Multi-period Dynamic Portfolio under Uncertain Exit Time with Novel Parametric Fuzzy Numbers

Zhihua Zhao, Zhi Li, Yongkang Yuan, Jinjia Lu

Abstract—**It is vital to address the multi-period dynamic portfolio problem under uncertain exit time to dynamically adjust investment strategy and withdraw investment timely. Given the complexity of a fuzzy environment, investors often make decisions based on uncertain information about the financial market. To quantify this kind of uncertain exit time, a novel class of fuzzy numbers with one parameter is defined for the first time, which reflects investors' willingness to exit and their risk attitudes. Using novel trapezoidal fuzzy numbers as special examples, their numerical characteristics (mean, variance and covariance) are strictly derived based on possibility theory; the trends of these characteristics are analyzed through parameter derivation; their arithmetic operations (addition, scalar multiplication) are defined and corresponding closure property is verified in novel trapezoidal fuzzy numbers. Referring to the properties of numerical characteristics in probability theory, our novel trapezoidal fuzzy numbers exhibit similar properties by rigorous mathematical derivation. Moreover, fuzzy possibilistic entropy with an adjustment coefficient is introduced to measure diversification in the traditional MV model. Consequently, we propose a mean-variance-entropy multi-period dynamic portfolio model with uncertain exit time described by novel fuzzy numbers with a parameter. In theory, our model offers two key advantages. (1) The model output can guarantee the diversification by adopting fuzzy entropy with an adjustment coefficient. (2) Investors can withdraw investments promptly according to exit will by setting the exit point of profit (EP) and exit point of loss (EL). Finally, a numerical example is provided to demonstrate the practical feasibility of our proposed model, offering more effective solutions for investors facing dynamic exit times.**

Index Terms— **portfolio, multi-period, possibility theory, novel fuzzy number, uncertain exit time**

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I. INTRODUCTION

Modern portfolio theory (MPT) by Markowitz [1] in the 1950s indisputably lays the cornerstone for modern finance. A mathematical idea was put forward by using expectation and variance of asset returns to quantify return and risk, respectively. However, the modern portfolio theory is not without its limitations, it has the following two aspects: (1) the mean-variance model in MPT is a static and simplified model that does not satisfy long-term investors' real demand; (2) it is easy for the MV model to yield extreme results, which is inconsistent with the principle of diversified investment. Therefore, numerous aspects in the portfolio field need to be improved and enriched, which attracts a number of scholars to study the hot topic.

A. Literature Review

In the complex financial market, the return and risk of financial products are uncertain. Quantifying uncertainty to construct a portfolio model is an important component. Earlier studies [2,3] on portfolios focused on the stochastic environment based on probability theory. This relies on capturing the probability distribution of asset returns as accurately as possible, which is very difficult for scholars and investors. In real investment, investors often predict the future asset return based on their own experience according to the real market information. That implies that another uncertainty with fuzzy information exists in the investment field. It was not until the fuzzy set theory proposed by Zadeh [4] that scholars began to pay more attention to portfolios in the fuzzy environment.

Subsequent to the emergence of fuzzy set theory, a multitude of scholars embarked on the exploration of the theory of fuzzy variables in relation to probability theory. Dubois and Prade [5] formulated the definition of the mean of fuzzy interval numbers and corroborated that the mean of the sum of two fuzzy numbers was commensurate with the sum of their individual means. Moreover, Carlsson and Fullér [6] devised the concepts of possibilistic variance and covariance of fuzzy numbers. Concerning the numerical characteristics of fuzzy variables, possibility theory was initially introduced by Zadeh [7] and subsequently furthered by Dubois and Prade [5]. Grounded on possibility theory, Carlsson and Fullér [6] defined the upper and lower possibilistic means of fuzzy numbers. Zhang [8] subsequently defined the upper and lower possibilistic variances and covariances and proceeded to deliberate on the properties of their numerical characteristics, which enriched the possibilistic theory.

An increasing number of scholars have directed their attention to the fuzzy portfolio problems. They substituted fuzzy mean and covariance for the random mean and covariance in Markowitz's M-V model. Wadata [9] and Ramaswamy [10] delved into portfolio models underpinned by fuzzy decision theory. Tanaka and Guo [11] put forward the portfolio model of fuzzy probability and examined the portfolio model with an exponential probability distribution. Carlsson [6] devised the possibility portfolio model without short selling, under the condition that the return was considered as trapezoidal fuzzy numbers. Khayamim et al. [12] utilized fuzzy logic and possibility theory to probe into the uncertain factors of investment psychology. Deng et al. [13] adopted the possibility means and possibility variances as measures for portfolio return and risk, respectively, to erect the mean-variance-efficient portfolio model. Deng and Geng [14] explored a novel two-parameter coherent fuzzy portfolio predicated on possibility theory.

To realistically simulate the actual financial market, it is of utmost necessity to transition the portfolio model from a single-period to a multi-period framework. In the context of multi-period models, transaction costs exert a profound impact on the model outcomes and thus cannot be readily overlooked. Arnott [15] and Yoshimoto [16] et al. provided empirical evidence demonstrating that disregarding transaction costs would culminate in inefficient portfolios. Lin et al. [17] contemplated the fuzzy portfolio problem under practical constraints, encompassing limitations on the number of assets, the minimum number of traders, and budgetary constraints. Mulvey [18] employed a piecewise linear function to approximate transaction costs; however, this approach proves inadequate for non-convex transaction costs that more closely mirror real-world scenarios. Yoshimoto et al. [16] explored the portfolio problem involving variable transaction costs. Konno [19] devised a branch and bound algorithm for ascertaining the optimal solution of the M-AD model with concave transaction costs, enabling the attainment of the global optimal solution. Peykani et al. [20] investigated a time-consistent multi-period rolling portfolio optimization problem within a fuzzy environment. Qian and Wang [21] introduced an augmented algorithm that amalgamates the parallel processing capabilities of PGAs with the multi-objective optimization prowess of NSGA-III, customized specifically for multi-period optimization design.

However, in real-world scenarios, a pre-determined investment plan may be terminated prematurely due to a plethora of factors. These encompass sudden substantial consumption or expenditure, the onset of serious illness or death, alterations in the market environment, and so forth. Yarri [22] incorporated the assumption of uncertain life into the life insurance model, and Hakansson [23] further extended Yarri's work [22] to the domain of the multi-period portfolio with an uncertain exit time, which was treated as a constant. Merton [24] investigated the problem of optimal investment and consumption in continuous time to maximize the expectation of the utility function, wherein the exit time was postulated to be the time of the first event of an independent Poisson process (thus, the investment period was exponential). Since then, the portfolio model with uncertain exit time has garnered increasing attention. Over the past decade, the number of related studies has been steadily on the rise. However, in essence, these studies have invariably been predicated on random environments. Given that fuzziness can characterize more complex and subjective elements of the real market, it is worth studying the uncertain exit time in a fuzzy environment.

Portfolio models can be categorized based on whether the exit time is independent of the asset return in each period. In the majority of situations, the probability distribution (or membership function) of the uncertain exit time is indeed independent of the asset return in each period. This implies that a low return does not exert any influence on the early-stage exit time.

The investor's early exit behavior is predominantly associated with several factors, including sudden large expenditures, serious illness, and alterations in the market environment. There has been extensive research on uncertain exit time within a random environment. For instance, Martellini and Urosevic [25] investigated the asset allocation problem when the exit time was either an exogenous or endogenous random variable by means of the static mean-variance model. Yi et al. [26] extended the multi-period portfolio problem with uncertain exit time to the scenario involving exogenous liabilities. In accordance with the concepts of mean and variance, Yao et al. [27] delved into the multi-period asset liability management problem with uncertain exit time and stochastic cash flow. Yi et al. [28] applied the average field formula to the M-V portfolio problem with uncertain exit time in discrete time. Cui et al. [29] proposed a two-dimensional average field formula and derived the optimal solution as well as the effective frontier analytical formula. Wu et al. [30] regarded the exit probability of each period as the conditional probability of market conditions. Ge et al. [31] tackled a multi-period weighted mean-variance portfolio selection problem with uncertain time horizons and stochastic cash flows in a Markov regime-switching market. Yao et al. [32] explored a dynamic trading problem involving transaction costs and uncertain exit times in a general Markov market, where the mean vector and covariance matrix of returns are contingent upon the states of the stochastic market, with market states undergoing regime switching within a time-varying state set.

B. Motivation

In order to ensure the diversification of investment, entropy has been introduced into the portfolio model as a measure of the dispersion degree. Shannon [33], drawing inspiration from thermodynamics, christened the average information devoid of redundancy as information entropy. Philippatos [34], Cheng [35], and Qu [36] probed into the stochastic entropy for portfolio problems. Xu [37] and Zhou [38] incorporated fuzzy stochastic entropy into portfolio problems. Mehmet and Osman [39] contrasted Shannon entropy with Gini-Simpson entropy and ascertained that the latter exhibited superior performance in the portfolio context. From the aforementioned studies, the focal points of modern portfolio problems can be discerned as follows: how to dynamically adjust investment strategies, when to withdraw investment, and how to ensure investment diversification during multi-period investment. However, the preponderance of research concerning uncertain exit time has been conducted within a stochastic environment. In this paper, fuzzy uncertain exit time is innovatively characterized by a novel class of one-parameter fuzzy numbers, which encapsulates investors' exit intentions and risk attitudes. A multi-period multi-objective portfolio model under uncertain exit time is devised to address the three aforementioned issues.

To more comprehensively organize the previous research articles and to distinctly compare the disparities between our study and prior works, we have compiled Table Ⅰ. As evident from Table Ⅰ, existing research articles typically take into account merely one or two of the following three factors: possibility theory, multi-period dynamics, and uncertain exit time. Prior to our current endeavor, no article had concurrently considered all three aspects. In contrast to the previous literature, our study offers the following notable advantages. (1) We have thoroughly accounted for the uncertain exit time and the multi-period nature in the context of fuzzy portfolio selection. Building upon traditional fuzzy numbers, we have defined a novel class of fuzzy numbers, and their numerical characteristics have been rigorously derived based on possibility theory. In the specific case of trapezoidal fuzzy numbers, we have defined operations such as number addition, scalar multiplication, and fuzzy addition, and have proven certain properties of the numerical characteristics. (2) Regarding the model, the novel fuzzy number is employed to characterize the uncertain exit time, and the fuzzy possibilistic entropy with an adjustment coefficient is utilized to depict the degree of investment diversification. Subsequently, the fuzzy convex programming method is applied to solve the model. (3) We consider preference for risk and investment diversification in the model, which can provide suitable investment schemes for different investors.

C. Organization

The rest of this paper is organized as follows. In Section 2, we will present the possibility theory of fuzzy numbers and the pertinent conclusions regarding trapezoidal fuzzy numbers. In Section 3, we shall define novel fuzzy numbers and derive their numerical characteristics along with relevant propositions. In Sections 4 and 5, the possibility mean-variance-entropy portfolio model will be formulated, and a numerical example will be provided to validate the feasibility and efficacy of the proposed model.

II. Preliminaries

A. Fuzzy Set Theory

A fuzzy set is an important way to describe fuzzy information. It was proposed by Zadeh [4] in 1965. The following is an introduction to fuzzy set and its properties.

Definition 2.1 [4] If A has a membership function $\mu_A(x): R \to [0,1]$, which is called a fuzzy set on the domain U, then it must satisfy the following conditions:

(a) A is normal, namely

$\exists x_0 \in U, \text{s.t.} \mu_A(x_0) = 1.$

(b) A is convex, namely

$\forall x_1, x_2 \in U, \gamma \in [0,1], \text{s.t.}$

$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}.$

(c) $\mu_A(x)$ is upper semi-continuous and it has boundary $\text{supp}A = \{x \in R \mid \mu_A(x) > 0\}$, which is called support set of fuzzy A.

According to the above definition, the membership functions of general LR-type fuzzy numbers $A = (a, b, \alpha, \beta)$ are as follows:

$$\mu_A(x) = \begin{cases} L_A(\frac{a-x}{\alpha}), & x \leq a \\ 1 & , a \leq x \leq b \\ R_A(\frac{x-b}{\beta}), & x \geq b \\ 0 & , \text{others}. \end{cases} \tag{1}$$

Remark 1: L_A, R_A represent the left and right membership functions of fuzzy number A respectively, which are continuous monotone decreasing functions and satisfy $L_A(0) = R_A(0) = 1,\ L_A(1) = R_A(1) = 0$.

Definition 2.2 [4] Let A be a fuzzy number. $\mu_A(x)$ is membership function of A. Then $[A]^\gamma = \{x \in R \mid \mu_A(x) \geq \gamma\} (\gamma \in [0,1])$ shows the γ-level set of A, which is written as $[A]^\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$ briefly.

Definition 2.3 [4] Let $A_1 = (a_1, b_1, \alpha_1, \beta_1)$ and $A_2 = (a_2, b_2, \alpha_2, \beta_2)$ be two fuzzy numbers. The corresponding γ-level sets are $[A_1]^\gamma = [\underline{a}_1(\gamma), \overline{a}_1(\gamma)]$ and $[A_2]^\gamma = [\underline{a}_2(\gamma), \overline{a}_2(\gamma)]$, respectively. Then we can define the fuzzy addition and scalar-multiplication.

(a) Fuzzy addition,

$$\begin{aligned} & A_1 + A_2 \\ &= (a_1, b_1, \alpha_1, \beta_1) + (a_2, b_2, \alpha_2, \beta_2) \\ &= (a_1 + a_2, b_1 + b_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2). \end{aligned}$$

The form of γ-level set,

$$[A_1 + A_2]^\gamma = [\underline{a}_1(\gamma) + \underline{a}_2(\gamma), \overline{a}_1(\gamma) + \overline{a}_2(\gamma)], \forall \gamma \in [0,1].$$

(b) Fuzzy scalar-multiplication,

$$\lambda A_1 = \begin{cases} (\lambda a_1, \lambda b_1, \lambda \alpha_1, \lambda \beta_1), & \lambda \geq 0, \\ (\lambda b_1, \lambda a_1, \lambda \beta_1, \lambda \alpha_1), & \lambda < 0. \end{cases}$$

B. The Possibility Theory of Fuzzy Number

In the fuzzy environment, the possibility theory was put forward by Zadeh [7] in 1978 and perfected and developed by Dubois and Prade [5]. Later, many scholars studied the portfolio model based on the possibility theory.

Definition 2.4 [7] Let A be a fuzzy number. $[A]^\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$ is γ-level set of A.

(a) Possibilistic mean of A is

$$E(A) = \int_0^1 \gamma[\underline{a}(\gamma) + \overline{a}(\gamma)] d\gamma. \tag{2}$$

(b) Possibilistic variance of A has two forms, the first form is

$$Var^*(A) = \frac{1}{2}\int_0^1 \gamma[\overline{a}(\gamma) - \underline{a}(\gamma)]^2 d\gamma. \tag{3}$$

(c) The second form is

$$Var(A) = \int_0^1 \gamma\{[E(A) - \underline{a}(\gamma)]^2 + [E(A) - \overline{a}(\gamma)]^2\} d\gamma. \tag{4}$$

Remark 2: Both of above definitions are common variances, of which the second is more common. Let

$[B]^{\gamma}=[\underline{b}(\gamma),\overline{b}(\gamma)]$ be the γ -level set of fuzzy number B. Possibilistic covariance of A and B has two forms, the first form is

$$Cov^*(A,B)=\frac{1}{2}\int_0^1 \gamma[\overline{a}(\gamma)-\underline{a}(\gamma)][\overline{b}(\gamma)-\underline{b}(\gamma)]d\gamma. \tag{5}$$

The second form is

$$\begin{aligned}Cov(A,B)=\int_0^1 \gamma[&(E(A)-\underline{a}(\gamma))(E(B)-\underline{b}(\gamma))\\&+(E(A)-\overline{a}(\gamma))(E(B)-\overline{b}(\gamma))]d\gamma.\end{aligned} \tag{6}$$

According to extension principle of Zadeh, if $A_i(i=1,\cdots,n)$ are fuzzy numbers, $\forall\lambda_i\in R$, then

$$E[\sum_{i=1}^n \lambda_i A_i]=\sum_{i=1}^n \lambda_i E(A_i), \tag{7}$$

$$Var[\sum_{i=1}^n \lambda_i A_i]=\sum_{i=1}^n \lambda_i^2 Var(A_i)+2\sum_{1\le i<j\le n}\lambda_i\lambda_j Cov(A_i,A_j). \tag{8}$$

C. Numerical Characteristics of Trapezoidal Fuzzy Numbers

If a fuzzy number A is called a trapezoidal fuzzy number with core interval $[a,b]$, left width α and right width β, then its membership function is expressed as follows:

$$\mu_A(x)=\begin{cases}1-\dfrac{a-x}{\alpha},\ a-\alpha\le x<a\\ \quad 1,\ a\le x<b\\ 1-\dfrac{x-b}{\beta},\ b\le x\le b+\beta\\ \quad 0,\ \text{others,}\end{cases} \tag{9}$$

that is, $A=(a,b,\alpha,\beta)$. Thus, the γ -level set of the trapezoidal fuzzy number A is

$$[a-(1-\gamma)\alpha,b+(1-\gamma)\beta],\forall\gamma\in[0,1]. \tag{10}$$

According to the possibility theory, numerical characteristics of trapezoidal fuzzy numbers are summarized as follows:

(a) Possibilistic mean:

$$E(A)=\frac{a+b}{2}+\frac{\beta-\alpha}{6}. \tag{11}$$

(b) Possibilistic variance:

$$Var^*(A)=(\frac{b-a}{2}+\frac{\alpha+\beta}{6})^2+\frac{(\alpha+\beta)^2}{72}, \tag{12}$$

$$Var(A)=(\frac{b-a}{2}+\frac{\alpha+\beta}{6})^2+\frac{(\alpha+\beta)^2}{72}+\frac{(\alpha-\beta)^2}{72}. \tag{13}$$

(c) Possibilistic covariance:

$$\begin{aligned}&Cov^*(A_1,A_2)\\&=(\frac{b_1-a_1}{2}+\frac{\alpha_1+\beta_1}{6})(\frac{b_2-a_2}{2}+\frac{\alpha_2+\beta_2}{6})\\&+\frac{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}{72},\end{aligned} \tag{14}$$

$$\begin{aligned}&Cov(A_1,A_2)\\&=(\frac{b_1-a_1}{2}+\frac{\alpha_1+\beta_1}{6})(\frac{b_2-a_2}{2}+\frac{\alpha_2+\beta_2}{6})\\&+\frac{\alpha_1\alpha_2+\beta_1\beta_2}{36}.\end{aligned} \tag{15}$$

Remark 3: Because the second variance is used by most scholars, and the influence of the difference $\alpha-\beta$ between left width and right width is considered more than that of the first one, the second variance and covariance will be used in the model demonstration in this paper.

III. New results of novel fuzzy numbers

In this section, we introduce conventional fuzzy numbers and their numerical characteristics based on possibility theory. In this section, we define a new type of fuzzy numbers based on it and use the general formula of possibility theory to deduce numerical characteristics. In the case of trapezoidal fuzzy numbers, we define the number addition, number multiplication, and fuzzy addition of the novel fuzzy numbers, and then prove some properties of the numerical characteristics of novel fuzzy numbers to ensure that the portfolio model and empirical research are feasible in theory.

A. Novel Fuzzy Numbers

Li et al. [40] proposed a novel concept that establishes a connection between the uncertain exit time and a novel type of fuzzy number. This concept places its emphasis on the determination of the investment's loss exit point and profit exit point, namely the Exit Point of Loss (EL) and the Exit Point of Profit (EP).In this paper, a novel fuzzy number is constructed by intercepting the original fuzzy number with a given intercepting level K (membership degree), so that the values of EL and EP can be obtained directly from the γ -level set. In this way, the change of EL and EP only depends on the given K value, which can be changed according to the risk attitude of investors.

The novel fuzzy number is a function with the parameter K of exit willing. In this function, on the one hand, we select the upper part of the membership function of fuzzy number according to investors' exit willing by using the horizontal straight line $y=K(K\in[0,1])$. On the other hand, we make the values of other parts of the function 0, which is generally a piecewise function. In this way can we cut the original fuzzy number with the same left and right height, which means that part of the risks and benefits is equally discarded. That is more reasonable and in line with the law of investment.

Definition 3.1 Let A be a general fuzzy number. And its γ-level set is $[A]^{\gamma}=[\underline{a}(\gamma),\overline{a}(\gamma)]$, $\forall\gamma\in[0,1]$. Now we set K to be the interception level, then the γ-level set of novel fuzzy number $A_{(K)}$ can be denoted by

$$[A_{(K)}]^{\gamma}=\begin{cases}[\underline{a}(K),\overline{a}(K)], & \gamma\in[0,K),\\ [\underline{a}(\gamma),\overline{a}(\gamma)], & \gamma\in[K,1].\end{cases}$$

Remark 4: If $K=0$, the novel fuzzy number degenerates to the corresponding conventional fuzzy number. If $K=1$, the novel fuzzy number degenerates to a clear number interval, that is the kernel of the corresponding general fuzzy number ($Ker\,A=\{x\mid x\in U,\mu_A(x)=1\}$).

B. Novel Trapezoidal Fuzzy Numbers

a. Novel trapezoidal fuzzy numbers and their numerical characteristics

Li [40] employed the improved fuzzy simulation to approximate the continuous membership function of fuzzy numbers to a piecewise - linear function, and subsequently approximated their mean value, variance, value at risk, and credibility measure. Inspired by this approach, we derived the explicit expression of the numerical characteristics of the novel trapezoidal fuzzy number. Based on the definition of novel fuzzy numbers, this section presents the definition, numerical characteristics, and properties of novel trapezoidal fuzzy numbers and conducts a comparative analysis with conventional trapezoidal fuzzy numbers.

Definition 3.2 Let A be a trapezoidal fuzzy number with core interval $[a,b]$, left width α and right width β. Now we set the interception level as $K(K\in[0,1])$. Then the membership function of the novel trapezoidal fuzzy number $A_{(K)}=(a,b,\alpha,\beta,K)$ can be denoted by

$$\mu_{A_{(K)}}(x)=\begin{cases}1-\dfrac{a-x}{\alpha}, & a-(1-K)\alpha\le x<a\\ 1, & a\le x<b\\ 1-\dfrac{x-b}{\beta}, & b\le x\le b+(1-K)\beta\\ 0, & \text{others.}\end{cases}\tag{16}$$

And we will write the novel trapezoidal fuzzy numbers as NTFN briefly in the following sections.

Comparison between the graph of the trapezoidal fuzzy numbers and corresponding novel trapezoidal fuzzy numbers is shown in the following figures.

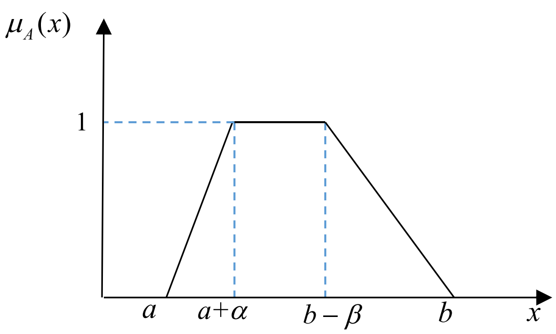

Fig. 1. Conventional trapezoidal fuzzy

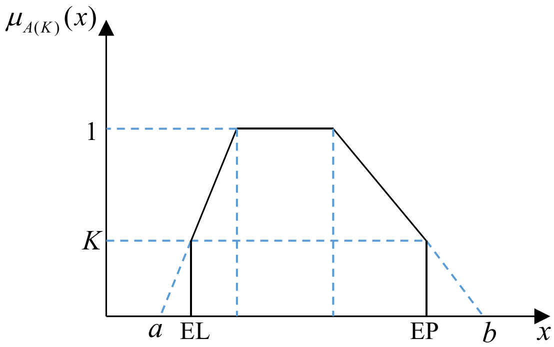

Fig. 2. Novel trapezoidal fuzzy number with exit point

According to the possibility theory, we can deduce some propositions of the novel trapezoidal fuzzy numbers $A_{(K)}=(a,b,\alpha,\beta,K)$.

Proposition 1: The γ-level set of $A_{(K)}$

$$[A_{(K)}]^{\gamma}=\begin{cases}[a-(1-K)\alpha,b+(1-K)\beta], & 0\le\gamma\le K,\\ [a-(1-\gamma)\alpha,b+(1-\gamma)\beta]\ , & K\le\gamma\le 1.\end{cases}\tag{17}$$

From the above equation, it is obvious that the γ-level set of $A_{(K)}$ is fixed if $0\le\gamma\le K$.

Proposition 2: The possibilistic mean of $A_{(K)}$ is

$$E(A_{(K)})=\frac{a+b}{2}+\frac{\beta-\alpha}{6}+K^3\frac{\alpha-\beta}{6}.$$

Proof. According to Eq. (2), it is easy to obtain the specific Equation (18) of $E(A_{(K)})$.

From the above equation, it is easily known that $E(A_{(K)})=\frac{a+b}{2}+\frac{\beta-\alpha}{6}$ if $K=0$ and $E(A_{(K)})=\frac{a+b}{2}$ if $K=1$, that is, in the case of a general fuzzy number and a clear number interval respectively. Additionally, as K increases in the interval $[0,1]$, $E(A_{(K)})$ decreases to $\frac{a+b}{2}$, which means that the influence of the difference between left and right width $\beta-\alpha$ on the possibilistic mean value of NTFN is reduced.

Proposition 3: The first form of possibilistic variance of $A_{(K)}$ is

$$Var^*(A_{(K)})=(\frac{b-a}{2}+\frac{\alpha+\beta}{6})^2+\frac{(\alpha+\beta)^2}{72}-\frac{K^3}{6}(b-a+\alpha+\beta)(\alpha+\beta)+\frac{K^4}{8}(\alpha+\beta)^2.$$

Proof. By Equation (3), we can deduce Equation (19).

Based on the above equation, we know that

$$Var^*\left(A_{(K)}\right)=\left(\frac{b-a}{2}\right)^2 \text{ if } K=0$$

and

$$Var^*\left(A_{(K)}\right)=(\frac{b-a}{2}+\frac{\alpha+\beta}{6})^2+\frac{(\alpha+\beta)^2}{72} \text{ if } K=1,$$

that is, in case of general fuzzy number and clear number interval respectively. In addition, for

$$\begin{aligned} &Var^*(A_{(K)})' \\ &=-\frac{1}{2}K^2\left(b-a+\alpha+\beta\right)\left(\alpha+\beta\right)+\frac{1}{2}K^3\left(\alpha+\beta\right)^2 \\ &=\frac{1}{2}K^2\left(\alpha+\beta\right)\left[(K-1)\left(\alpha+\beta\right)-(b-a)\right] \le 0, \end{aligned}$$

$Var^*\left(A_{(K)}\right)$ decreases to $\left(\frac{b-a}{2}\right)^2$ as K increases in the interval $[0,1]$.

Proposition 4: The second form of possibilistic variance of $A_{(K)}$ is

$$\begin{aligned} &Var^*(A_{(K)})= \\ &\frac{1}{4}(\alpha^2+\beta^2)+\frac{1}{4}K^4(\alpha^2+\beta^2) \\ &+\frac{1}{2}[(E(A_{(K)})-a+\alpha)^2+(E(A_{(K)})-b-\beta)^2] \\ &+\frac{1}{3}(K^3+2)[\beta(E(A_{(K)})-b-\beta)-\alpha(E(A_{(K)})-a+\alpha)]. \end{aligned}$$

Proof. By Equation (4), it is easily to obtain Equation (20). Now let us calculate the two terms in the above equation respectively. The first term is as in Equation (21); The second term is as in Equation (22). Finally, we sum up the two terms, we can obtain Equation (23). According to the above derivation, we continue to analyze the variation trend of the second form variance of $A_{(K)}$ in the domain. Before analyzing, it is primary to derivate $Var(A_{(K)})$ for the parameter. Its derivative for parameter is in Equation (24).

By the above equation, we discuss the $Var(A_{(K)})$ in three cases about α and β :

(a) If $\alpha=\beta=0$, NTFN degenerates to a clear number interval, and at this time, NTFN is a constant which is independent of intercept level K . So $Var(A_{(K)})$ is fixed whatever K varies into $[0,1]$. In this case, $Var(A_{(K)})=\frac{(b-a)^2}{4}$.

(b) If $\alpha=\beta\neq 0$, $Var(A_{(K)})'$ can be written as $K^2[K\cdot 2\alpha^2-\alpha(b-a)-2\alpha^2]$ briefly. It is easily known that its zero point is $K=\frac{\alpha(b-a)+2\alpha^2}{2\alpha^2}\ge 1$ and $Var(A_{(K)})'\le 0$ when $K\in[0,1]$. Hence, $Var(A_{(K)})$ is monotone decreasing in $K\in[0,1]$ from $(\frac{b-a}{2}+\frac{\alpha}{3})^2+\frac{1}{18}\alpha^2\left(K=0\right)$ to $\frac{(b-a)^2}{4}\left(K=1\right)$.

(c) If $\alpha\neq\beta$, we set $F(K)$ as

$$\begin{aligned} F(K)=K(\alpha^2+\beta^2)-\frac{1}{2}(\alpha+\beta)^2 \\ -\frac{1}{2}(\alpha+\beta)(b-a)-\frac{1}{3}(\alpha-\beta)^2-\frac{1}{6}K^3(\alpha-\beta)^2. \end{aligned}$$

that is, $\mathrm{Var}(A_{(K)})'=K^2F(K)$. Then the derivative of $F(K)$ is denoted by $F'(K)=\alpha^2+\beta^2-\frac{1}{2}K^2(\alpha-\beta)^2$. It is obvious that the zero point of $F'(K)$ is $K=\pm\sqrt{\frac{2(\alpha^2+\beta^2)}{(\alpha-\beta)^2}}\notin\left[0,1\right]$, so $F'(K)>0$ in $[0,1]$. Hence, $F(K)$ and $Var(A_{(K)})'=K^2F(K)$ are strictly monotone increasing in $[0,1]$. However, for $Var(A_{(K)})'|_{K=1}=-\frac{1}{2}(\alpha+\beta)(b-a)\le 0$, so $Var(A_{(K)})'\le 0$ if $K\in[0,1]$. Therefore, $Var(A_{(K)})$ is monotone decreasing from $(\frac{b-a}{2}+\frac{\alpha+\beta}{6})^2+\frac{1}{36}(\alpha^2+\beta^2)$ to $\frac{(b-a)^2}{4}$ in $\left[0,1\right]$.

Proposition 5. The possibilistic covariance between $A_{1(K)}$ and $A_{2(K)}$ is in Equation (25).

b. Some properties about numerical characteristics of novel trapezoidal fuzzy number

Li [40] defined new fuzzy numbers but did not define arithmetic operations for fuzzy numbers, nor prove the relevant properties of mean value, variance, etc. All of these are the necessary basis to ensure the theoretical feasibility of the fuzzy portfolio model. Therefore, we define the number addition, number multiplication, and fuzzy addition of the novel trapezoidal fuzzy numbers, according to the definitions of the arithmetic operation of the conventional fuzzy numbers. On this basis, the properties of the mean, variance, and covariance of the novel trapezoidal fuzzy numbers are given and proved, which keeps the consistency with the properties of the mean, variance and covariance of random variables in probability theory.

$A_{(K)}=(a,b,\alpha,\beta,K)$ is a NTFN and $[\underline{a}(\gamma),\overline{a}(\gamma)]$ is its γ -level set. We set $m,n,l\ge 0$, $K\in[0,1]$ is given real number. In order to define and calculate expediently, we set the same K of all the NTFNs in the same equation. Then we define some arithmetic operations as follows, which lays a theoretical foundation for the fuzzy portfolio model and empirical calculation.

Definition 3.3 Number addition,

$$A_{(K)}+l=(a+l,b+l,\alpha,\beta,K). \tag{26}$$

The form of γ - level set is denoted by

$$[A_{(K)}+l]^{\gamma}=[\underline{a}(\gamma)+l,\overline{a}(\gamma)+l],\ \forall\gamma\in[0,1]. \quad (27)$$

Definition 3.4 Number multiplication,

$$nA_{(K)}=(na,nb,n\alpha,n\beta,K). \quad (28)$$

The form of γ - level set is denoted by

$$[nA_{(K)}]^{\gamma}=[n\underline{a}(\gamma),n\overline{a}(\gamma)],\ \forall\gamma\in[0,1]. \quad (29)$$

Definition 3.5 Fuzzy addition, let $A_{1(K)}=(a_1,b_1,\alpha_1,\beta_1,K), A_{2(K)}=(a_2,b_2,\alpha_2,\beta_2,K)$ are two NTFNs, their γ - level sets are $[\underline{a}_1(\gamma),\overline{a}_1(\gamma)],[\underline{a}_2(\gamma),\overline{a}_2(\gamma)]$. Then addition of NTFN is defined by

$$A_{1(K)}+A_{2(K)}=(a_1+a_2,b_1+b_2,\alpha_1+\alpha_2,\beta_1+\beta_2,K). \quad (30)$$

The form of γ - level set is denoted by

$$\begin{aligned}&[A_{1(K)}+A_{2(K)}]^{\gamma}\\&=[\underline{a}_1(\gamma)+\underline{a}_2(\gamma),\overline{a}_1(\gamma)+\overline{a}_2(\gamma)],\ \forall\gamma\in[0,1].\end{aligned} \quad (31)$$

According to the above definitions, it is obviously known that the NTFN after arithmetic operation is also a NTFN, which means arithmetic operation is closed in NTFN. Additionally, based on the properties of the numerical characteristics of random variables in probability theory, we summarize the arithmetic operation properties of the numerical characteristics of fuzzy numbers as follow.

Proposition 6: $E(A_{(K)}+l)=E(A_{(K)})+l.$

Proposition 7: $E(nA_{(K)})=nE(A_{(K)}).$

Proposition 8: $E(A_{1(K)}+A_{2(K)})=E(A_{1(K)})+E(A_{2(K)}).$

Proposition 9: $Var(A_{(K)}+l)=Var(A_{(K)}).$

Proposition 10: $Var(nA_{(K)})=n^2Var(A_{(K)}).$

Proposition 11:

$$\begin{aligned}&Var(A_{1(K)}+A_{2(K)})=Var(A_{1(K)})\\&\quad+Var(A_{2(K)})+2Cov(A_{1(K)},A_{2(K)}).\end{aligned}$$

Proposition 12: $Cov(A_{1(K)},A_{2(K)})=Cov(A_{2(K)},A_{1(K)}).$

Proposition 13:

$$Cov(A_{1(K)}+n,A_{2(K)}+l)=Cov(A_{1(K)},A_{2(K)}).$$

Proposition 14:

$$Cov(nA_{1(K)},lA_{2(K)})=nlCov(A_{1(K)},A_{2(K)}).$$

Proposition 15:

$$\begin{aligned}&Cov(A_{1(K)},A_{2(K)}+A_{3(K)})\\&=Cov(A_{1(K)},A_{2(K)})+Cov(A_{1(K)},A_{3(K)}).\end{aligned}$$

The above proposition implies that the numerical characteristics of novel fuzzy numbers bear resemblance to those of random variables, thereby providing a theoretical foundation for the establishment of a novel fuzzy portfolio model.

IV. FUZZY PORTFOLIO MODEL

In this section, a portfolio model is constructed in the fuzzy environment. We use the novel trapezoidal fuzzy numbers to describe the uncertain exit time and add the fuzzy possibilistic entropy by Zhang et al. [41] to measure the degree of investment diversification. In addition, the weights of investors' preference for two objectives of risk and entropy are introduced in our model.

A. *Possibilistic Multi-Period Mean-variance-entropy Model*

The concept of utilizing the mean and variance to quantify returns and risks has its roots in Markowitz's M-V model. Nevertheless, since returns and risks cannot simultaneously attain their optimal values, we typically aim to minimize the portfolio risk when the upper limit of returns is specified or maximize the portfolio return when the lower limit of risk is set. Additionally, taking into account investors' subjective factors, the asset return is treated as a fuzzy number. Consequently, a multi-period fuzzy portfolio model is put forward in this section. Prior to constructing the model, for the sake of clarity in illustration, we present some symbols in Table Ⅱ.

a. Fuzzy possibilistic entropy

Shannon entropy, Yager entropy, and proportion entropy are commonly employed to measure the degree of investment diversification. Nevertheless, the aforementioned entropies are solely associated with the investment proportion, failing to take into account the specific impact of the investment proportion when the rate of return is lower than the return of the risk-free asset. Zhang et al. [40] posited that the higher the ratio of return to risk, the greater the corresponding asset proportion. Consequently, they proposed a fuzzy possibilistic entropy and extended it to the multi-period scenario. Hence, we utilize this type of entropy to measure the degree of portfolio diversification. The fuzzy possibilistic entropy is denoted as in Equation (32).

Remark 5: In Equation (32), $\varepsilon=\dfrac{\max\{E(r_{t,i})-r_f(t),0\}}{Var(r_{t,i})}$ is an enough small positive number, which represents the part of the $i-$th asset over the risk-free asset in the *t*-period. Besides, $\theta(x_{t,i})=\dfrac{\dfrac{\max\{E(r_{t,i})-r_f(t),0\}}{Var(r_{t,i})}}{\sum_{i=1}^{n}\dfrac{\max\{E(r_{t,i})-r_f(t),0\}}{Var(r_{t,i})}}$ is an adjustment coefficient of $x_{t,i}$. By the above equation, it is found that $x_{t,i}=0$ if $\mathrm{E}(r_{t,i})<r_f(t)$, which is in accordance with the principle that investors never invest the asset whose return is lower than the risk-free return.

b. Model construction

In our proposed model, we set risk and entropy as two objectives when the portfolio return is given. In addition, the return of each period is required to exceed the given minimum expected return. Then we can construct the portfolio model as in Equation (33).

B. Multi-period Mean-variance-entropy Portfolio Model with Uncertain Exit Time

a. Application of novel fuzzy numbers with uncertain exit time

The returns are regarded as novel trapezoidal fuzzy numbers, which describe uncertain exit time in investment. Besides, investors always adjust investment proportion according to market information. And at the same time, transaction costs are considered into the multi-period portfolio model as a non-negligible factor. Before constructing models, firstly we rewrite and replenish some symbols.

By the propositions in Section 3, we can induce the equations about mean, variance and covariance in the fuzzy portfolio model. The possibilistic mean of the returns under investment proportion x_t is denoted by

$$\begin{aligned} &E(R_{t(K)})=E(\sum_{i=1}^{n}x_{t,i}r_{t,i(K)})=\sum_{i=1}^{n}x_{t,i}E(r_{t,i(K)}) \\ &=\sum_{i=1}^{n}x_{t,i}(\frac{a_{t,i}+b_{t,i}}{2}+\frac{\beta_{t,i}-\alpha_{t,i}}{6}+K^3\cdot\frac{\alpha_{t,i}-\beta_{t,i}}{6}). \end{aligned} \tag{34}$$

The possibilistic variance of the returns under investment proportion x_t is denoted by

$$\begin{aligned} &Var(R_{t(K)}) \\ &=\sum_{i=1}^{n}x_{t,i}^2 Var(r_{t,i(K)}) \\ &+2\sum_{1\le i<j\le n}x_{t,i}x_{t,j}Cov(r_{t,i(K)},r_{t,j(K)}),\ t=1,2,\dots,T. \end{aligned} \tag{35}$$

Supposed the transaction costs are V-type functions, so the transaction costs of the $i-$th risk asset in the *t*-period are $c_{t,i}\left|x_{t,i}-x_{t-1,i}\right|$. Then in the *t*-period, the total transaction costs of investment are

$$C_t=\sum_{i=1}^{n}c_{t,i}\left|x_{t,i}-x_{t-1,i}\right|,\ t=1,2,\dots,T.$$

So, the net return in the *t*-period is

$$E_t(R_{t(K)})=\sum_{i=1}^{n}x_{t,i}E(r_{t,i(K)})-\sum_{i=1}^{n}c_{t,i}\left|x_{t,i}-x_{t-1,i}\right|,\ t=1,2,\dots,T.$$

Besides, the recursion equation of wealth is denoted by

$$W_{t+1}=W_t(1+E_t(R_{t(K)})),\ t=1,2,\dots,T.$$

b. Construction and analysis of model

Based on the above analysis, we construct the multi-period mean-variance-entropy portfolio model under uncertain exit time with a parameter as in Equation (36).

Remark 6: The return covariance of any two assets in the same period is denoted by Equation (37). The adjustment coefficient of $x_{t,i}$ is denoted by Equation (38).

It is noted that there are two mutually restrictive objectives in the model. Minimizing risk means holding as a high proportion as possible for the minimum risk assets, which is in contradiction with maximizing the degree of diversification of investment. In order to find a Pareto solution, we use the fuzzy convex programming method [42] to deal with this model. We need to find the maximum values of two objective functions ($V(x),P(x)$) in the same feasible region X_0, that is, we first solve the following four models:

$$\begin{cases}\min\ V(x)\\ \text{s.t.}\ \ x\in X_0\end{cases} \tag{39}$$

$$\begin{cases}\max\ V(x)\\ \text{s.t.}\ \ x\in X_0\end{cases} \tag{40}$$

$$\begin{cases}\min\ P(x)\\ \text{s.t.}\ \ x\in X_0\end{cases} \tag{41}$$

$$\begin{cases}\max\ P(x)\\ \text{s.t.}\ \ x\in X_0\end{cases} \tag{42}$$

For given different parameter K values, we can obtain corresponding optimal solutions of above four models. We write them as $V_{(K)}^{\min},V_{(K)}^{\max},P_{(K)}^{\min},P_{(K)}^{\max}$, respectively. Then two membership functions are set to measure investors' degree of satisfaction to two objectives $V(x),P(x)$ by

$$\mu_{V(K)}(x)=\begin{cases}0,\ \ V(x)\ge V_{(K)}^{\max}\\ \dfrac{V_{(K)}^{\max}-V(x)}{V_{(K)}^{\max}-V_{(K)}^{\min}},\ \ V_{(K)}^{\min}\le V(x)\le V_{(K)}^{\max}\\ 1,\ \ V(x)\le V_{(K)}^{\min}\end{cases} \tag{43}$$

and

$$\mu_{P(K)}(x)=\begin{cases}0,\ \ P(x)\le P_{(K)}^{\min}\\ \dfrac{P(x)-P_{(K)}^{\min}}{P_{(K)}^{\max}-P_{(K)}^{\min}},\ \ P_{(K)}^{\min}\le P(x)\le P_{(K)}^{\max}\\ 1,\ \ P(x)\ge P_{(K)}^{\max}.\end{cases} \tag{44}$$

Finally, we can use fuzzy convex programming to transform the bi-objective model into the single objective model as follow:

$$\begin{cases}\max\ \ u_{(K)}(x)=\alpha_V\mu_{V(K)}(x)+\alpha_P\mu_{P(K)}(x)\\ \text{s.t.}\ \ \ x\in X_0,\\ \qquad \alpha_V+\alpha_P=1,\\ \qquad \alpha_V,\alpha_P\ge 0.\end{cases} \tag{45}$$

Remark 7: α_V,α_P represent the preference degree coefficients of minimizing the multi-period cumulative risk and maximizing the degree of diversification of investment.

V. Numerical Example

In this section, the multi-period mean-variance-entropy portfolio model is applied to analyze portfolios with real market data. We suppose investors choose five kinds of risk assets in the market. There are five kinds of stocks selected from the Shenzhen Stock Exchange. The codes are SZ000418, SZ000429, SZ000061, SZ000830, and SZ000860, which are abbreviated by stocks 1-5, respectively. The initial wealth is set to $W_1 = 10000$, and investors intend to make three periods investment decisions. We collect the closing price of each quarter from January 2013 to December 2018 and calculate the returns of each quarter. Then we use the approximate method to get the trapezoidal fuzzy numbers distribution of asset return in each period.

Under different parameters K values, the three periods returns and their means, variances, and entropies of five stocks are shown in Tables IV-VI. From these tables, we can see that the average return of each stock is monotonically decreasing or increasing about K, which depends on whether the value of $\beta-\alpha$ is positive or negative. The variance of return rate decreases when the value of K increases. These results are consistent with the analysis in Section 3. Additionally, because of the decrease of variance, most of the entropy increases monotonically.

After obtaining the data, we normalize the entropy of five stocks in each period to [0.3,1.9] and set $\varepsilon = 1\times 10^{-9}$ in order to ensure the validity of the total entropy equation of investment. According to the historical performance of the market, we set the minimum expected return in each period as $\mu_1 = 0.04, \mu_2 = 0.07, \mu_3 = 0.09$, the risk-free return rate of each period as $r_f(1) = r_f(2) = 0.012, r_f(3) = 0.01$, the upper and lower limits of the investment proportion of each stock in each period as $l_{t,i} = 0.1, u_{t,i} = 0.6 \quad (i = 1,\ldots,5; t = 1,2,3)$. Besides, the unit transaction costs of five stocks are all $c_{t,i} = 0.003 (i = 1,2,\ldots,5; t = 1,2,3)$.

According to the models in Section 4, corresponding optimal portfolio selection solutions are solved based on investors' different exit methods. These are shown in Tables VII-XI.

Based on the above results, we analyze the variance, entropy and final wealth with different values of K, α_V, α_P, respectively. The results are shown in Figures 3-5.

In Figure 3, at the same preference levels of α_V, α_P, with the increase of the K value, the variance significantly decreases. This proves that the stronger the investors' exit willing is, the lower the risk is. On the other hand, with the increase of preference for risk minimization α_V, the variance is significantly decreasing.

TABLE XI

COMPARISON UNDER DIFFERENT K, α_V, α_P

K	α_V	α_P	$u_{(K)}$	variance	entropy	wealth
0	1	0	1.000	0.0224	-4.315	12130
	0.5	0.5	0.745	0.0341	-3.772	12130
	0	1	1.000	0.0350	-3.744	12130
0.3	1	0	1.000	0.0216	-4.325	12130
	0.5	0.5	0.795	0.0277	-3.878	12358
	0	1	1.000	0.0372	-3.730	12687
0.5	1	0	1.000	0.0189	-4.290	12130
	0.5	0.5	0.740	0.0295	-3.810	12130
	0	1	1.000	0.0305	-3.773	12130
0.7	1	0	1.000	0.0144	-4.190	12149
	0.5	0.5	0.738	0.0233	-3.703	12130
	0	1	1.000	0.0244	-3.661	12130

This proves that the preference coefficient we set has a significant impact on the results. We can provide different portfolio selections for investors based on their preferences for risk and diversification.

In Figure 4, at the same preference levels of α_V, α_P, with the increase of the intercept level K, the entropy increases slowly. This shows that the stronger the investors' exit willing is, the higher the portfolio diversification degree is, but the change is slight. On the other hand, at the same K value, with the preference of minimizing the degree of investment dispersion α_P increasing, the entropy increases significantly. This proves that the preference coefficient we set has a significant impact on the results. We can provide different portfolio choices for investors based on their preference for risk and diversification.

In Figure 5, the values of wealth are significantly higher than others when $K = 0.3$. But the ultimate wealth doesn't change significantly in other cases, because it is not the main goal of our model. However, it can be seen from the figure that in two cases, the return rate increases significantly when K is close to 0.3. This shows that a moderate willingness to exit helps to improve the final wealth. On the one hand, if investors seldom or never withdraw investments or adjust investment decisions, they may lose the opportunity of getting profit from other assets and bear the huge losses of holding assets. On the other hand, if investors have strong exit willing, they will withdraw investment prematurely and lose the opportunity to acquire long-term returns.

After analyzing three important results respectively, we can summarize the following conclusion holistically. First of all, at the same degree of preference, with the increase of K from 0 to 1, the variance decreases significantly, the entropy increases slowly, and the final wealth is unchanged, which verifies the feasibility and rationality of our model. However, the increase of K will also bring other disadvantaged results. That will result in the waste of working capital, too frequent adjustments in investment proportion, high transaction costs, and so on. Besides, the actual investment time will be far less than the fixed investment period. The results of the model also show that moderate exit willing can improve the wealth of multi-period investment, but the performance in risk or investment diversification is slightly worse. From the perspective of different preference degrees, the preference coefficient we set does have a significant impact on the results. To sum up, our model results show the advantages and disadvantages of investment schemes under different exit willing. We can provide different portfolio selections based on their different exit willing and the preference degree for risk and diversification of investment.

VI. Conclusion

In this paper, a novel type of fuzzy number is defined on the basis of traditional fuzzy numbers, and its numerical characteristics are deduced according to the possibility theory. For trapezoidal fuzzy numbers, we define number addition, scalar multiplication, and fuzzy addition, and prove certain properties of the numerical characteristics. Regarding the model, the novel fuzzy number is employed to describe the uncertain exit time, and the fuzzy possibilistic entropy with an adjustment coefficient is utilized to characterize the degree of investment diversification. Subsequently, the fuzzy convex programming method is applied to solve the model. Moreover, we take into account the preferences for risk and investment diversification within the model, which enables us to offer suitable investment schemes for different investors.

The main contribution of this paper lies in studying the uncertain exit time within a fuzzy environment. Our model primarily addresses the following three issues: (1) how to make dynamic adjustments to the investment proportion in multi - period investment; (2) how to quantify the uncertain exit time; (3) how to guarantee investment diversification. To some degree, our model is capable of customizing investment schemes based on investors' preferences.

In future research on portfolios within a fuzzy environment, on one hand, scholars could take into account an increasing number of constraints in the model. This would bring portfolio decisions closer to the actual investment environment. On the other hand, researchers can explore other types of novel fuzzy numbers that can effectively reflect and gauge investors' psychological attitudes. This approach will offer investors a more satisfactory portfolio selection.

TABLE I
Comparison of literature features

Features	Risk measure	Environment	Possibility theory	Multi-period dynamic	Uncertain exit time
Yao et al. (2013)	Mean-variance	Random	×	√	×
Arash et al. (2018)	Mean-variance	Fuzzy	√	×	×
Liesiö et al. (2020)	Mean-variance	Random	×	×	×
Deng et al. (2021)	Mean-variance-efficiency	Fuzzy	√	×	×
Yao et al. (2022)	Mean-variance	Random	×	√	√
Deng and Geng (2023)	Mean-variance	Fuzzy	√	×	×
Peykani et al. (2023)	Mean-entropy	Fuzzy	×	√	×
Ge et al. (2023)	Mean-variance	Random	×	√	√
Qian and Wang (2024)	Mean-variance	Random	×	√	×
Our approach	**Mean-variance-entropy**	**Fuzzy**	√	√	√

TABLE II
Illustration of symbols

Symbol	Illustration
$x_{t,i}$	The investment proportion of the $i-\mathrm{th}$ risk asset in the *t*-period
x_t	The *t*-period investment proportion vector, that is, $x_t=(x_{t,1},x_{t,2},\ldots,x_{t,n})$
$r_{t,i}$	The fuzzy rate of return of the $i-\mathrm{th}$ risk asset in the *t*-period
R_t	The fuzzy rate of return of the *t*-period investment x_t
$r_f(t)$	The *t*-period risk-free investment rate of return
W_t	The wealth value at the beginning of the *t*-period
μ_t	The minimum expected rate of return in the *t*-period
$l_{t,i},u_{t,i}$	The investment proportion upper and lower limits of the *i*-kind risk asset in the *t*-period

TABLE III
Rewrite and replenish of symbols

Symbol	Illustration

$r_{t,i(K)}$	The fuzzy rate of return of the *i*-kind risk asset in the *t*-period with quitting intention K
$R_{t(K)}$	The fuzzy rate of return of the *t*-period investment x_t with quitting intention K
$c_{t,i}$	The transaction expenses of the *i*-kind risk asset in the *t*-period

TABLE IV
RETURN OF FIVE STOCKS AND THEIR MEAN, VARIANCE AND ENTROPY UNDER DIFFERENT VALUES OF $K(t=1)$

number	$A=(a,b,\alpha,\beta)$	index	$K=0$	$K=0.3$	$K=0.5$	$K=0.7$
1	(0.0589,0.1081,0.1536,0.0423)	mean	0.0649	0.0654	0.0673	0.0713
		variance	0.0040	0.0038	0.0032	0.0022
		entropy	13.300	14.150	17.280	26.410
2	(0.0091,0.0338,0.0797,0.1360)	mean	0.0308	0.0306	0.0297	0.0276
		variance	0.0030	0.0028	0.0023	0.0014
		entropy	6.2200	6.5600	7.7200	10.990
3	(0.0249,0.1564,0.1589,0.2077)	mean	0.0988	0.0985	0.0977	0.0957
		variance	0.0180	0.0173	0.0152	0.0116
		entropy	4.8200	5.0000	5.6300	7.2700
4	(-0.0089,0.0659,0.1231,0.0955)	mean	0.0239	0.0240	0.0245	0.0258
		variance	0.0061	0.0059	0.0052	0.0039
		entropy	1.9400	2.0400	2.4200	3.4700
5	(-0.0590,0.0551,0.0120,0.1701)	mean	0.0244	0.0237	0.0211	0.0154
		variance	0.0084	0.0082	0.0074	0.0061
		entropy	1.4700	1.4300	1.2300	0.5500

TABLE V
RETURN OF FIVE STOCKS AND THEIR MEAN, VARIANCE AND ENTROPY UNDER DIFFERENT VALUES OF $K(t=2)$

number	return $A=(a,b,\alpha,\beta)$	index	$K=0$	$K=0.3$	$K=0.5$	$K=0.7$
1	(0.0459,0.2990,0.0699,0.0297)	mean	0.1658	0.1659	0.1666	0.1680
		variance	0.0207	0.0205	0.0200	0.0189
		entropy	7.4400	7.5100	7.7300	8.2500
2	(0.0066,0.1945,0.1062,0.0642)	Mea n	0.0936	0.0938	0.0945	0.0960
		variance	0.0154	0.0152	0.0143	0.0128
		entropy	5.3000	5.4000	5.7500	6.5800
3	(-0.0241,0.0850,0.1561,0.1067)	mean	0.0222	0.0224	0.0232	0.0250
		variance	0.0107	0.0103	0.0092	0.0071
		entropy	0.9600	1.0100	1.2300	1.8300
4	(0.0040,0.0878,0.1828,0.1944)	mean	0.0478	0.0478	0.0476	0.0471
		variance	0.0130	0.0123	0.0105	0.0073
		entropy	2.7600	2.9000	3.4100	4.8300
5	(0.0528,0.1101,0.2100,0.0922)	mean	0.0618	0.0623	0.0643	0.0685
		variance	0.0077	0.0073	0.0061	0.0041
		entropy	6.4700	6.9200	8.6200	13.880

TABLE VI
RETURN OF FIVE STOCKS AND THEIR MEAN, VARIANCE AND ENTROPY UNDER DIFFERENT VALUES OF $K(t=3)$

number	return $A=(a,b,\alpha,\beta)$	index	$K=0$	$K=0.3$	$K=0.5$	$K=0.7$
1	(-0.0145,0.0690,0.1052,0.2612)	mean	0.0533	0.0526	0.0500	0.0443
		variance	0.0128	0.0121	0.0102	0.0071
		entropy	3.3900	3.5100	3.9100	4.8300
2	(0.0152,0.0799,0.0869,0.0243)	mean	0.0371	0.0374	0.0384	0.0407
		variance	0.0028	0.0027	0.0025	0.0020
		entropy	9.6300	10.030	11.470	15.170
3	(0.0818,0.1910,0.0942,0.0316)	mean	0.1260	0.1263	0.1273	0.1296
		variance	0.0060	0.0059	0.0055	0.0047
		entropy	19.390	19.850	21.470	25.320
4	(0.0460,0.2118,0.1175,0.3273)	mean	0.1639	0.1629	0.1595	0.1519
		variance	0.0280	0.0269	0.0236	0.0179
		entropy	5.4900	5.6800	6.3300	7.9100
5	(0.0210,0.1266,0.0738,0.2629)	mean	0.1053	0.1045	0.1014	0.0945
		variance	0.0139	0.0133	0.0115	0.0084
		entropy	6.8400	7.1000	7.9600	10.040

TABLE VII
OPTIMAL SOLUTIONS WHEN $K=0$

model	$u_{(K)}$	variance	entropy	wealth	t	stock 1	stock 2	stock 3	stock 4	stock 5
(39)	—	0.0224	—	12130	$t=1$	0.1969	0.5031	0.1000	0.1000	0.1000
					$t=2$	0.1271	0.1000	0.1000	0.1000	0.5729
					$t=3$	0.1000	0.3087	0.3913	0.1000	0.1000
(40)	—	0.0467	—	13508	$t=1$	0.1000	0.6000	0.1000	0.1000	0.1000
					$t=2$	0.6000	0.1000	0.1000	0.1000	0.1000
					$t=3$	0.1000	0.1000	0.1000	0.6000	0.1000
(41)	—	—	-4.690	12521	$t=1$	0.2957	0.2774	0.2269	0.1000	0.1000
					$t=2$	0.2550	0.2490	0.1000	0.1391	0.2569
					$t=3$	0.1000	0.2835	0.2998	0.1234	0.1933
(42)	—	—	-3.744	12130	$t=1$	0.1000	0.1000	0.1877	0.1000	0.5123
					$t=2$	0.2550	0.1000	0.4450	0.1000	0.1000
					$t=3$	0.4474	0.1000	0.1000	0.2526	0.1000
(45) $\alpha_V=1, \alpha_P=0$	1.000	0.0224	-4.315	12130	$t=1$	0.1969	0.5031	0.1000	0.1000	0.1000
					$t=2$	0.1271	0.1000	0.1000	0.1000	0.5729
					$t=3$	0.1000	0.3087	0.3913	0.1000	0.1000
(45) $\alpha_V=\alpha_P=0.5$	0.745	0.0341	-3.772	12130	$t=1$	0.1000	0.1000	0.1905	0.5095	0.1000
					$t=2$	0.2549	0.1000	0.4451	0.1000	0.1000
					$t=3$	0.4473	0.1000	0.1000	0.2527	0.1000
(45) $\alpha_V=0, \alpha_P=1$	1.000	0.0350	-3.744	12130	$t=1$	0.1000	0.1000	0.1877	0.1000	0.5123
					$t=2$	0.2550	0.1000	0.4450	0.1000	0.1000
					$t=3$	0.4473	0.1000	0.1000	0.2527	0.1000

TABLE VIII

OPTIMAL SOLUTIONS WHEN $K=0.3$

model	$u_{(K)}$	variance	entropy	wealth	t	stock 1	stock 2	stock 3	stock 4	stock 5
(39)	—	0.0216	—	12130	$t=1$	0.2001	0.4999	0.1000	0.1000	0.1000
					$t=2$	0.1238	0.1000	0.1000	0.1000	0.5762
					$t=3$	0.1000	0.3081	0.3919	0.1000	0.1000
(40)	—	0.0453	—	13500	$t=1$	0.1000	0.1000	0.6000	0.1000	0.1000
					$t=2$	0.6000	0.1000	0.1000	0.1000	0.1000
					$t=3$	0.1000	0.1000	0.1000	0.6000	0.1000
(41)	—	—	-4.720	12516	$t=1$	0.2958	0.2782	0.2259	0.1000	0.1000
					$t=2$	0.2537	0.2476	0.1000	0.1433	0.2554
					$t=3$	0.1000	0.2842	0.2890	0.1229	0.1939
(42)	—	—	-3.730	12687	$t=1$	0.1000	0.1000	0.1926	0.1000	0.5074
					$t=2$	0.6000	0.1000	0.1000	0.1000	0.1000
					$t=3$	0.4528	0.1000	0.1000	0.2472	0.1000
(45) $\alpha_V=1, \alpha_P=0$	1.000	0.0216	-4.325	12130	$t=1$	0.2001	0.4999	0.1000	0.1000	0.1000
					$t=2$	0.1238	0.1000	0.1000	0.1000	0.5762
					$t=3$	0.1000	0.3081	0.3919	0.1000	0.1000
(45) $\alpha_V=\alpha_P=0.5$	0.795	0.0277	-3.878	12358	$t=1$	0.1000	0.1000	0.1908	0.5092	0.1000
					$t=2$	0.2534	0.1000	0.4466	0.1000	0.1000
					$t=3$	0.1000	0.1000	0.6000	0.1000	0.1000
(45) $\alpha_V=0, \alpha_P=1$	1.000	0.0372	-3.730	12687	$t=1$	0.1000	0.1000	0.1926	0.1000	0.5074
					$t=2$	0.6000	0.1000	0.1000	0.1000	0.1000
					$t=3$	0.4528	0.1000	0.1000	0.2472	0.1000

TABLE IX

OPTIMAL SOLUTIONS WHEN $K=0.5$

model	$u_{(K)}$	variance	entropy	wealth	t	stock 1	stock 2	stock 3	stock 4	stock 5
(39)	—	0.0189	—	12130	$t=1$	0.2104	0.4896	0.1000	0.1000	0.1000
					$t=2$	0.1116	0.1000	0.1000	0.1000	0.5884
					$t=3$	0.1000	0.3060	0.3940	0.1000	0.1000
(40)	—	0.0409	—	13473	$t=1$	0.1000	0.1000	0.6000	0.1000	0.1000
					$t=2$	0.6000	0.1000	0.1000	0.1000	0.1000
					$t=3$	0.1000	0.1000	0.1000	0.6000	0.1000
(41)	—	—	-4.718	12513	$t=1$	0.2964	0.2799	0.2236	0.1000	0.1000
					$t=2$	0.2566	0.2421	0.1000	0.1467	0.2546
					$t=3$	0.1000	0.2868	0.2963	0.1217	0.1951
(42)	—	—	-3.773	12130	$t=1$	0.1000	0.1000	0.2099	0.1000	0.4901
					$t=2$	0.2472	0.1000	0.4528	0.1000	0.1000
					$t=3$	0.4223	0.1000	0.1000	0.2777	0.1000
(45) $\alpha_V=1$, $\alpha_P=0$	1.000	0.0189	-4.290	12130	$t=1$	0.2104	0.4896	0.1000	0.1000	0.1000
					$t=2$	0.1116	0.1000	0.1000	0.1000	0.5884
					$t=3$	0.1000	0.3060	0.3940	0.1000	0.1000
(45) $\alpha_V=\alpha_P$ $=0.5$	0.740	0.0295	-3.810	12130	$t=1$	0.1000	0.1000	0.1920	0.5080	0.1000
					$t=2$	0.2480	0.1000	0.4520	0.1000	0.1000
					$t=3$	0.4224	0.1000	0.1000	0.2776	0.1000
(45) $\alpha_V=0$, $\alpha_P=1$	1.000	0.0305	-3.773	12130	$t=1$	0.1000	0.1000	0.2099	0.1000	0.4901
					$t=2$	0.2472	0.1000	0.4528	0.1000	0.1000
					$t=3$	0.4223	0.1000	0.1000	0.2777	0.1000

TABLE X

OPTIMAL SOLUTIONS WHEN $K=0.7$

model	$u_{(K)}$	variance	entropy	wealth	t	stock 1	stock 2	stock 3	stock 4	stock 5
(39)	—	0.0144	—	12149	$t=1$	0.2287	0.4713	0.1000	0.1000	0.1000
					$t=2$	0.1000	0.1000	0.1000	0.1000	0.6000
					$t=3$	0.1000	0.3012	0.3988	0.1000	0.1000
(40)	—	0.0331	—	13410	$t=1$	0.1000	0.1000	0.6000	0.1000	0.1000
					$t=2$	0.6000	0.1000	0.1000	0.1000	0.1000
					$t=3$	0.1000	0.1000	0.1000	0.6000	0.1000
(41)	—	—	-4.566	12502	$t=1$	0.2976	0.2824	0.2176	0.1024	0.1000
					$t=2$	0.2570	0.2214	0.1000	0.1510	0.2706
					$t=3$	0.1000	0.2856	0.2918	0.1253	0.1973
(42)	—	—	-3.661	12130	$t=1$	0.1000	0.1000	0.2457	0.1000	0.4543
					$t=2$	0.2338	0.1000	0.4662	0.1000	0.1000
					$t=3$	0.3775	0.1000	0.1000	0.3225	0.1000
(45) $\alpha_V=1$, $\alpha_P=0$	1.000	0.0144	-4.190	12149	$t=1$	0.2287	0.4713	0.1000	0.1000	0.1000
					$t=2$	0.1000	0.1000	0.1000	0.1000	0.6000
					$t=3$	0.1000	0.3012	0.3988	0.1000	0.1000
(45) $\alpha_V=\alpha_P$ $=0.5$	0.738	0.0233	-3.703	12130	$t=1$	0.1000	0.1000	0.1949	0.5051	0.1000
					$t=2$	0.2359	0.1000	0.4641	0.1000	0.1000
					$t=3$	0.3776	0.1000	0.1000	0.3224	0.1000
(45) $\alpha_V=0$ $\alpha_P=1$	1.000	0.0244	-3.661	12130	$t=1$	0.1000	0.1000	0.2457	0.1000	0.4543
					$t=2$	0.2338	0.1000	0.4662	0.1000	0.1000
					$t=3$	0.3775	0.1000	0.1000	0.3225	0.1000

$$
\begin{aligned}
&E(A_{(K)})\\
&=\int_0^1 \gamma(\underline{a}(\gamma)+\overline{a}(\gamma))d\gamma\\
&=\int_0^K \gamma(a+b+(1-K)(\beta-\alpha))d\gamma+\int_K^1 \gamma(a+b+(1-\gamma)(\beta-\alpha))d\gamma\\
&=\frac{1}{2}K^2(a+b+(1-K)(\beta-\alpha))+\frac{1}{3}(\alpha-\beta)(1-K^3)+\frac{1}{2}(a+b+\beta-\alpha)(1-K^2)\\
&=\frac{b+a}{2}+\frac{\beta-\alpha}{6}+K^3\frac{\alpha-\beta}{6}.
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&Var^*(A_{(K)})\\
&=\frac{1}{2}\int_0^1 \gamma(\overline{a}(\gamma)-\underline{a}(\gamma))^2 d\gamma\\
&=\frac{1}{2}\left[\int_0^K \gamma(b-a+(1-K)(\alpha+\beta))^2 d\gamma+\int_K^1 \gamma(b-a+(1-\gamma)(\alpha+\beta))^2 d\gamma\right]\\
&=\frac{1}{2}[\frac{1}{2}K^2(b-a+(1-K)(\alpha+\beta))^2+\frac{1}{2}(1-K^2)(b-a+\alpha+\beta)^2\\
&\quad-\frac{2}{3}(1-K^3)(b-a+\alpha+\beta)(\alpha+\beta)+\frac{1}{4}(1-K^4)(\alpha+\beta)^2]\\
&=(\frac{b-a}{2}+\frac{\alpha+\beta}{6})^2+\frac{(\alpha+\beta)^2}{72}-\frac{K^3}{6}(b-a+\alpha+\beta)(\alpha+\beta)+\frac{K^4}{8}(\alpha+\beta)^2.
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
Var(A_{(K)})&=\int_0^1 \gamma[(E(A_{(K)})-\underline{a}(\gamma))^2+(E(A)-\overline{a}(\gamma))^2]d\gamma\\
&=\int_0^K \gamma[(E(A_{(K)})-a+(1-K)\alpha)^2+(E(A_{(K)})-b-(1-K)\beta)^2]d\gamma\\
&\quad+\int_K^1 \gamma[(E(A_{(K)})-a+(1-\gamma)\alpha)^2+(E(A_{(K)})-b-(1-\gamma)\beta)^2]d\gamma.
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\int_0^K \gamma[(E(A_{(K)})-a+(1-K)\alpha)^2+(E(A_{(K)})-b-(1-K)\beta)^2]d\gamma\\
&=\frac{1}{2}K^2[2(E(A_{(K)}))^2-2E(A_{(K)})(a+b+(1-K)(\beta-\alpha))+a^2+b^2\\
&\qquad-2(1-K)a\alpha+2(1-K)b\beta+(1-K)^2\alpha^2+(1-K)^2\beta^2]\\
&=K^2[(E(A_{(K)}))^2-E(A_{(K)})(a+b+\beta-\alpha)+\frac{1}{2}(a^2+b^2+\alpha^2+\beta^2)-a\alpha+b\beta]\\
&\quad+K^3[E(A_{(K)})(\beta-\alpha)+a\alpha-b\beta-\alpha^2-\beta^2]+K^4[\frac{1}{2}(\alpha^2+\beta^2)].
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&Var(A_{(K)})\\
&=\frac{1}{2}[(E(A_{(K)})-a+\alpha)^2+(E(A_{(K)})-b-\beta)^2]\\
&\quad+\frac{1}{3}[-2\alpha(E(A_{(K)})-a+\alpha)^2+2\beta(E(A_{(K)})-b-\beta)^2]\\
&\quad+\frac{1}{4}(\alpha^2+\beta^2)+\frac{1}{3}K^3[\beta(E(A_{(K)})-b-\beta)-\alpha(E(A_{(K)})-a+\alpha)]+\frac{1}{4}K^4(\alpha^2+\beta^2)\\
&=\frac{1}{4}(\alpha^2+\beta^2)+\frac{1}{2}[(E(A_{(K)})-a+\alpha)^2+(E(A_{(K)})-b-\beta)^2]\\
&\quad+\frac{1}{3}(K^3+2)[\beta(E(A_{(K)})-b-\beta)-\alpha(E(A_{(K)})-a+\alpha)]+\frac{1}{4}K^4(\alpha^2+\beta^2).
\end{aligned}
\tag{23}
$$

$$\begin{aligned}
&\int_K^1 \gamma\left[(E(A_{(K)})-a+(1-\gamma)\alpha)^2+(E(A_{(K)})-b-(1-\gamma)\beta)^2\right]d\gamma \\
&=\int_K^1 \gamma[(E(A_{(K)})-a+\alpha-\gamma\alpha)^2+(E(A_{(K)})-b-\beta+\gamma\beta)^2]d\gamma \\
&=\int_K^1 \begin{matrix}[\gamma((E(A_{(K)})-a+\alpha)^2+(E(A_{(K)})-b-\beta)^2) \\ +\gamma^2(-2\alpha(E(A_{(K)})-a+\alpha)+2\beta(E(A_{(K)})-b-\beta))+\gamma^3(\alpha^2+\beta^2)]\end{matrix} \; d\gamma \\
&=\frac{1}{2}(1-K^2)[(E(A_{(K)})-a+\alpha)^2+(E(A_{(K)})-b-\beta)^2] \\
&\quad+\frac{1}{3}(1-K^3)[-2\alpha(E(A_{(K)})-a+\alpha)+2\beta(E(A_{(K)})-b-\beta)]+\frac{1}{4}(1-K^4)(\alpha^2+\beta^2) \\
&=\frac{1}{2}[(E(A_{(K)})-a+\alpha)^2+(E(A_{(K)})-b-\beta)^2] \\
&\quad+\frac{1}{3}[-2\alpha(E(A_{(K)})-a+\alpha)+2\beta(E(A_{(K)})-b-\beta)] \\
&\quad+\frac{1}{4}(\alpha^2+\beta^2)+K^2[-\frac{1}{2}((E(A_{(K)})-a+\alpha)^2+(E(A_{(K)})-b-\beta)^2)] \\
&\quad+K^3[\frac{2}{3}(E(A_{(K)})(\alpha-\beta)-a\alpha+b\beta+\alpha^2+\beta^2)]+K^4[-\frac{1}{4}(\alpha^2+\beta^2)].
\end{aligned} \tag{22}$$

$$\begin{aligned}
&Var(A_{(K)})' \\
&=K^3(\alpha^2+\beta^2)+\frac{K^2}{2}(\alpha-\beta)(2E(A_{(K)})-a-b+\alpha-\beta) \\
&\quad+K^2[\beta(E(A_{(K)})-b-\beta)-\alpha(E(A_{(K)})-a+\alpha)]+\frac{1}{3}(K^3+2)(\beta-\alpha)\cdot\frac{1}{2}K^2(\alpha-\beta) \\
&=K^2[K(\alpha^2+\beta^2)-\frac{1}{2}(\alpha+\beta)^2-\frac{1}{2}(\alpha+\beta)(b-a)-\frac{1}{3}(\alpha-\beta)^2-\frac{1}{6}K^3(\alpha-\beta)^2].
\end{aligned} \tag{24}$$

$$\begin{aligned}
&Cov(A_{1(K)},A_{2(K)}) \\
&=\frac{1}{2}[(E(A_{1(K)})-a_1+\alpha_1)(E(A_{2(K)})-a_2+\alpha_2)+(E(A_{1(K)})-b_1-\beta_1)(E(A_{2(K)})-b_2-\beta_2)] \\
&\quad+\frac{1}{6}(K^3+2)[\beta_1(E(A_{2(K)})-b_2-\beta_2)+\beta_2(E(A_{1(K)})-b_1-\beta_1) \\
&\quad-\alpha_1(E(A_{2(K)})-a_2+\alpha_2)-\alpha_2(E(A_{1(K)})-a_1+\alpha_1)] \\
&\quad+\frac{1}{4}(1+K^4)(\alpha_1\alpha_2+\beta_1\beta_2).
\end{aligned} \tag{25}$$

$$PE_m(x)=-\sum_{t=1}^{T}\sum_{i=1}^{n}\left[\frac{x_{t,i}\theta(x_{t,i})}{2}\ln\left(\varepsilon+\frac{x_{t,i}\theta(x_{t,i})}{2}\right)+\left(1-\frac{x_{t,i}\theta(x_{t,i})}{2}\right)\ln\left(1-\frac{x_{t,i}\theta(x_{t,i})}{2}\right)\right]. \tag{32}$$

$$\begin{cases}
\min \; \sum_{t=1}^{T} Var(R_t)=\sum_{t=1}^{T}\left[\sum_{i=1}^{n}x_{t,i}^2 Var(r_{t,i})+2\sum_{1\le i<j\le n}x_{t,i}x_{t,j}Cov(r_{t,i},r_{t,j})\right] \\
\max \; PE_m(x)=-\sum_{t=1}^{T}\sum_{i=1}^{n}\left[\frac{x_{t,i}\theta(x_{t,i})}{2}\ln\left(\varepsilon+\frac{x_{t,i}\theta(x_{t,i})}{2}\right)+\left(1-\frac{x_{t,i}\theta(x_{t,i})}{2}\right)\ln\left(1-\frac{x_{t,i}\theta(x_{t,i})}{2}\right)\right] \\
\text{s.t.} \quad W_{t+1}=W_t(1+E(R_t)), \\
\qquad \sum_{i=1}^{n}x_{t,i}=1, E(R_t)=\sum_{i=1}^{n}x_{t,i}E(r_{t,i})\ge\mu_t, \\
\qquad x_{0,i}=0,\; 0\le l_{t,i}\le x_{t,i}\le u_{t,i}\le 1, \\
\qquad i=1,2,\cdots,n, t=1,2,\cdots,T.
\end{cases} \tag{33}$$

$$\left\{\begin{array}{ll} \min & \sum_{t=1}^{T} Var(R_{t(K)}) = \sum_{t=1}^{T}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} x_{t,i}x_{t,j}Cov(r_{t,i(K)}, r_{t,j(K)})\right] \\ \max & PE_m(x) = -\sum_{t=1}^{T}\sum_{i=1}^{n}\left[\begin{array}{l}\frac{x_{t,i}\theta(x_{t,i})}{2}\ln\left(\varepsilon + \frac{x_{t,i}\theta(x_{t,i})}{2}\right) \\ +\left(1-\frac{x_{t,i}\theta(x_{t,i})}{2}\right)\ln\left(1-\frac{x_{t,i}\theta(x_{t,i})}{2}\right)\end{array}\right] \\ \text{s.t.} & W_{t+1} = W_t(1+E_t(R_{t(K)})), \\ & E_t(R_{t(K)}) = \sum_{i=1}^{n} x_{t,i}\left(\frac{a_{t,i}+b_{t,i}}{2} + \frac{\beta_{t,i}-\alpha_{t,i}}{6} + K^3\cdot\frac{\alpha_{t,i}-\beta_{t,i}}{6}\right) - \sum_{i=1}^{n} c_{t,i}\left|x_{t,i}-x_{t-1,i}\right| \ge \mu_t, \\ & \sum_{i=1}^{n} x_{t,i} = 1, \\ & x_{0,i} = 0, \\ & 0 \le l_{t,i} \le x_{t,i} \le u_{t,i} \le 1, \\ & i = 1,2,\cdots,n, t = 1,2,\cdots,T. \end{array}\right. \tag{36}$$

$$\begin{aligned} & Cov(r_{t,i(K)}, r_{t,j(K)}) \\ = & \frac{1}{2}\left[(E(r_{t,i(K)}) - a_{t,i} + \alpha_{t,i})(E(r_{t,j(K)}) - a_{t,j} + \alpha_{t,j}) + (E(r_{t,i(K)}) - b_{t,i} - \beta_{t,i})(E(r_{t,j(K)}) - b_{t,j} - \beta_{t,j})\right] \\ & +\frac{1}{6}(K^3+2)[\beta_{t,i}(E(r_{t,j(K)}) - b_{t,j} - \beta_{t,j}) + \beta_{t,j}(E(r_{t,i(K)}) - b_{t,i} - \beta_{t,i}) \\ & -\alpha_{t,i}(E(r_{t,j(K)}) - a_{t,j} + \alpha_{t,j}) - \alpha_{t,j}(E(r_{t,i(K)}) - a_{t,i} + \alpha_{t,i})] \\ & +\frac{1}{4}(1+K^4)(\alpha_{t,i}\alpha_{t,j} + \beta_{t,i}\beta_{t,j}). \end{aligned} \tag{37}$$

$$\theta(x_{t,i}) = \frac{\dfrac{\max\{\frac{a_{t,i}+b_{t,i}}{2} + \frac{\beta_{t,i}-\alpha_{t,i}}{6} + K^3\cdot\frac{\alpha_{t,i}-\beta_{t,i}}{6} - r_f(t), 0\}}{Var(r_{t,i(K)})}}{\sum_{i=1}^{n}\dfrac{\max\{\frac{a_{t,i}+b_{t,i}}{2} + \frac{\beta_{t,i}-\alpha_{t,i}}{6} + K^3\cdot\frac{\alpha_{t,i}-\beta_{t,i}}{6} - r_f(t), 0\}}{Var(r_{t,i(K)})}}. \tag{38}$$

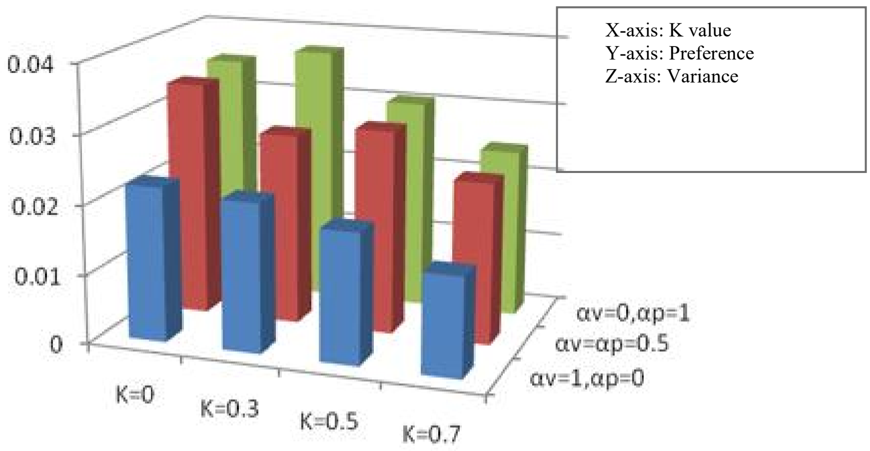

Fig. 3. Comparison of variance under different K, α_V, α_P in Model (45)

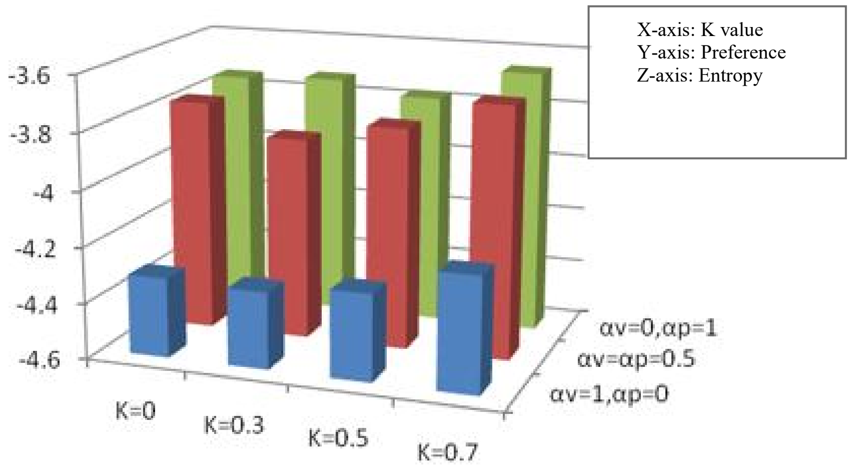

Fig. 4. Comparison of entropy under different K, α_V, α_P in Model (45)

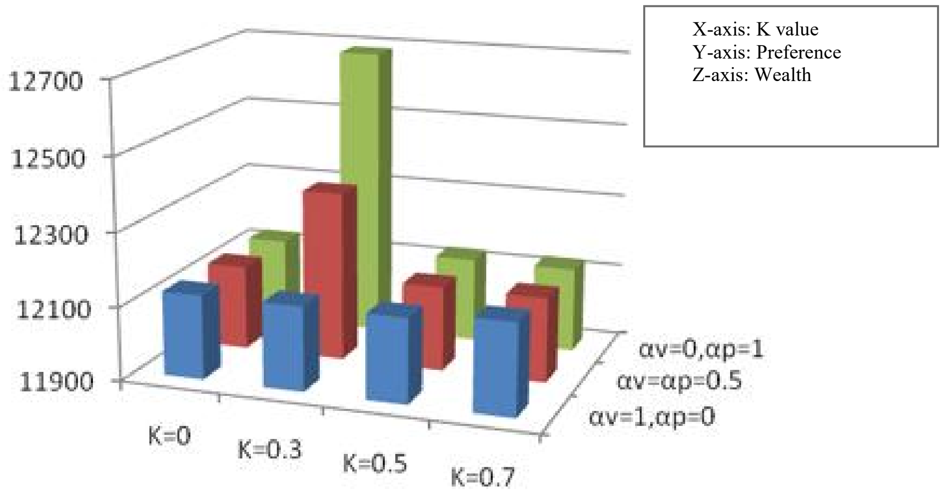

Fig. 5. Comparison of wealth under different K, α_V, α_P in Model (45)

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