Optimal Radio Label of the Cartesian Product of Pyramid Network and Even Path

Linlin Cui, Feng Li

Abstract—**The rapid advancement of wireless communication technology has driven substantial enhancements in data transmission speed, network coverage, and system reliability. Nevertheless, the scarcity of available spectrum resources presents a critical challenge in contemporary network design. To overcome this limitation, we propose to formalize the network frequency assignment problem as a graph-vertex labeling optimization task. By implementing distance-labeling constraints where each network station is assigned a unique label based on spatial separation we establish an optimized spectrum allocation framework that enhances spectrum management efficiency. This methodology effectively maximizes the utilization of scarce spectral resources. In this paper, we investigate the optimal radio labeling of the Cartesian product graph formed by an n-dimensional pyramid network and an m-order even path. We determine the optimal assignment strategy and establish a lower bound for the Cartesian product. Finally, we compare the performance of the proposed network model with the traditional grid network structure by numerical simulation.**

Index Terms—**frequency assignment, optimal radio label, pyramid network, even path, Cartesian product.**

I. Introduction

FREQUENCY resource planning is fundamentally a complex combinatorial optimization problem. It involves efficient allocation a limited number of frequency resources among multiple users or applications. This allocation process requires a careful balance between maximizing spectrum utilization and overall network performance. In practice, the number of available frequencies is often significantly less than the wireless communication of users or applications. This supply-demand imbalance often necessitates assigning the same or adjacent frequencies to different users. This assignment strategy inevitably leads to interference, which directly impacts communication quality. Therefore, effective frequency resource planning must consider multiple factors and identify optimal allocation strategies to minimize interference and improve spectral efficiency. To address this challenge, Hale [1] first proposed in 1980 to transform the network frequency assignment problem into the graph vertex labeling problem. The goal is to find the optimal tag value to reduce network interference and achieve interference-free communication. In this approach, a graph simulates a network topology where vertices represent base stations and edges represent potential frequency conflicts or adjacencies. Let $G=(V,E)$ be a graph, where V is the non-empty set of vertices and E is the set of edges. For any two vertices u and v in G, the shortest distance between them is denoted by $dist(u,v)$, often simplified as $d(u,v)$. The diameter of the graph, representing the maximum shortest distance between any two vertices in G, is denoted by

$$diam(G)=max\{d(u,v),u,v\in V(G)\}.$$

In addition to the studies of general graphs, people have shown a very strong interest in many types of special graphs, which have unique structures and properties. Let G be an ordered sequence of nonempty alternating points and edges called a path, that is, $W=\{v_0e_1v_1e_2...v_{i-1}e_iv_i\}$, where v_0 is the starting vertex and v_i is the ending vertex. If W contains m vertices and no duplicate vertices, we call this path, denoted P_m.

The explosion of users has led to an exponential increase in the size of the network. Simple graph models are difficult to simulate the increasingly large and complex network structures. The inability to effectively describe the relationships between the vertices and the data transmission path has prompted researchers to explore more effective modeling methods. In 1959, Sabidussi [2] first proposed the concept, which provided a new way to solve the modeling problem of large complex networks. As an important type of product, Cartesian product has been widely concerned for its simple and intuitive form, easy to understand and implement. Let graphs $A=(V_1,E_1)$ and $B=(V_2,E_2)$ be two simple undirected graphs, and the Cartesian product graph of graphs A and B is expressed as $G=A\square B$. Choosing any two vertices a_1b_1 and a_2b_2 in G, where $a_1a_2\in V(A)$, $b_1b_2\in V(B)$. If two vertices are adjacent if and only if $a_1=b_1, a_2b_2\in E(B)$ or $a_2=b_2, a_1b_1\in E(A)$, graphs A and B are called factor graphs of G. G retains the features of the factor graphs, but also expands the size and complexity of the graph. Let the Cartesian product of a-order P_a and b-order P_b be denoted as $G(a,b)=P_a\square P_b$. Based on the grid network, Dyer and Rosenfel [3] proposed the concept of a n-dimensional pyramid network. Let $PN(n)$ be a fixed structure based on a mesh network. It provides a communication model that converges faster than an n-dimensional grid. More studies on product graphs are detailed in the references [4]–[6].

In order to more intuitively understand the construction mode and characteristics of Cartesian product. The Cartesian product graph of the prismatic graph and the 4-order path is constructed, as shown in Figure.1. The red part is the structure of the cubic graph of the factor graph, and the blue part is the structure of the factor graph path.

Radio label is also called multi-level distance label, according to the distance between the vertices in the graph and diameter to label each vertex. It is to assign a unique label to each vertex, which is not a single value but composed of multiple parts. In the 1980s, Roberts [7] proposed the

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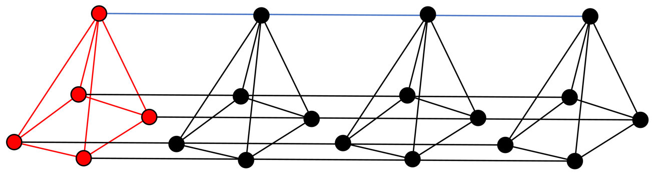

Fig. 1. The Cartesian product graph $G = PN(1)\Box P_4$

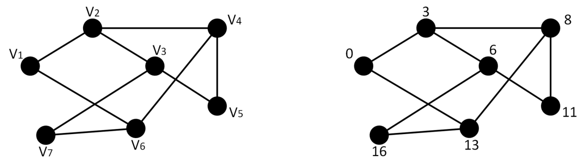

Fig. 2. Example of radio labeling for a 7-order simple graph

general properties, constraints and objectives of the problem. Since then, researchers have studied this issue in more depth. Griggs and Yeh [8] proposed an important improvement in the late 20th century, the 2-distance labeling method. Even between adjacent sites, there may be some weak interference. In order to effectively avoid this interference, it is necessary to ensure that adjacent stations are assigned to different channels. If two sites are adjacent, they are represented in the graph as a pair of vertices with a distance of 2. This representation clearly reflects the potential for interference between adjacent sites. Thus, the concept of radio labels [9] is extended. The radio label of a graph G is a mapping function as follows:

$$\omega: \ V(G) \to Z^{+} \cup 0,$$

such that

$$|\omega(b) - \omega(a)| \geq diam(G) + 1 - d(a,b), \ \forall a,b \in V(G).$$

Where $\omega(b)$ is the multi-level distance label of vertex b. $diam(G)$ denotes the diameter of the graph G. $d(a,b)$ denotes the shortest distance between vertices a and b. $|\omega(b) - \omega(a)|$ denotes the span between two vertices. The radio label of the graph G is the maximum span of ω, namely

$$span(\omega) = max\{|\omega(b) - \omega(a)|, \ \ \forall a,b \in V(G)\} \ ,$$

The optimal radio label is denoted $rn(G)$. It is the smallest value of all the spans in the graph G, namely

$$rn(G) = min\{span(\omega)\}.$$

In order to better understand the definition and mapping function of radio labels, we construct a 7-order simple graph as shown in Figure.2. Left figure is the original vertex label, and the right figure is the radio label.

The complex topology of product graph makes it an ideal model to simulate network environment, especially wireless channel assignment. Determining its radio number is a complex problem, which requires complex algorithms and mathematical tools. For specific types of product graphs, mathematical results have been obtained to determine the radio number, which lays a theoretical foundation for the design and understanding of efficient wireless networks. Ahmad et al. studied the radio labels of some ladder related graphs and determined the radio numbers of mongolian tent graphs, diamond graphs, fan graphs and double fans in [10]. Naseem et al. further studied the wedge sum of graphs in [11]. Jha and Klavžar et al. in [12] described an optimal $L(d,1)$-labeling for the direct product and the Cartesian product of a cycle. Zhang in [13] the optimum $L(3,2,1)$- label for the logarithmic tree. $L(3,2,1)$-labeling completely determines complete m-ary trees, spiders and banana trees. Shao and Yeh in [14] to study the floor plan of the $L(2,1)$-labeling. Chang et al. in [15] give a necessary and sufficient condition for the n-fold $L(2,1)$-label for the Cartesian product of paths and cycles. Bantva and Liu in [16] studied the radio labeling of the Cartesian product graph of two trees. Bantva et al. in [17] studied radio labeling of two-branch trees. Sethuraman and Nithya in [18] study the radio labeling of a special class of split graphs, called biconvex split graphs of diameter 3. In addition, the radio number of a biconvex split graph with three vertices of maximum degree with disjoint independent neighbors is determined. Mari and Jeyaraj in [19] aimed to determine the radio number $rn(c)$ for a particular family of structure graphs with diameter 3. ELrokh et al. in [20] studied the radio numbers of triangular snake graphs and double triangular snake graphs. Qi et al. in [21] studied the radio label of strong product graphs of complete graphs K_3 and n-order paths. PK and Kola in [22] studied the radio labeling of lexicographic product graphs of n-order paths and m-order cycles. Hong and Li in [23] determined the optimal radio labeling for the strong product graph. Cui and Li in [24] studied the radio labeling of the Cartesian product of the middle graph of the cycle $M_c(m)$ and the star S_n. Determined the optimal radio label of this 3D pattern, which is constructed from the Cartesian product of the middle of cycle and the star.

Although there have been a lot of studies on radio labeling of Cartesian product graphs. There are few studies on simulating network frequency assignment with Cartesian product graphs of pyramid network and other special graphs. In the next section, we study the radio labeling of the Cartesian product of pyramid network and even path and give a lower bound.

II. Main Results

In order to ensure the reliability of the conclusion of this paper, the necessary definition and lemma are given. They are the basis of the proof in this paper and help readers to understand the research methods and conclusions.

Definition 2.1 Let the n-dimensional pyramid network $PN(n)$ be a fixed structure based on a mesh network whose vertex set is $V(PN(n)) = \{(x,y,i) : 1 \leq x,y \leq 2^i, 0 \leq i \leq n\}$. For each fixed i $(0 \leq i \leq n)$. The subgraph derived from V_i is the mesh network $G(2^i,2^i)$. Since $|V_i| = 2^i \times 2^i = 4^i$, the number of vertices of $PN(n)$ is

$$V(PN(n)) = 4^0 + 4^1 + ... + 4^i + ... + 4^n = \frac{1}{3}(4^{n+1} - 1) \quad (1)$$

Because the subgraph derived from V_i is a mesh network $G(2^i,2^i)$, the number of edges in level i is $2^{i+1}(2^i - 1)$, and the number of edges between level i and level $i-1$ is $|V_i|$.

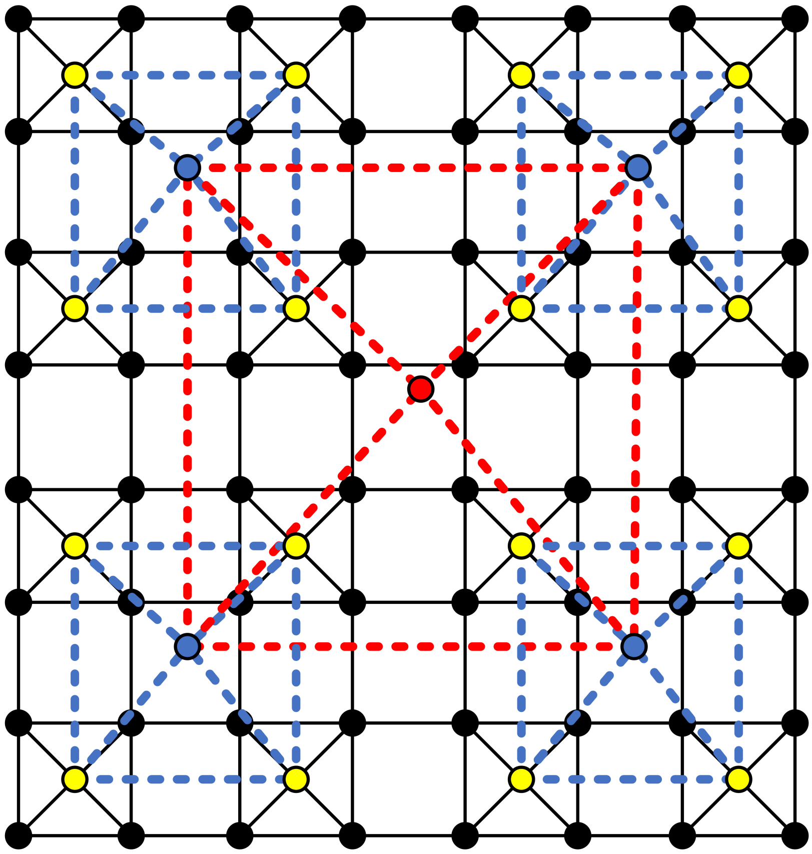

Fig. 3. The example of a 3-dimensional pyramid network, with red, blue, and yellow denoting the central vertices of the first-level, second-level and third-level pyramid networks.

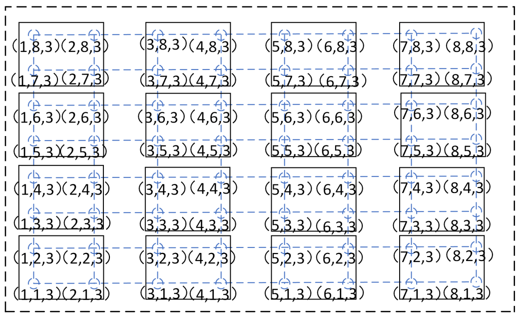

Fig. 4. The original vertex sorting in the third layer of a 3-dimensional pyramid network.

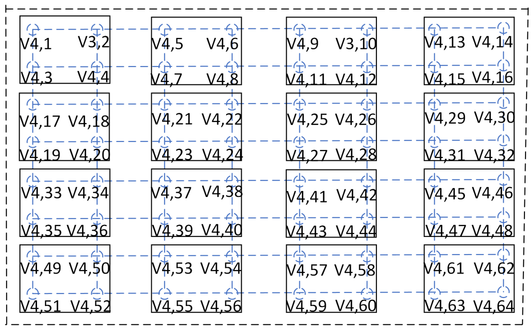

Fig. 5. The new vertex sorting in the third layer of a 3-dimensional pyramid network.

Therefore, the number of sides of $PN(n)$ is

$$\begin{aligned}\varepsilon(PN(n)) &= \sum_{i=0}^{n} 2^{i+1}\left(2^i-1\right)+\sum_{i=1}^{n} 4i \\ &= 4^{n+1}-2^{n+2} \\ &= 4(4^n-2^n).\end{aligned} \quad (2)$$

Definition 2.2 Let $PN(n)$ be an n-dimensional pyramid network whose vertex set $V(PN(n)) = \{(x,y,i) : 1 \le x,y \le 2^i, 0 \le i \le n\}$. The vertex subset $V_i = \{x,y,i) : 1 \le x,y \le 2^i\}$ is called the vertex set at level i. The subgraph derived from V_i is the mesh network $G(2^i,2^i)$. Each $(x,y,i) \in V_i$ is adjacent to four vertices in V_{i+1}, that is,

$$\begin{aligned}&(2x-1,\ 2y,\ i+1),(2x,2y,i+1).\\ &(2x-1,\ 2y-1,\ i+1),(2x,\ 2y-1,\ i+1).\end{aligned} \quad (3)$$

Let the vertex set of the n-dimensional pyramid network be $V(PN((n)) = (V_{i,j} : 1 \le i \le n+1, 1 \le j \le 4^{i-1}$. Let the root vertex be $V_{1,1}$ since V_i is adjacent to four vertices in V_{i+1}. Therefore, the vertices in V_i are taken as the central vertices of V_{i+1} and reordered every four for a group, that is

$$\begin{aligned}&\{V_{1,1}, V_{i,2}, V_{i,5}, V_{i,6}, \cdots, V_{i,4^i-3}, V_{i,4^i-2}\}\\ &\{V_{i,3}, V_{i,4}, V_{i,7}, V_{i,8}, \cdots, V_{i,4^i-1}, V_{i,4^i}\}\end{aligned} \quad (4)$$

To better understand the sorting process in Definition 2.2, the following Figure.3 shows the the 3-dimensional pyramid network $PN(3)$. Let's take the vertices of the third layer of a 3-dimensional pyramid network as an example. There are 64 vertices in the third layer, which we sort in groups of four. The Figure.3 shows a 3-dimensional pyramid network, the Figure.4 shows the original vertex ordering, and the Figure.5 shows the reordered vertices.

Lemma 2.3([25]) Let G_1 and G_2 be two simple undirected graphs, then the diameter of the Cartesian product graph G of G_1 and G_2 is

$$diam(G) = diam(G_1) + diam(G_2). \tag{5}$$

Lemma 2.4([3]) For an n-dimensional pyramid network. Its maximum distance is at the two diagonal vertices of the n-level. Then, the diameter of the n-dimensional pyramid network is $diam(PN(n)) = 2n$.

The following theorem follows directly from Definition 2.2 and Lemma 2.4

Theorem 2.5 Let $PN(n)$ be an n-dimensional pyramid network with $V_{1,1}$ as the root vertex. Choose any two adjacent vertices V_{i_1,j_1}, V_{i_2,j_2} in $PN(n)$, where $2 \le i \le n+1$.Then the distance between two vertices is

$$d(V_{i_1,j_1}, V_{i_2,j_2}) = \begin{cases} 2, \ if \ i_1 = i_2 \ and \ j_1 \ mod \ 4 \equiv 0,2; \\ 2n, \ if \ |i_1 - i_2| = 1, j_1 \ or \ j_2 = 1; \\ 1, \ others. \end{cases} \tag{6}$$

According to Theorem 2.6 and Lemma 2.4, we now give a lower bound on the number of radios of an n-dimensional pyramid network. As shown in the following corollary.

Corollary 2.6 Let $PN(n)$ be an n-dimensional pyramid network, then $rn(PN(n)) \ge 6n \times (4^{n-1}+1) + \frac{2n^2-3n+3}{3} \times (4^{n-1} - 1)$.

Proof. Let $V_{1,1}$ be the root vertex of $PN(n)$, and be the minimum radio label of $PN(n)$ satisfying $\omega(V_{1,1}) = 0$. According to the definition of the pyramid network, the root vertex is adjacent to the four vertices of the level 1. Obviously $d(V_{1,1}, V_{2,1}) = 1$. Suppose $\omega(V_{2,1}) \ge \omega(V_{1,1})$, such that

$$\begin{aligned} \omega(V_{2,1}) &\ge \omega(V_{1,1}) + diam(PN(n)) + 1 - d(V_{1,1}, V_{2,1}) \\ &\ge 2n. \end{aligned} \tag{7}$$

Suppose that the vertex $V_{2,2} \in V(PN(n))$ is adjacent to $V_{2,1}$. By Theorem 2.5, we can get that $d(V_{2,1}, V_{2,2}) = 1$. Assume that $\omega(V_{2,2}) \ge \omega(V_{2,1})$, makes

$$\begin{aligned} \omega(V_{2,2}) &\ge \omega(V_{2,1}) + diam(PN(n)) + 1 - d(V_{2,1}, V_{2,2}) \\ &\ge 4n. \end{aligned} \tag{8}$$

Similarly, choose vertex $V_{2,3} \in V(PN(n))$. Because vertices $V_{2,2}$ and $V_{2,3}$ are diagonal vertices. Therefore, $d(V_{2,2}, V_{2,3}) = 2$, such that

$$\begin{aligned} \omega(V_{2,3}) &\ge \omega(V_{2,2}) + diam(PN(n)) + 1 - d(V_{2,2}, V_{2,3}) \\ &\ge 6n - 1. \end{aligned} \tag{9}$$

Now the vertex $V_{2,4} \in V(PN(n))$ and $(V_{2,3}, V_{2,4}) \in E(PN(n))$. It follows from Theorem 2.6 that $d(V_{2,3}, V_{2,4}) = 1$ such that

$$\begin{aligned} \omega(V_{2,4}) &\ge \omega(V_{2,3}) + diam(PN(n)) + 1 - d(V_{2,3}, V_{2,4}) \\ &\ge 8n - 1. \end{aligned} \tag{10}$$

For vertex $V_{3,1}$ satisfies $(V_{2,4}, V_{3,1}) \in E(PN(n))$. Since $V_{2,4}$ and $V_{3,1}$ are not on the same level, $d(V_{2,4}, V_{3,1}) = 2i = 4$, such that

$$\begin{aligned} \omega(V_{3,1}) &\ge \omega(V_{2,4}) + diam(PN(n)) + 1 - d(V_{2,4}, V_{3,1}) \\ &\ge 10n - 5. \end{aligned} \tag{11}$$

After repeated iterations, we can get the maximum radio label of $PN(n)$. Let $\omega(V_{n+1,4^{n+1}})$ be the maximum radio label of $PN(n)$, that is, $\omega_{max} = \omega(V_{n+1,4^{n+1}})$ such that

$$\begin{aligned} \omega(V_{n+1,4^{n+1}}) &\ge \omega(V_{n+1,4^{n+1}-1}) + diam(PN(n)) + 1 \\ &\quad - d(V_{n+1,4^{n+1}-1}, V_{n+1,4^{n+1}}) \\ &\ge 6n \times (4^{n-1}+1) + \frac{2n^2-3n+3}{3} \times (4^{n-1} - 1). \end{aligned} \tag{12}$$

Definition 2.7 Let $G = PN(n) \Box P_m$ be a Cartesian product graph of an n-dimensional pyramid network and an m-order path, and $t(i)$ be a subpyramid network of G, $i \in [1, m]$. Let $V_{1,1}(i)$ be the root vertex of the subpyramid network. When m is even, we denote the subgraph formed by the pyramid network $t(j)$ and $t(j + \frac{m}{2})$ as $G'(j)$, where $j \in [1, \frac{m}{2}]$. The G cocontains $\frac{m}{2}$ subgraphs $G'(j)$.

Theorem 2.8 Let $G = PN(n) \Box \ P_m$ be the Cartesian product graph of an n-dimensional pyramid network and an m-order path. Then, the diameter of G is $diam(G) = 2n + m - 1$.

Proof. According to Lemma 2.4, we can directly obtain that the diameter of the pyramid net $PN(n)$ is $diam(PN(n)) = 2n$.

For an m-order path, it is a sequence with m vertices and $m-1$ edges. The maximum distance is the distance between the starting vertex and the ending vertex. Therefore, the diameter of m-order path is $diam(P_m) = m - 1$.

According to Lemma 2.3, we can get that the diameter of the Cartesian product graph G of an n-order pyramid network and an m-order path is

$$\begin{aligned} diam(G) &= diam(PN(n)) + diam(P_m) \\ &= 2n + m - 1. \end{aligned} \tag{13}$$

In order to better understand the diameter of the n-dimensional pyramid network, we constructed the 2-dimensional pyramid network as shown in Figure.6. The pair of diagonal vertices whose dimension is denoted by the red vertex is $(V_{3,6}, V_{3,14})$, $(V_{3,1}, V_{3,9})$. Choosing the diagonal vertex pair $(V_{3,1}, V_{3,9})$ as an example. The blue path marks the dimensional path toward $V_{3,1} \to V_{2,1} \to V_{1,1} \to V_{2,4} \to V_{3,9}$. Thus, we can get that the diameter of the 2-dimensional pyramid network is $diam(PN(2)) = 4$.

From Theorem 2.5 and Definition 2.7, we can directly derive the following corollary.

Corollary 2.9 Let $G'(j) \in PN(n) \Box P_m$, Choosing two adjacent vertices of $V_{x_1,y_1}(j)$, $V_{x_2,y_2}(j + \frac{m}{2})$, where $n \ge 2$, $j \in [1, \frac{m}{2}]$, $V_{x_1,y_1}(j) \in t(j + \frac{m}{2})$, $V_{x_2,y_2}(j + \frac{m}{2})$. Then

$$d(V_{x_1,y_1}(j), V_{x_2,y_2}(j + \frac{m}{2})) = \begin{cases} \frac{m}{2} + 2, \ if \ x_1 = x_2, y_1 \ mod \ 4 \equiv 0,2; \\ \frac{m}{2} + 2n, \ if \ |x_1 - x_2| = 1, y_1 \ or \ y_2 = 1; \\ \frac{m}{2} + 1, \ others. \end{cases} \tag{14}$$

Corollary 2.10 Let $G'(j) \subset PN(n) \Box P_m$ be a subgraph of G, where $j \in [1, \frac{m}{2}]$, m is even. Then the radio number of the subgraph $G'(j)$ is $rn(G'(j)) \ge q + 20n - \frac{4n^2}{3} + \frac{9m}{2} - 13 + 4^{n-1} \times (8n + 2m - 5) + 4^n \times (\frac{4n^2}{3} - 2n + 2)$.

Proof. For a root vertex $V_{1,1}(j)$ of the subgraph $G'(j)$ is its minimum radio label, let $\omega(V_{1,1}(j)) = q$. And $V_{1,1}(j + \frac{m}{2}) \in V(G'(j))$, such that the edge between the two vertices $(V_{1,1}(j), V_{1,1}(j + \frac{m}{2})) \in E(G'(j))$. According to the definition

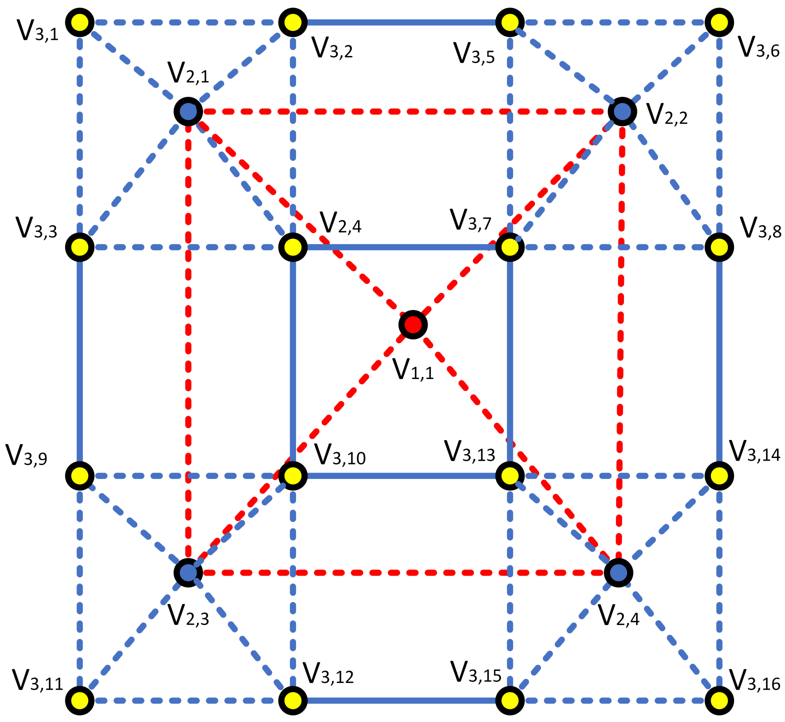

Fig. 6. Example of diagonal vertices in a 2-dimensional pyramid network

of radio label, such that

$$\omega(V_{1,1}(j+\frac{m}{2})) \geq \omega(V_{1,1}(j)) + diam(G) + 1 \\ - d(V_{1,1}(j), V_{1,1}(j+\frac{m}{2})), \tag{15}$$

Since $V_{1,1}(j)$ is the smallest radio label of the subgraph $G'(j)$. It is obvious that both vertices are the root vertices of the subpyramid network. According to the equation (14) , $d(V_{1,1}(j+\frac{m}{2})) = \frac{m}{2}$. Such that

$$\omega\left(V_{1,1}\left(j+\frac{m}{2}\right)\right) \geq q+2n+m-\frac{m}{2} \\ \geq q+2n+\frac{m}{2}. \tag{16}$$

Now, let $V_{2,2}(j) \in V(G'(j))$, it is clear that $d(V_{1,1}(j+\frac{m}{2}), V_{2,2}(j)) = \frac{m}{2}+1$. By the definition of the radio label, suppose $\omega(V_{2,2}(j)) \geq \omega(V_{1,1}(j+\frac{m}{2}))$, such that

$$\omega(V_{2,2}(j)) \geq \omega\left(V_{1,1}\left(j+\frac{m}{2}\right)\right) + diam(G) + 1 \\ - d\left(V_{1,1}\left(j+\frac{m}{2}\right), V_{2,2}(j)\right) \\ \geq q+2n+\frac{m}{2}+2n+m-\frac{m}{2}-1 \\ \geq q+4n+m-1. \tag{17}$$

Similarly, for vertices $V_{2,3}(j+\frac{m}{2}) \in V(G'(j))$. Since $V_{2,2}$ and $V_{2,3}$ are diagonal vertices in the subpyramid network, $d(V_{2,2}(j), V_{2,3}(j+\frac{m}{2})) = \frac{m}{2}+1$. Suppose by definition that $\omega(V_{2,3}(j+\frac{m}{2})) \geq \omega(V_{2,2}(j))$ such that

$$\omega\left(V_{2,3}\left(j+\frac{m}{2}\right)\right) \geq \omega(V_{2,2}(j)) + diam(G) + 1 \\ - d\left(V_{2,2}(j), V_{2,3}\left(j+\frac{m}{2}\right)\right) \\ \geq q+4n+m-1+2n+m-\frac{m}{2}-2 \\ \geq q+6n+\frac{3m}{2}-2. \tag{18}$$

For the vertex $V_{2,3}(j) \in V(G'(j))$, it follows from Corollary 2.9 that$d(V_{2,2}(j+\frac{m}{2}), V_{2,3}(j)) = \frac{m}{2}+2$, such that

$$\omega(V_{2,3}(j)) \geq \omega(V_{2,2}(j+\frac{m}{2})) + diam(G) + 1 \\ - d(V_{2,2}(j+\frac{m}{2}), V_{2,3}(j)) \\ \geq q+8n+2m-4. \tag{19}$$

By the definition of an n-dimensional pyramid net $PN(n)$, the maximum radio label of a subgraph $G'(j)$ is

$$\omega(V_{n+1,4^{n+1}}(j+\frac{m}{2})) \geq \omega(V_{n+1,4^{n+1}-1}(j)) + diam(G) + 1 \\ - d(V_{n+1,4^{n+1}-1}(j), V_{n+1,4^{n+1}}(j+\frac{m}{2})) \\ \geq q+18n+\frac{9m}{2}+4^{n-1}\times(8n+2m-5) \\ +\frac{2}{3}\times 4^{n-1}\times(2n^2-3n+3) \\ \geq q+20n-\frac{4n^2}{3}+\frac{9m}{2}-13+4^{n-1}\times \\ (8n+2m-5)+4^n\times(\frac{4n^2}{3}-2n+2). \tag{20}$$

According to the results of Definition 2.7 and Corollary 2.10 , we can conclude that the radio number of the Cartesian product graph $G = PN(n)\square P_m$, where m is even. The corollary is as follows:

Corollary 2.11 Let $G = PN(n)\square P_m$ be a Cartesian product graph of an n-dimensional pyramid network and an m-order even path. Then the radio number of G is $rn(G) \geq \frac{21mn+5m^2}{2} - \frac{2mn^2}{3} - 7m - n + 4^{n-1}\times(4mn+m^2-\frac{5m}{2}) + 4^n\times(\frac{2mn^2}{3}-mn+m)$.

Proof. Let $V_{1,1}(j)$ be the smallest radio label in $G'(j)$ such that $\omega(V_{1,1}(j)) = 0$. From Definition 2.7 , G contains a total of $\frac{m}{2}$ subgraphs of G$'(j)$. It follows from the result of

Corollary 2.10 that the maximum radio label of $G'(j)$ is

$$rn(G'(j)) \geq 20n - \frac{4n^2}{3} + \frac{9m}{2} - 13 + 4^{n-1} \times (8n + 2m - 5) + 4^n \times (\frac{4n^2}{3} - 2n + 2). \quad (21)$$

Supposing that the maximum radio label of the subgraph $G'(j)$ is the starting vertex, that is, $\omega_{min} = \omega(V_{n+1,4^{n+1}}(j + \frac{m}{2}))$. According to the topology of G, the distance between the root vertices of two adjacent subgraphs $G'(j)$ is $\frac{m}{2} + n$. The radio number of G is

$$\begin{aligned} rn(G) &\geq \frac{m}{2} \times rn(G'(j)) + (\frac{m}{2} - 1) \times (n + \frac{m}{2}) \\ &\geq \frac{m}{2}(20n - \frac{4n^2}{3} + \frac{9m}{2} - 13 + 4^{n-1} \times (8n + 2m - 5) \\ &\quad + 4^n \times (\frac{4n^2}{3} - 2n + 2)) + (\frac{m}{2} - 1) \times (n + \frac{m}{2}) \\ &\geq \frac{21mn + 5m^2}{2} - \frac{2mn^2}{3} - 7m - n + 4^{n-1} \times (4mn \\ &\quad + m^2 - \frac{5m}{2}) + 4^n \times (\frac{2mn^2}{3} - mn + m). \end{aligned} \quad (22)$$

According to Corollary 2.11, we give a special case. When $n = 1$, $PN(1)$ is a prismatic graph. The following corollary gives a lower bound on the radio labeling of the Cartesian product of the prismatic graph $PN(1)$ and the even path.

Corollary 2.12 Let $G = PN(1) \square P_m$ be a Cartesian product graph of prismatic graph and m-order even path. Then the radio number of G is $rn(G) \geq m^2 + \frac{27m}{2}$.

III. Application and Numerical Simulation

In practical applications, wireless communication networks typically consist of multiple radio stations and must handle variety of complex communication scenarios. These scenarios exhibit diverse requirements, including varying data transfer rates, reliability needs, and anti-interference capabilities. To meet these specific requirements and maximize the overall performance of wireless communication systems, researchers have focused on developing and refining adaptive channel assignment strategies. The central concept of this strategy is to dynamically adjust the channel assignment scheme based on real-time monitoring of the wireless channel state. This entails the system continuously monitoring the availability and quality of each channel, considering key parameters such as signal-to-noise ratio, interference levels, and multipath fading. Leveraging this real-time monitoring data, the adaptive channel assignment strategy can intelligently allocate available channels to the radio stations with the greatest need, thereby optimizing resource utilization and enhancing communication efficiency. This real-time monitoring and dynamic adjustment mechanism effectively improves communication quality, significantly reduces interference between radio stations, and ultimately enhances overall spectrum utilization. The key advantage of this strategy lies in its ability to flexibly adapt to the evolving wireless environment, ensuring the stability and reliability of the wireless communication system.

The pyramidal network structure is extremely stable and has a strong support capacity. Its structural stability stems from its unique architectural design. Figure.7 shows the number of vertices in an n-dimensional pyramid network; Figure.8 shows the radio number of an n-dimensional pyramid network. Figure.9 represents a lower bound on the radio labeling of the Cartesian product graph of a prism and an even path. Figure.10 shows the radio number of the Cartesian product graph of an n-dimensional pyramid network and an m-order even path. The radio number represents the communication efficiency of the network, and the smaller the value, the higher the efficiency.

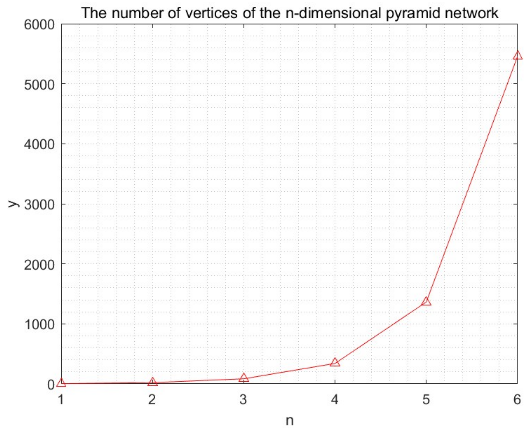

Fig. 7. The number of vertices in an n-dimensional pyramid network

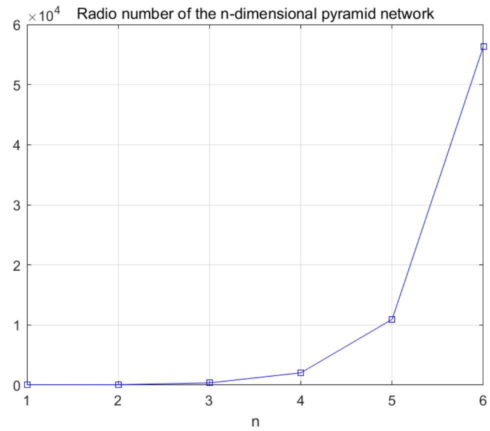

Fig. 8. The radio number of the n-dimensional pyramid network

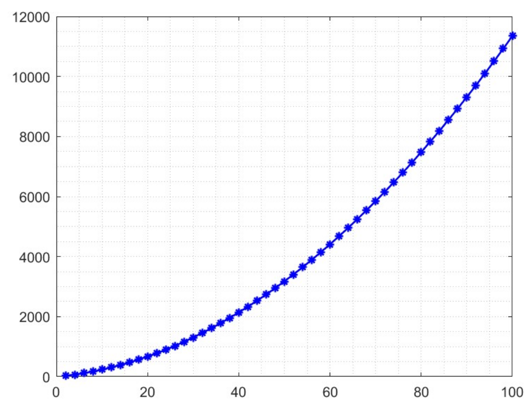

Fig. 9. The radio number of the Cartesian product of the prismatic graph and even path

Figure.11 compares with the existing network models, so

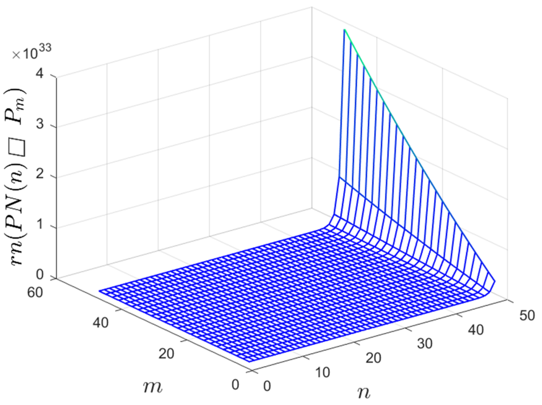

Fig. 10. Radio number of Cartesian product graphs of n-dimensional pyramid network and m-order even path

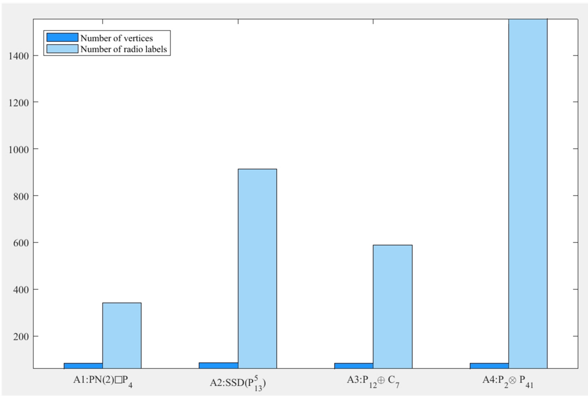

Fig. 11. Comparison of radio labels of the four models

as to more intuitively show its performance advantages. In order to make a more in-depth comparison, we use computer simulation technology to compare the radio numbers of several different topologies in references [21], [22], [26] with the same number of vertices. In Figure.9, the dark blue bars represent the number of vertices in the model, and the light blue bars represent the number of radios in the model. A1-A5 correspond to different models in references [21], [22], [26]. Where A1 represents the Cartesian product network model of n-dimensional pyramid network and m-order even path. A2 stands for supersub-division of path graph $SSD(P_{13})$ network model, where $\alpha = 5$; A3 represents the lexicographic product network model of 12-order path and 7-order cycle; A4 represents the strong product network model of the 2-order and 41-order paths.

Through careful analysis and comparison of these data, we can clearly see that the model proposed in this paper has a lower radio number when the number of vertices is close. This directly indicates that compared with the existing network model, the results of this paper achieve higher communication efficiency, which means that data transmission and information exchange can be more efficient under the same conditions. The reduction in the number of radios means fewer frequency collisions and more spectrum utilization, which improves overall network performance.

In wireless communication networks, channel resources are inherently limited, and inter-channel interference remains an unavoidable phenomenon. Such interference can severely degrade signal transmission quality, necessitating careful consideration of the channel interference factor and its impact on overall network efficiency during performance evaluations. To address this challenge systematically, we employ graph-theoretical topology to model communication relationships among radio stations. Through optimization of vertex labeling schemes, we aim to achieve optimal channel allocation strategies. This approach enables maximal reduction of channel usage while improving network resource efficiency and minimizing inter-channel interference.

This study focuses on a 3D network architecture featuring an n-dimensional pyramidal topology, selected for its capacity to effectively model complex connectivity patterns in wireless communications. The hierarchical structure provides an ideal framework for developing optimal channel allocation strategies. Our primary objective is to design an implementation strategy that minimizes channel utilization while enhancing overall network performance through optimized channel assignment.

IV. Conclusion

In this paper, the network topology of the Cartesian product of n-dimensional pyramid network and even-order path is studied. The optimal frequency assignment scheme and its lower bound are determined. The results show that this model is especially suitable for the management of high-traffic wireless communication networks. Effectively enhance connectivity and stability, and perform better than existing models. It is especially suitable for large-scale network deployment. Future research will be extended to Cartesian products of other graphs to further optimize the results.

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