Application of Differential Transform Method to Solve Fractional Relaxation Oscillation Equation

Pratiksha Devshali, Geeta Arora

Abstract—**In this work, the defining equation of relaxation oscillation processes significant in physics, biology and engineering, called the fractional relaxation oscillation equation is studied. We have solved this benchmark equation by the fractional differential transform, which is a semi-analytic method and has advantages over many numerical and analytic methods. To further explain the general solution, four specific examples are presented. The solutions obtained using the fractional differential transform method are compared with the solutions by wavelet collocation method and residual power series method, and with the available exact solutions as well. The graphical and tabular comparison of the solutions and the error analysis establishes the fact that the fractional differential transform method gives more accurate results for the fractional relaxation oscillation equation.**

Index Terms—**Fractional relaxation oscillation equation, Differential transform method, DTM, FROE.**

I. Introduction

A Particular kind of nonlinear oscillation known as relaxation oscillation happens in systems where energy builds up gradually and is discharged rapidly, creating a recurring cycle of accumulation and discharge. This process of relaxation oscillation is significant in the phenomenon where the physical system tends to return to equilibrium after being disturbed. Because of its special qualities, including its capacity to generate steady, repeating oscillations under specific circumstances, this process is significant in a variety of engineering applications.

A mathematical model known as the relaxation oscillation equation explains how systems that display nonlinear periodic oscillations with discrete periods of gradual energy or charge collection and fast release, behave. In many applications in engineering where energy or a system variable alternates between two states, this kind of oscillatory behavior is essential.

Such process is modeled as an ordinary linear differential equation of order 1 or 2. For instance

$$D^1 y(x) + Ay(x) = f(x)$$

is a relaxation equation, and

$$D^2 y(x) + Ay(x) = f(x)$$

is an oscillation equation, where x represents the time and A is a positive constant representing damping or restoring. The function $f(x)$ is the external forcing function that drives the system, and $y(x)$ may represent voltage, current, displacement, temperature etc.

In the recent times, the classical differential equations have been replaced by the fractional differential equations, as the latter have proved to be a better model. The fractional models use fractional derivatives to characterize systems that are not entirely represented by integer-order models due to memory effects or anomalous diffusion. Derivatives of non-integer orders make it possible to describe complex systems ([1], [2], [3]) more accurately and flexibly because the current state depends on the system's complete history as well as its immediate past.

Fractional calculus is quite old yet a new field in mathematics. It involves the study of functions with their fractional derivatives and fractional integrals. Its name does not bear literal meaning and it can be attributed to the question arisen by L'Hopital to Leibniz [4] on the notation of ordinary derivative. L'Hopital was inquisitive for the value of $\frac{d^n x}{dx^n}$ when $n = \frac{1}{2}$. As $\frac{1}{2}$ is a fraction, the name of this branch of calculus evolved to be fractional calculus whereas in actual scenario n can be any real or complex number. The fractional derivative can be defined in many non unique ways (see [5], [6], [7]) and the most popular are the Caputo's [8] and Riemann Liouville's ([9], [10]) definitions.

Francesko Mainardi [11] in 1995, came up with the fractional analogy of the relaxation oscillation equation known as the Fractional relaxation oscillation equation (FROE). It is one of the simplest yet significant fractional order differential equations in order to understand the viscoelastic systems and many biological and engineering processes. The process of relaxation oscillation in many branches of physics and biology ([12], [13], [14]) can better be explained by FROE because it can capture memory and hereditary properties of the system.

For any system involving some kind of dependence on memory and its hereditary properties, let the system's response be $y(x)$, $D^\alpha y(x)$ be the Caputo's fractional derivative, where α is the arbitrary order of the derivative. Let $f(x)$ be the external forcing function and A is a damping coefficient, then the FROE is given as:

$$D^\alpha y(x) + Ay(x) = f(x) \quad x > 0; \quad y^{(k)}(0) = 0, \qquad (1)$$

$$k = 0, 1, 2, \ldots, n-1 \qquad (2)$$

where $n - 1 < \alpha \leq n$.

For $0 < \alpha \leq 1$ this equation is called fractional relaxation equation (FRE) and for $1 < \alpha \leq 2$ it is fractional oscillation equation (FOE) specifically. The $0 < \alpha \leq 1$ system is categorized [11] as "ultraslow processes" and $1 < \alpha \leq 2$ system, as "intermediate processes" by Mainardi. The ultraslow processes exhibit slow decay and no oscillations while the intermediate processes are about damped oscillations.

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Dr. Pratiksha is Assistant Professor in the Department of Applied Science, Symbiosis Institute of Technology, Symbiosis International (Deemed) University, Lavale, Pune, India. (e-mail: text2pratiksha@gmail.com).

Dr. Geeta Arora is Professor at the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Punjab, India. (e-mail: geetadma@gmail.com).

The analytic solution of the FROE in terms of Green's function [4] is given as

$$y(x) = \int_0^x G_2(x-\tau) f(\tau) d\tau, \tag{3}$$

$$G_2(x) = x^{\alpha-1} E_{\alpha,\alpha}(-Ax^\alpha).$$

The FROE has recently been solved by some authors, (see [1], [2], [3], [15], [16], [17]) using various numerical and analytic methods employing the wavelets, transforms, Hilfer derivative etc. However, the solution of the FROE by semi-analytic technique of differential transform is not there in the literature. This work is an attempt to solve the equation by using fractional DTM.

Section II is devoted to a brief account of fractional differential transform method (DTM). Then the method has been implemented on the general FROE in the Section III. The Section IV contains the numerical examples with tabular and pictorial comparison of the solutions along with the error analysis. The discussions and conclusions are recorded in Section V.

II. Fractional DTM

In 1986, Zhou [18] came up with differential transform method for the first time, for his work on electric circuit analysis. This method resulted in a power series solution of the differential equation. The convergence of the series solution by DTM has been studied by Odibat et al. in [19]. The differential transform of the k^{th} order of a function $u(x)$ is given by

$$U(k) = \frac{1}{k!}\left(\frac{d^k u(x)}{dx^k}\right)_{x=x_0}$$

and inverse differential transform of $U(k)$ is

$$u(x) = \sum_{k=0}^{\infty} U(k)(x-x_0)^k.$$

To solve the fractional differential equations, the DTM also got modified. The fractional differential transform [20] of the k^{th} order of function $u(x)$ involving the Caputo's fractional derivative of order α where $(n-1 < \alpha \leq n)$ is given as,

$$U_\alpha(k) = \frac{1}{\Gamma(\alpha k+1)}\left[\left(D_{x_0}^{\alpha}\right)^k u(x)\right]_{x=x_0} \tag{4}$$

and inverse differential transform of $U_\alpha(k)$ is given as

$$u(x) = \sum_{k=0}^{\infty} U_\alpha(k)(x-x_0)^{k\alpha}. \tag{5}$$

Since this work will incorporate fractional differential transform (DT) only, instead of $U_\alpha(k)$ for the k^{th} fractional differential transform of $u(x)$, $U(k)$ will be used.

Fractional differential transforms of the initial conditions are given as

$$U(k) = \begin{cases} \frac{1}{(k\alpha)!}\left[\frac{d^{k\alpha} u(x)}{dx^{k\alpha}}\right]_{x=x_0}, & \text{if } k\alpha \in \mathbb{Z}^+, 0 \leq k \leq \left(\frac{n}{\alpha}\right) - 1. \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

A. Prerequisites

The basic results from [21], [22], [23] are used to solve FROE. For the entirety of the paper, the results are presented with the proofs. It is assumed that $U(k)$, $V(k)$ and $W(k)$ are the fractional differential transforms of $u(x)$, $v(x)$ and $w(x)$ respectively.

Proposition. If $u(x) = v(x) \pm w(x)$ then $U(k) = V(k) \pm W(k)$.

Proof: Using Equation (5) the following can be written:

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} V(k)(x-x_0)^{k\alpha} \pm \sum_{k=0}^{\infty} W(k)(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} [V(k) \pm W(k)](x-x_0)^{k\alpha}. \end{aligned}$$

Thus by definition of differential transform, $U(k) = V(k) \pm W(k)$. This proposition implies that DT preserves linearity. ■

Proposition. If $u(x) = cv(x)$ then $U(k) = cV(k)$.

Proof: Using Equation (5)

$$\begin{aligned} u(x) &= c\sum_{k=0}^{\infty} V(k)(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} cV(k)(x-x_0)^{k\alpha}. \end{aligned}$$

Therefore, $U(k) = cV(k)$. This proposition implies that DT preserves scalar multiplication. ■

Proposition. If $u(x) = v(x)w(x)$ then $U(k) = \Sigma_{r=0}^{k} V(r)W(k-r)$.

Proof: Using the Equation (5)

$$\begin{aligned} u(x) &= v(x)w(x) \\ &= \sum_{k=0}^{\infty} V(k)(x-x_0)^{k\alpha} . \sum_{k=0}^{\infty} W(k)(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty}\sum_{r=0}^{k} V(r)W(k-r)(x-x_0)^{k\alpha}. \end{aligned}$$

Thus, $U(k) = \Sigma_{r=0}^{k} V(r)W(k-r)$. This proposition implies that DT of a product of two functions results in the convolution of their transformed functions. ■

Proposition. If $u(x) = D_{x_0}^{\alpha} v(x)$ then

$$U(k) = \frac{\Gamma(\alpha k+\alpha+1)}{\Gamma(\alpha k+1)} V(k+1).$$

Proof: Employing Equation (4),

$$\begin{aligned} U(k) &= \frac{1}{\Gamma(\alpha k+1)}\left(D_{x_0}^{\alpha}\right)^k \left[D_{x_0}^{\alpha} v(x)\right]_{x=x_0} \\ &= \frac{1}{\Gamma(\alpha k+1)}\left[\left(D_{x_0}^{\alpha}\right)^{k+1} v(x)\right]_{x=x_0} \\ &= \frac{\Gamma(\alpha k+\alpha+1)}{\Gamma(\alpha k+1)\Gamma(\alpha k+\alpha+1)}\left[\left(D_{x_0}^{\alpha}\right)^{k+1} v(x)\right]_{x=x_0} \\ &= \frac{\Gamma(\alpha k+\alpha+1)}{\Gamma(\alpha k+1)} V(k+1) \end{aligned}$$

Therefore, $U(k) = \frac{\Gamma(\alpha k+\alpha+1)}{\Gamma(\alpha k+1)} V(k+1)$. This proposition gives the DT of Caputo's derivative of a function. ■

Proposition. If $u(x) = (x - x_0)^q; \quad q = n\alpha; \quad n \in \mathbb{Z}$ then $U(k) = \delta\left(k - \frac{q}{\alpha}\right)$.

Proof: Here $\delta(k)$ is Kronecker delta function defined as

$$\delta(k-n) = \begin{cases} 1, & \text{if } k = n. \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

By Expression (5), $u(x) = (x - x_0)^q$ can be written as,

$$u(x) = \sum_{k=0}^{\infty} (x - x_0)^{\alpha k} \delta\left(k - \frac{q}{\alpha}\right).$$

Therefore, $U(k) = \delta\left(k - \frac{q}{\alpha}\right)$. This proposition gives the DT of a power function. ■

Proposition. If $u(x) = D^{\xi}_{x_0} v(x), m - 1 < \xi \leq m$ then

$$U(k) = \frac{\Gamma(\alpha k + \xi + 1)}{\Gamma(\alpha k + 1)} V\left(k + \frac{\xi}{\alpha}\right).$$

Proof: From Equation (4)

$$\begin{aligned} U(k) &= \frac{1}{\Gamma(\alpha k+1)} \left(D^{\alpha}_{x_0}\right)^k \left[D^{\xi}_{x_0} v(x)\right]_{x=x_0} \\ &= \frac{1}{\Gamma(\alpha k+1)} \left[\left(D_{x_0}\right)^{\alpha k+\xi} v(x)\right]_{x=x_0} \\ &= \frac{\Gamma(\alpha k+\xi+1)}{\Gamma(\alpha k+1)\Gamma(\alpha k+\xi+1)} \left[\left(D_{x_0}\right)^{\alpha k+\xi} v(x)\right]_{x=x_0} \\ &= \frac{\Gamma(\alpha k+\xi+1)}{\Gamma(\alpha k+1)} V\left(k + \frac{\xi}{\alpha}\right). \end{aligned}$$

Therefore, $U(k) = \frac{\Gamma(\alpha k+\xi+1)}{\Gamma(\alpha k+1)} V\left(k + \frac{\xi}{\alpha}\right)$. This proposition gives the DT of a general derivative of a function. ■

III. Implementation of method

Consider Equation (1) with the general initial conditions. Taking fractional differential transform (DT) on both sides and let

$$DT(f(x)) = F(k)$$
$$DT(y(x)) = Y(k).$$

Applying the proposition of linearity:

$$DT(Av(x) \pm Bw(x)) = AV(k) \pm BW(k)$$

and proposition on the Caputo's derivative of a function:

$$DT(D^{\alpha}_{x_0} y(x)) = \frac{\Gamma(\alpha k + \alpha + 1)}{\Gamma(\alpha k + 1)} Y(k+1),$$

the transformed equation can be written as,

$$\frac{\Gamma(\alpha k + \alpha + 1)}{\Gamma(\alpha k + 1)} Y(k+1) + AY(k) = F(k).$$

In the recursive form, it can be re-written as,

$$Y(k+1) = \frac{\Gamma(\alpha k + 1)\left(F(k) - AY(k)\right)}{\Gamma(\alpha k + \alpha + 1)}. \tag{8}$$

From this recursive expression and from the initial value $Y(0)$; the values or expressions of $Y(1), Y(2), Y(3), \ldots$ etc. are obtained. Then the approximate series solution or the m^{th} truncated solution of the FROE is obtained as:

$$y(x) = \Sigma^{m}_{k=0} Y(k)(x - x_0)^{k\alpha}. \tag{9}$$

For the available exact solutions, the absolute error can be found as

$$\text{Error} = |\text{Exact Solution} - m^{th}\text{Truncated Solution}|.$$

And the relative error is calculated as,

$$\text{Error} = \frac{|\text{Exact Solution} - m^{th}\text{Truncated Solution}|}{|\text{Exact Solution}|}.$$

The series solution is truncated at m for which the errors are minimum or are in the desired range. In the next section four FROEs are solved and the error analysis of the solutions is discussed in the Section (IV-E).

IV. Numerical examples

To show the effectiveness of the method, four examples are discussed. The solution obtained by the fractional differential transform method has been compared with the available solution by other methods.

A. Example 1

Consider a fractional relaxation equation from [15], with $\alpha = 0.5$ known as Mainardi's "ultraslow process" [11].

$$D^{0.5} y(x) + y(x) = 0$$

with initial condition $y(0) = 1$. This is the generalized FRO equation (1) with values $A = 1, f(x) = 0$.
Taking differential transform on both sides and applying the propositions from Section (II). The transformed equation can be written as

$$\frac{\Gamma(0.5k + 1.5)}{\Gamma(0.5k + 1)} Y(k+1) + Y(k) = 0.$$

It can be rewritten as,

$$Y(k+1) = -\frac{\Gamma(0.5k + 1)}{\Gamma(0.5k + 1.5)} Y(k). \tag{10}$$

Using Equation (6), the transformed initial condition is

$$Y(0) = 1.$$

The approximate solution for $m = 21$ by inverse fractional differential transform (see 5) is given as,

$$y(x) = Y(0) + Y(1)x^{0.5} + Y(2)x^1 + \ldots + Y(21)x^{10.5}.$$

Substituting the values $Y(0), Y(1), Y(2) \ldots$ etc. the following expression is obtained.

$$\begin{aligned} y(x) &= 1 - 1.12837916714x^{0.5} + 1x - 0.7522527780x^{1.5} \\ &+ 0.4999999999x^2 - 0.3009011111x^{2.5} + 0.1666666665x^3 \\ &- 0.0859717459x^{3.5} + 0.0416666665x^4 - 0.0191048323x^{4.5} \\ &+ 0.0083333332x^5 - 0.0034736058x^{5.5} + 0.0013888888x^6 - \\ &0.0005344008x^{6.5} + 0.0001984126x^7 - 0.0000712534x^{7.5} \\ &+ 0.0000248015x^8 - 0.0000083827x^{8.5} + 0.0000027357x^9 \\ &- 0.00000088238549x^{9.5} + 0.00000028254315x^{10} \\ &- 0.000000086163217x^{10.5}. \end{aligned}$$

This solution is compared with the available results from the wavelet collocation method and exact solution as recorded in [15], in Table (I). The last column depicts the absolute error in the exact solution and the solution by fractional DTM.

TABLE I
COMPARISON OF SOLUTIONS BY FRACTIONAL DTM WITH EXISTING SOLUTIONS FOR EXAMPLE IV-A

x	Wavelet collocation method [15]	Exact solution(A)	Fractional DTM(F)	Error($\|A-F\|$)
0.0	1.0000000	1.0000000	1.0000000	0
0.1	0.7235784	0.7235784	0.7235784	1.3611e-013
0.2	0.6437882	0.6437882	0.6437882	9.9298e-013
0.3	0.5920184	0.5920184	0.5920184	1.8488e-012
0.4	0.5536062	0.5536062	0.5536062	2.2161e-012
0.5	0.5231565	0.5231565	0.5231565	3.9851e-011
0.6	0.4980245	0.4980245	0.4980245	2.3950e-010
0.7	0.4767027	0.4767027	0.4767027	1.0558e-009
0.8	0.4582460	0.4582460	0.4582460	3.8471e-009
0.9	0.4420214	0.4420214	0.4420213	1.2184e-008
1.0	0.4275835	0.4275835	0.4275835	3.4561e-008

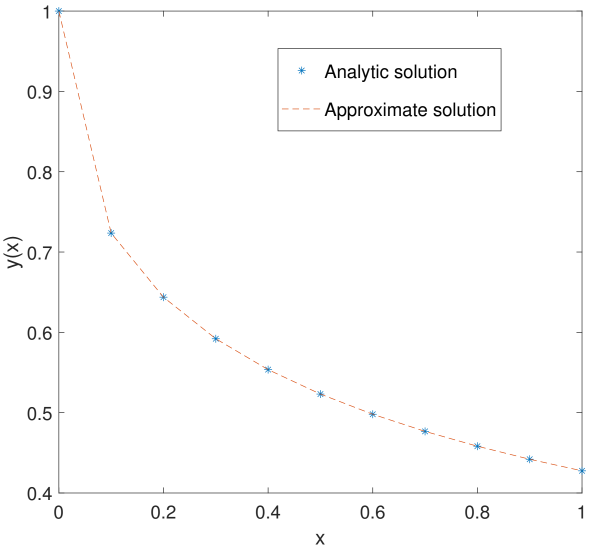

Fig. 1. Comparison of analytic and approximate solutions of Example IV-A

From the Table (I), following inferences can be made:

- The results by fractional DTM are very close to the exact solution, but there are small deviations for larger values of x.
- The errors are increasing from 1.36×10^{-13} at $x = 0.1$ to 3.45×10^{-8} at $x = 1.0$.
- The error is growing gradually but it is too small to impact the practical scenario.

B. Example 2

Consider a fractional oscillation equation from [15] with $\alpha = 1.5$ depicted as an "intermediate process" by Mainardi [11].

$$D^{3/2}y(x) + y(x) = 0$$

with initial conditions $y(0) = 1$ and $y'(0) = 0$ i.e. $A = 1, f(x) = 0$ in generalized FRO equation (1).
Using equation 8, the fractional differential transform of the equation can be written as,

$$\frac{\Gamma(1.5k+2.5)}{\Gamma(1.5k+1)}Y(k+1) + Y(k) = 0.$$

It can be simplified as,

$$Y(k+1) = -\frac{\Gamma(1.5k+1)}{\Gamma(1.5k+2.5)}Y(k). \tag{11}$$

The initial conditions after applying differential transform is

$$Y(0) = 1.$$

The approximate solution considering $m = 14$ in equation (9), is

$$y(x) = Y(0) + Y(1)x^{1.5} + Y(2)x^3 + \ldots + Y(14)x^{21}.$$

Substituting the value of $Y(0), Y(1), Y(2), \ldots$ etc. the approximate solution can be written as,

$$\begin{aligned} y(x) = {} & 1 - 0.752252778063675x^{1.5} + 0.166666666666666x^3 \\ & - 0.0191048324587600x^{4.5} + 0.00138888888888888x^6 \\ & - 7.12534543916457e-5x^{7.5} + 2.75573192239858e-6x^9 \\ & - 8.40376876209885e-8x^{10.5} + 2.087675698786e-9x^{12} \\ & - 4.33044445067896e-11x^{13.5} + 7.64716373181981e-13x^{15} \\ & - 1.16774718055184e-14x^{16.5} + 1.5619206968586e-16x^{18} \\ & - 1.84971338370751e-18x^{19.5} + 1.95729410633912e-20x^{21} \end{aligned} \tag{12}$$

The Table (II) explains and supports the efficiency and applicability of fractional DTM. The more the number of terms in the right hand side of equation (12) the lesser will be the deviation from the exact solution [15]. However, how many terms one should take out of the infinite series is still an open question- seeking a general answer. Currently a trial and error approach is used to select the number of terms.
The following inferences can be made from the Table (II):

- The error due to fractional DTM is either zero or very small (1.1102×10^{-16}).
- Both the methods are consistent throughout $0.0 < x < 1.0$.
- The fractional DTM gives near perfect accuracy, thus making it suitable for problems that need higher precision.

The visual depiction of the similar output of the methods through the FRE and FOE examples are given in Figure (1) and (2).

TABLE II
COMPARISON OF SOLUTION BY FRACTIONAL DTM WITH EXISTING SOLUTIONS FOR EXAMPLE IV-B

x	Wavelet collocation method [15]	Exact solution(A)	Fractional DTM(F)	Error($\|A-F\|$)
0.0	1.0000000	1.0000000	1.0000000	0
0.1	0.9763777	0.9763777	0.9763777	0
0.2	0.9340362	0.9340362	0.9340362	0
0.3	0.8808084	0.8808084	0.8808084	1.1102e-016
0.4	0.8200563	0.8200563	0.8200563	0
0.5	0.7540488	0.7540488	0.7540488	0
0.6	0.6845298	0.6845298	0.6845298	0
0.7	0.6129215	0.6129215	0.6129215	0
0.8	0.5404169	0.5404169	0.5404169	1.1102e-016
0.9	0.4680306	0.4680306	0.4680306	0
1.0	0.3966293	0.3966293	0.3966293	0

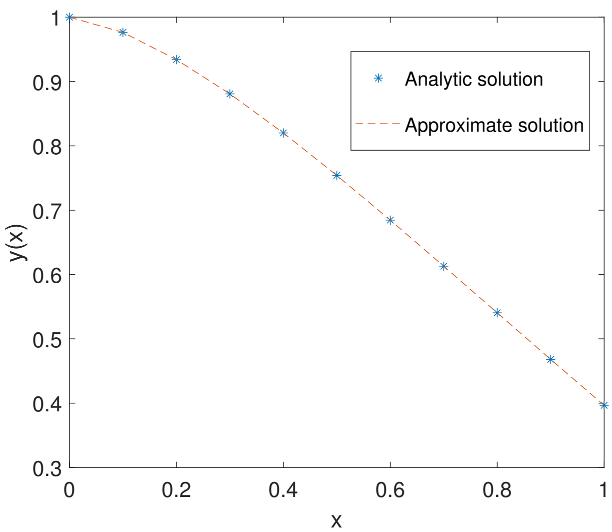

Fig. 2. Comparison of analytic and approximate solutions of Example IV-B

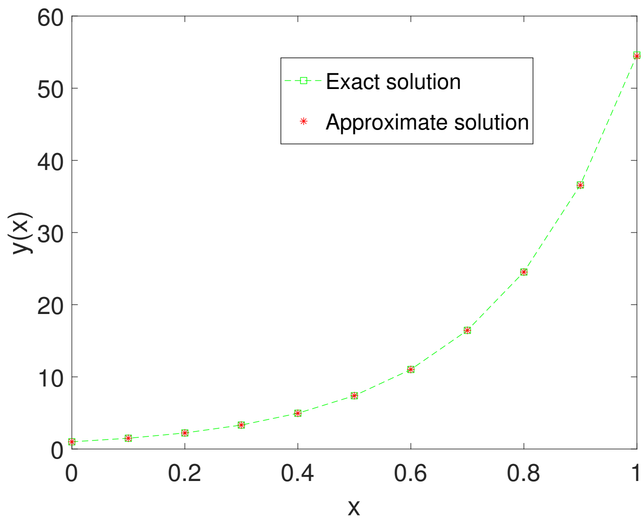

Fig. 3. Exact and approximate solutions in Example IV-C for m=10

C. Example 3

Consider a fractional relaxation oscillation from [16].

$$D^{\alpha}y(x) - 4y(x) = 0$$

with initial conditions $y(0) = 1$ i.e. $A = -4, f(x) = 0$ in equation (1).
Using the propositions in section II, the fractional differential transform of the equation can be written as:

$$Y(k+1) = -4\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}Y(k). \tag{13}$$

The initial conditions imply

$$Y(0) = 1.$$

The relation (13) gives

$$Y(1) = \frac{4}{\Gamma(\alpha+1)}, Y(2) = \frac{4^2}{\Gamma(2\alpha+1)}, Y(3) = \frac{4^3}{\Gamma(3\alpha+1)},$$

and so on.
Using the inverse differential transform one can obtain,

$$y(x) = \sum_{n=0}^{\infty} \frac{4^n x^{n\alpha}}{\Gamma(n\alpha+1)}.$$

The truncated solution can thus be written as,

$$y(x) = \sum_{n=0}^{m} \frac{4^n x^{n\alpha}}{\Gamma(n\alpha+1)}.$$

For $\alpha = 1$ the $y(x) = e^{4x}$. Since the exact solution is available in [4], [15] for $\alpha = 1$, so the solution due to DTM is checked for $\alpha = 1$ and with various trials of m as recorded in Table (III) and Table (IV).

The following inferences can be made from the Table (IV):

- Both RPSM and Fractional DTM give the results very close to the exact solution.
- The fractional DTM has minimal error at small values of x and it is increasing gradually from 0 to 1.05×10^{-7} as x increases from 0.0 to 1.0.
- The error is yet quite small ($< 10^{-6}$), that indicates good precision of the method.
- The results by RPSM are slightly less precise as compared to the fractional DTM, although the difference is negligible for practical purposes.
- Both methods are consistent with the exact solution for smaller values of x and the slight deviations at greater values of x do not affect the general accuracy.

D. Example 4

Consider another fractional relaxation oscillation equation from [16]

$$D^{\alpha}y(x) - y(x) = 1$$

with initial conditions $y(0) = 0$ i.e. $A = -1, f(x) = 1$ in generalized FRO equation (1).

TABLE III
COMPARISON OF SOLUTIONS OF FROE IN EXAMPLE IV-C FOR $m = 10$

x	Exact solution(A)	Residual Power Series Method([16])	Fractional DTM(F)	Error($\|A-F\|$)
0.0	1.000000000	1.0000000	1.0000000	0
0.1	1.491824698	1.4918246	1.491824698	1.09E-12
0.2	2.225540928	2.2255409	2.225540926	2.30E-09
0.3	3.320116923	3.3201167	3.320116716	2.07E-07
0.4	4.953032424	4.9530273	4.953027348	5.08E-06
0.5	7.389056099	7.3889947	7.388994709	6.14E-05
0.6	11.02317638	11.0227019	11.02270198	0.000474398
0.7	16.44464677	16.4419542	16.44195426	0.00269251
0.8	24.5325302	24.5203334	24.52033341	0.012196784
0.9	36.59823444	36.5517073	36.5517073	0.046527141
1.0	54.59815003	54.4431040	54.44310406	0.155045977

TABLE IV
COMPARISON OF SOLUTIONS OF FROE IN EXAMPLE IV-C FOR $m = 20$

x	Exact solution(A)	Residual Power Series Method([16])	Fractional DTM(F)	Error($\|A-F\|$)
0.0	1.000000000	1.0000000	1.000000000	0
0.1	1.491824698	1.4918246	1.491824698	2.22E-16
0.2	2.225540928	2.2255409	2.225540928	0
0.3	3.320116923	3.3201169	3.320116923	4.44E-16
0.4	4.953032424	4.9530324	4.953032424	0
0.5	7.389056099	7.3890560	7.389056099	4.62E-14
0.6	11.02317638	11.0231763	11.02317638	2.12E-12
0.7	16.44464677	16.4446467	16.44464677	5.51E-11
0.8	24.5325302	24.5325301	24.5325302	9.28E-10
0.9	36.59823444	36.5982344	36.59823443	1.12E-08
1.0	54.59815003	54.5981499	54.59814993	1.05E-07

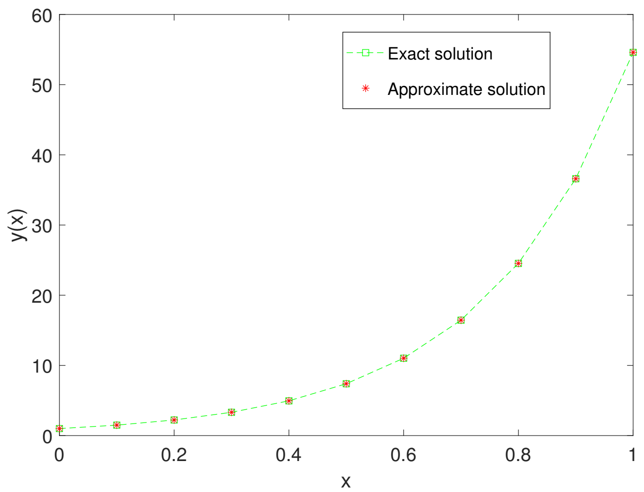

Fig. 4. Exact and approximate solutions in Example IV-C for m=15

Using propositions in section II, the fractional differential transform of the equation can be written as,

$$Y(k+1) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}(\delta(k)+Y(k)). \quad (14)$$

The initial condition after the proposed transformation becomes,

$$Y(0) = 0.$$

The relation (14) gives

$$Y(1) = \frac{1}{\Gamma(\alpha+1)}, Y(2) = \frac{1}{\Gamma(2\alpha+1)}, Y(3) = \frac{1}{\Gamma(3\alpha+1)},$$

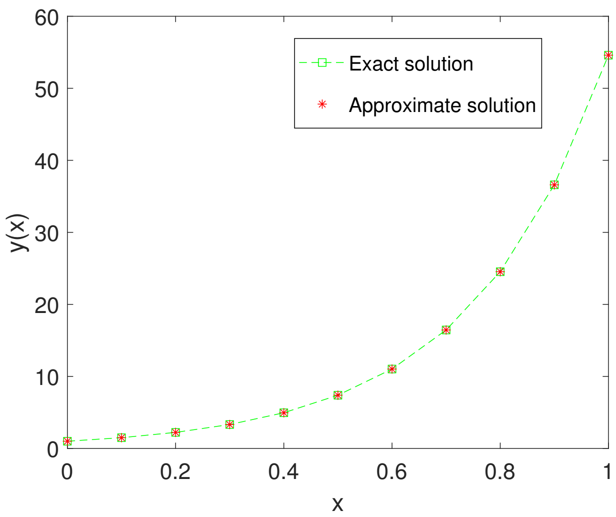

Fig. 5. Exact and approximate solutions in Example IV-C for m=20

and so on.
Using the inverse differential transform one can obtain

$$y(x) = \sum_{n=1}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

The approximate solution can be obtained by truncating the series as,

$$y(x) = \sum_{n=1}^{m} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

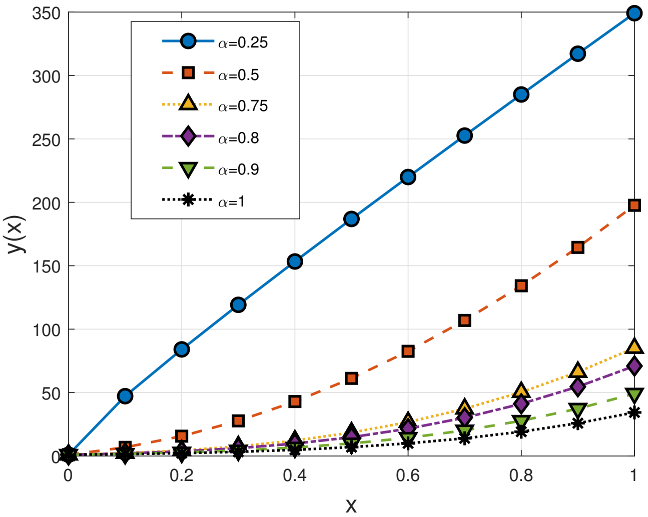

Fig. 6. Response of the system at different values of α in Example IV-C

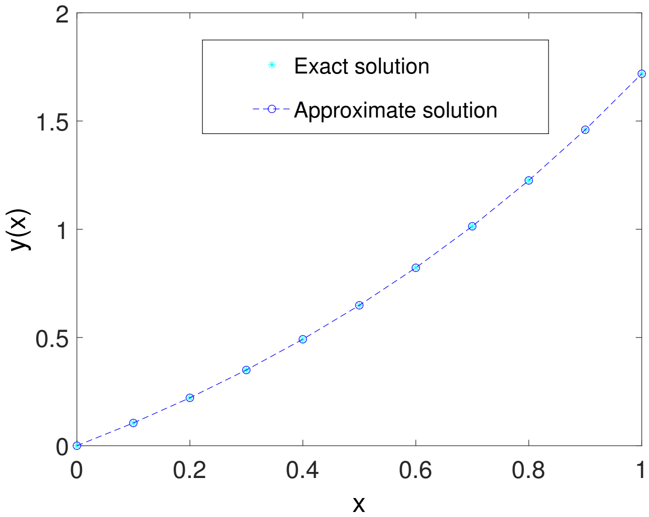

Fig. 8. Exact and approximate solutions of Example IV-D for m=15

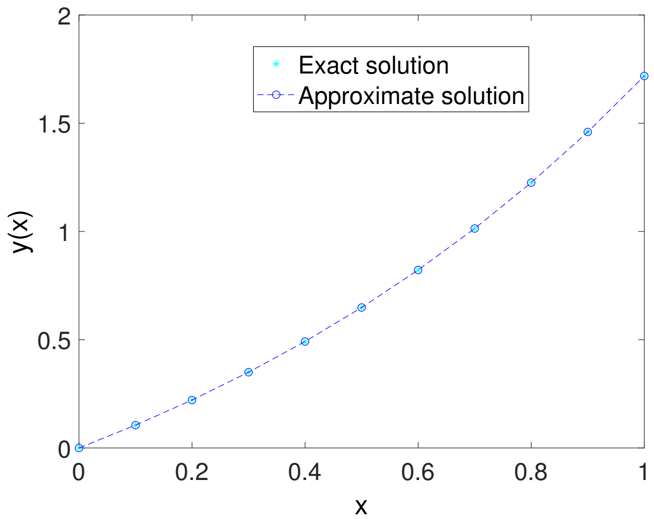

Fig. 7. Exact and approximate solutions of Example IV-D for m=10

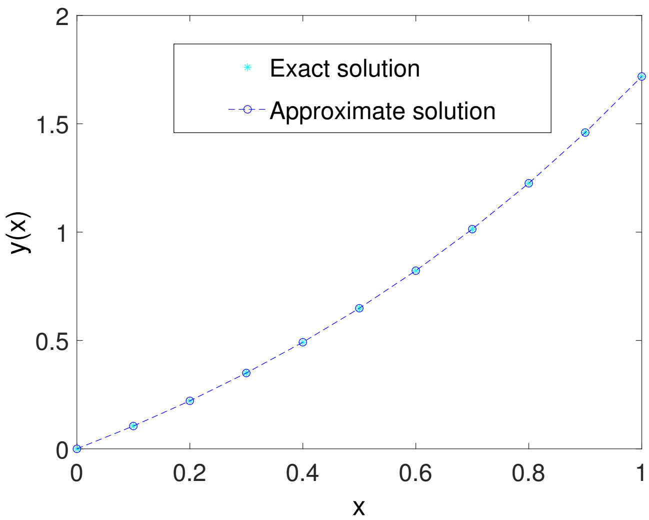

Fig. 9. Exact and approximate solutions of Example IV-D for m=20

For $\alpha = 1$ the $y(x) = e^x - 1$. For the available exact solution [4], [15], the solutions due to DTM are recorded in the Table V and VI. The visual depiction of the similar output of the methods through two FROE examples are given in Figure (3, 4, 5) and (7, 8, 9).

It can be observed from the Table (VI) that

- The minimal error values show that the RPSM and FDTM results are very close to the exact solution.
- The absolute errors are of the order 10^{-16} or 10^{-17} suggesting high accuracy of the fractional DTM.
- On all the points $0.0 \leq x \leq 1.0$, the results by fractional DTM consistently match the exact solution.
- The RPSM slighly deviates from the exact solution at some points.
- The errors are very small and don't grow as 'x' grows. That indicates the stable behavior of the method.

E. Error Analysis

From the Figure (11), it is observed that the as n increases, the absolute error decreases, validating that

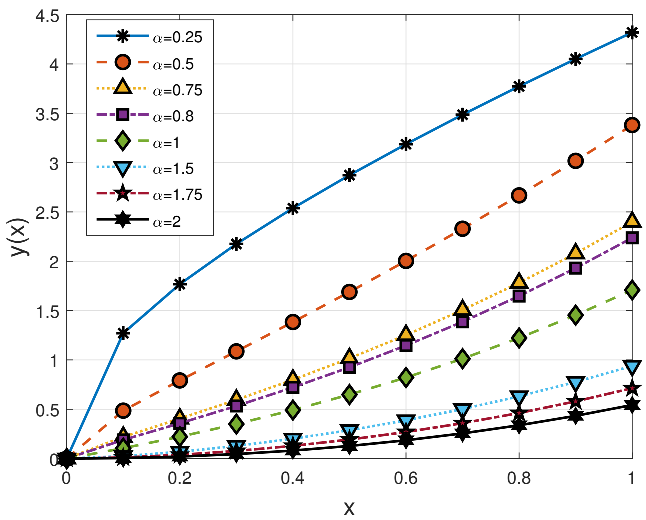

Fig. 10. Response of the system at different values of α for Example IV-D

TABLE V
COMPARISON OF SOLUTIONS OF FROE IN EXAMPLE IV-D FOR $m = 15$

x	Exact solution(A)	Residual Power Series Method([16])	Fractional DTM(F)	Error($\|A - F\|$)
0.0	0	0	0	0
0.1	0.105170918	0.1051709	0.105170918	8.33E-17
0.2	0.221402758	0.2214027	0.221402758	0
0.3	0.349858808	0.3498588	0.349858808	5.55E-17
0.4	0.491824698	0.4918246	0.491824698	0
0.5	0.648721271	0.6487212	0.648721271	1.11E-16
0.6	0.8221188	0.8221188	0.8221188	1.11E-16
0.7	1.013752707	1.0137527	1.013752707	8.88E-16
0.8	1.225540928	1.2255409	1.225540928	2.00E-15
0.9	1.459603111	1.4596031	1.459603111	9.10E-15
1.0	1.718281828	1.7182818	1.718281828	5.08E-14

TABLE VI
COMPARISON OF SOLUTIONS OF FROE IN EXAMPLE IV-D FOR $m = 20$

x	Exact solution(A)	Residual Power Series Method([16])	Fractional DTM(F)	Error($\|A - F\|$)
0.0	0	0	0	0
0.1	0.105170918	0.1051709	0.105170918	8.33E-17
0.2	0.221402758	0.2214027	0.221402758	0
0.3	0.349858808	0.3498588	0.349858808	5.55E-17
0.4	0.491824698	0.4918246	0.491824698	0
0.5	0.648721271	0.6487212	0.648721271	1.11E-16
0.6	0.8221188	0.8221188	0.8221188	1.11E-16
0.7	1.013752707	1.0137527	1.013752707	6.66E-16
0.8	1.225540928	1.2255409	1.225540928	6.66E-16
0.9	1.459603111	1.4596031	1.459603111	2.22E-16
1.0	1.718281828	1.7182818	1.718281828	0

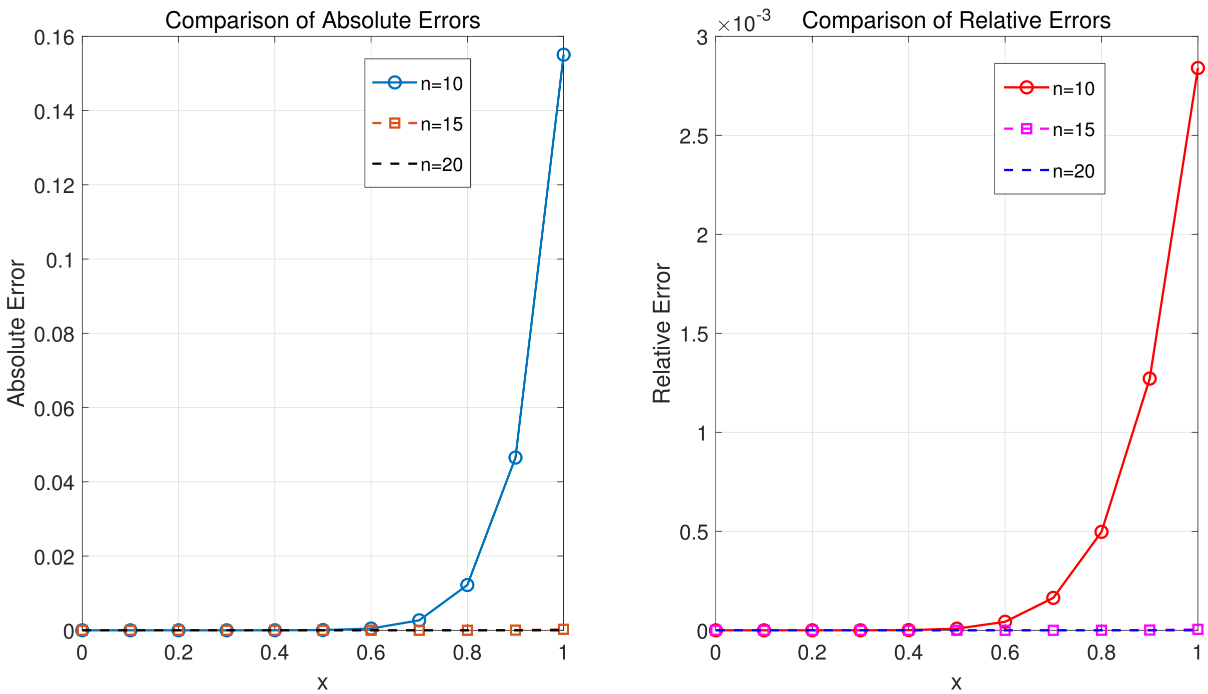

Fig. 11. Absolute and relative errors observed in Example IV-C

higher n leads to better approximation. For $n = 20$ the error is negligible for small values of x, inferring excellent convergence.
For $n = 10$, the relative error is higher for $x > 0.5$ and as n increases there is an evident improvement. For larger values of x, larger values of n are essential to retain the accuracy.
Similarly from the Figure (12), it is noticed that the absolute error decreases for all values of x as n increases. Moreover the decreasing relative error confirms the improvement in approximation with increasing n.
The absolute and relative error plots of Figure (11, 12), confirm the convergence of the series solution to the

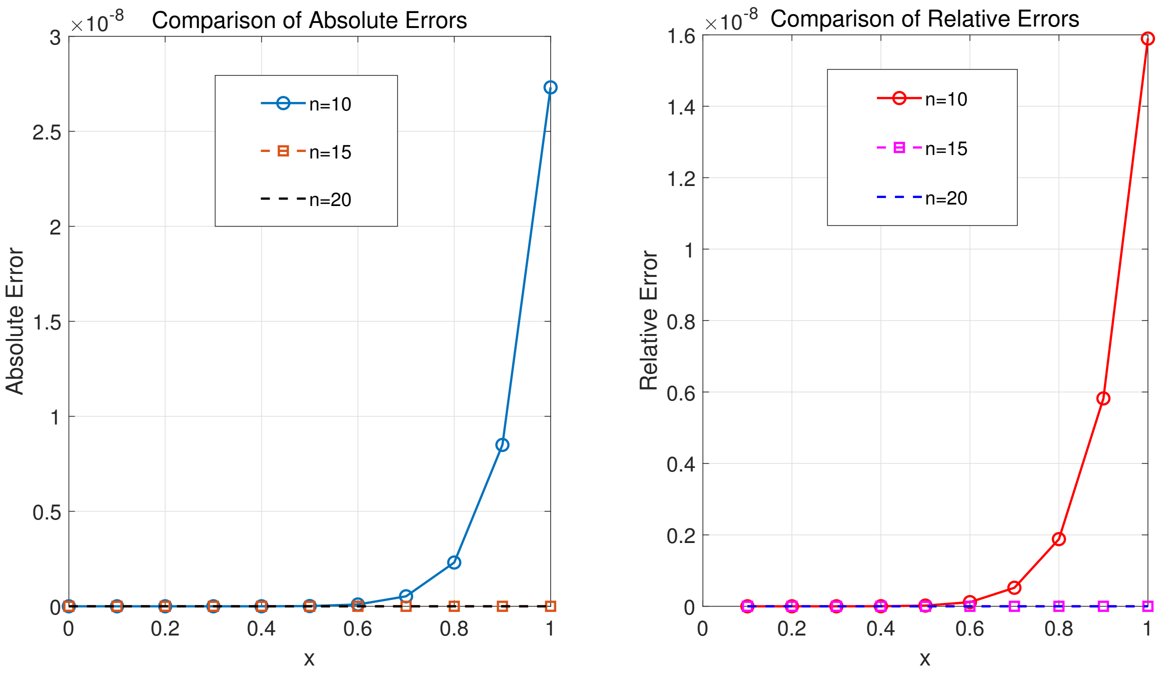

Fig. 12. Absolute and relative errors observed in Example IV-D

exact solution as n is increased. The error reduction can significantly be observed for the larger values of x.

V. Discussion and Conclusion

Any physical systems involving relaxation, diffusion, oscillations and wave propagation is governed by a benchmark equation called the fractional relaxation oscillation equation. In this work, the solution of this equation is discussed using the fractional differential transform method. This semi-analytic method can solve the fractional differential equations up to the desired level of accuracy. A comparison of the solutions of numerical examples cited in the literature, is done with the solutions by two popular methods called wavelet collocation method (IV-A, IV-B) and the residual power series method (IV-C, IV-D).

The Figures (6) and (10) depict the behavior of the response of the systems stated in Examples (IV-C)) and (IV-D) respectively, for different fractional orders and only four terms in the series solution. As α increases, the response $y(x)$ is slowing down. In Figure (10), the $\alpha = 1$ curve, almost separates the slow-growing curves ($\alpha > 1$) and the fast-growing curves ($\alpha < 1$). It shows that the lower value of α depicts a system with fast changes and higher value of α corresponds to the slower dynamics.

The results by fractional DTM are comparable to the exact solution at $\alpha = 1$. The best attributes of this method are its semi-analytic nature, ease to understand and quick to program. The errors can be reduced to as low as required, by increasing the number of terms in the solution series. Thus, from the inferences of Tables (I, II, IV, VI) it can be concluded that the fractional differential transform method offers slightly better precision compared to wavelet collocation method and the residual power series method.

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