^u - S)$. We have $|S| = 4$ and $Df_5^u - S$ contains three isolated vertices $v_1,\ v_3$ and v_5. Hence, $m(Df_5^u - S) = 1$. Therefore $DI_2(Df_5^u) = |S| + m(Df_5^u - S) = 5$.

Subcase:(c) $n = 6$.
We have $S_1 = \{x, u, v_2, v_5\}$ is a 2-dominating set of Df_6^u with minimum cardinality. Now, $Df_6^u - S_1$ contains two components: one of order 1 and other of order 2. So, $m(Df_6^u - S_1) = 2$. Hence $|S_1| + m(Df_6^u - S_1) = 6$. Also, $S_2 = \{x,\ u,\ v_1,\ v_3,\ v_5\}$, $S_3 = \{x,\ u,\ v_2,\ v_4,\ v_6\}$ and $S_4 = \{x,\ u,\ v_2,\ v_4,\ v_5\}$ are 2-dominating sets of Df_6^u with cardinality $|S_2| = |S_3| = |S_4| = 5$. Then $Df_6^u - S_2$, $Df_6^u - S_3$ and $Df_6^u - S_4$ contain two isolated vertices. Thus, $m(Df_6^u - S_2) = m(Df_6^u - S_3) = m(Df_6^u - S_4) = 1$. Hence $|S_2| + m(Df_6^u - S_2) = |S_3| + m(Df_6^u - S_3) = |S_4| + m(Df_6^u - S_4) = 6$. Therefore, S_1, S_2, S_3 and S_4 are

DI_2-sets of Df_6^u. Thus, $DI_2(Df_6^u) = 6$.
Hence from all the above three subcases, we get $DI_2(Df_n^u) = \left\lfloor \frac{n}{3} \right\rfloor + 4$ for $4 \le n \le 6$.
Case:(iii) $n > 6$.
When $n \equiv 0(mod\ 3)$, S can be chosen as $\{x, u, u_{i+1}/i = 3m + 1\ for\ m = 0, 1, 2, ...and\ i + 1 \le n\}$. When $n \equiv 1(mod\ 3)$ and $n \equiv 2(mod\ 3)$, the DI_2-set can be chosen as either $S = \{x, u, u_n, u_{i+1}/i = 3m + 1\ for\ m = 0, 1, 2, ...\ and\ i + 1 \le n\}$ or $S = \{x, u, u_{n-1}, u_{i+1}/i = 3m + 1\ for\ m = 0, 1, 2, ...and\ i + 1 \le n\}$
From the above two subcases, $|S| = \left\lceil \frac{n}{3} \right\rceil + 2$ and $m(Df_n^u - S) = 2$. Thus, the above described sets S form the DI_2-set of Df_n^u. Hence, $DI_2(Df_n^u) = |S| + m(Df_n^u - S) = \left\lceil \frac{n}{3} \right\rceil + 4$. ∎

IX. Conclusion

This article introduces the concept of 2-domination integrity as a new measure of vulnerability in graphs. Also, an algorithm to find DI_2 of a graph is developed. This vulnerability parameter is determined for standard graphs and for certain special graphs, friendship graph, bistar, flower and coconut tree. Further, some theorems in determining the 2-domination integrity of graphs obtained by vertex switching of few graphs are included. Studying various graph vulnerability parameters is of great significance to network designers in order to construct a stable network where reconstruction occurs even after the disruption.

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