The Steiner Wiener Index of Bicyclic Graphs

Xiao Wang, Xian Ya Geng

***Abstract*—Let $G = (V, E)$ be a simple graph, and the Steiner k-Wiener index be the sum of all Steiner distances over sets of k vertices of G. In this paper, the minimum value of the Steiner k-Wiener indexin bicyclic graphs is examined and the bicyclic graphs with the minimum Steiner k-Wiener index are identified.**

***Index Terms*—Steiner k-Wiener index, bicyclic graphs, Steiner distance, Wiener index.**

I. Introduction

As one of the most significant molecular topological indices, the Wiener index was introduced in 1947 by Harold Wiener [2] to investigate distance problems among atoms within molecules. Subsequently, it has garnered widespread attention in graph theory, leading to numerous results, as noted in relevant surveys [21], recent papers [7, 14, 15, 17, 19, 20], and the references cited therein.

The concept of the Steiner distance in a graph, which was introduced by Chartrand *et al.* [4] in 1989, represents a natural extension of the classical graph distance concept. For a graph G and a set of vertices $S \subseteq V(G)$, the Steiner distance $d_G(S)$ represents the sum of distances among the k vertices in S within the graph G. This can be written as $d_G(S) = min\{|E(T)| : T \text{ is a subtree of } G, S \subseteq V(T)\}$. For measurement methods, computation, and applications of Steiner distance in combinatorial optimization, please refer to [3, 5, 8, 10, 13, 26, 27].

In 2016, Li *et al.* [9] extended the Wiener index to the Steiner k-Wiener index using the definition of Steiner distance. For a positive integer k ($2 \le k \le n-1$), the Steiner k-Wiener index of graph G is defined as the sum of the Steiner distances of all k-subsets S in G, and is denoted as:

$$SW_k(G) = \sum_{S \subseteq V(G), |S|=k} d(S) \tag{1}$$

When $k = 2$, the Steiner 2-Wiener index is the Wiener index. Furthermore, Li *et al.* [9] also derived equations for calculating the Steiner k-Wiener indices for certain special graph classes and identified trees with the maximum and minimum Steiner k-Wiener indices in tree graphs, which were designated as paths and star graphs, respectively. In 2018, Lu *et al.* [6] established a tight lower bound for the Steiner k-Wiener index in an ensemble of fixed diameter trees and obtained the corresponding extremal graph. In 2021, Lai *et al.* [11] determined the upper and lower bounds on the Steiner k-Wiener index for unicyclic graphs when $k = n - 1$. In 2022, Fan *et al.* [12] used a graph transformation to derive the upper and lower bounds of the Steiner k-Wiener index for unicyclic graphs when $3 \le k \le n-2$, Simultaneously, the tree structure ranked second in the Steiner k-Wiener index ranking was precisely defined. For further research on the Steiner k-Wiener indices, the reader is referred to [16, 22–25, 28].

Inspired by the aforementioned works, this paper we primarily investigate the utilization of specific graph transformations in bicircular graphs with n vertices, these transformations alter the values of the Steiner k-Wiener index, leading to the determination of the graphs with the minimum Steiner k-Wiener index.

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II. Preliminaries

The definitions and notations used in this text can be found in [1]. Let G be a simple graph, whose vertices are denoted as $V(G)$ and the edge set as $E(G)$. For any vertices u and v in $V(G)$, the distance $d_G(u, v)$refers to the number of edges in the shortest path that links u and v within G. The Wiener index $W(G)$ of G is defined as:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) \tag{2}$$

A simple graph with n vertices and $n + 1$ edges is called a bicyclic graph. Depending on the relative positional relationship between the two fundamental cycles, bicyclic graphs can be categorized into three distinct types.

(I) $C_n(p, q)$ is composed of two disjoint cycles C_p and C_q share a common vertex;

(II) $C_n(p, l, q)$ is formed by two disjoint cycles C_p and C_q connected by a path of length at least 1;

(III) $P_n(k, l, m)$ is composed of two disjoint cycles C_{l+k} and C_{l+m} that share a path of length l.

The graphs $C_n(p, q)$, $C_n(p, l, q)$ and $P_n(k, l, m)$ (where $k + m - l - 1 \le n$) shown in Figure 1 correspond to the bicyclic graph types (I)∼(III) mentioned above, respectively.

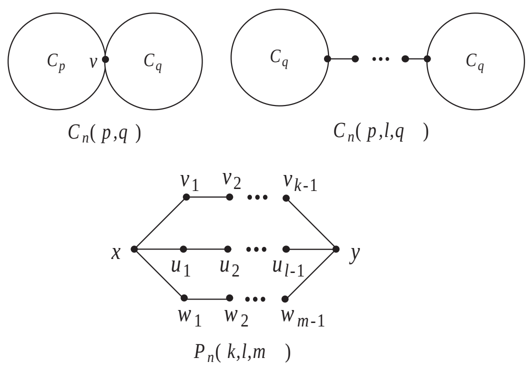

Fig. 1: The graphs $C_n(p, q)$, $C_n(p, l, q)$, and $P_n(k, l, m)$.

Lemma 2.1: [12] Let $k \in \mathbb{Z}$, $3 \leq k \leq n-2$, if $G \in UC(n)$ with $(n \geq 6)$. Then

$$SW_k(G) \geq SW_k(C_3(S_{n-2}))$$

with the equality holding if and only if $G \cong C_3(S_{n-2})$.

Lemma 2.2: [12] If G is a non-trivial connected graph, T_m is a non-trivial tree with m vertices, and $V(G) \cap V(T_m) = \{u\}$.Then

$$SW_k(GuT_m) \geq SW_k(GuS_m)$$

with the equality holding if and only if $GuT_m \cong GuS_m$.

Lemma 2.3: [18] Let $u \in V(G)$, H_1 and H_2 denote two connected subgraphs of G, and $V(H_1) \cap V(H_2) = \{u\}$. Then

$$\begin{aligned} SW_k(G) =& SW_k(H_1) + SW_k(H_2) \\ &+ \sum_{j=1}^{k-j} \binom{|V(H_2)|-1}{k-j} SW_k(H_1, u) \\ &+ \sum_{j=1}^{k-1} \binom{|V(H_1)|-1}{j} SW_k(H_2, u) \\ &+ \sum_{j=1}^{k-2} \binom{|V(H_2)|-1}{k-1-j} SW_{j+1}(H_1, u) \\ &+ \sum_{j=1}^{k-2} \binom{|V(H_1)|-1}{j} SW_{k-j}(H_2, u) \end{aligned} \quad (3)$$

where $SW_{k+1}(G, u)$ is the sum of Steiner distances of all $k+1$ element subsets S containing u in $V(G)$.

III. Lower bound of the Steiner k-Wiener index of bicyclic graphs

Theorem 3.1: Let H_1 and H_2 be two non-trivial connected graphs, and T_m be a non-trivial tree with m vertices, where $V(H_1) \cap V(T_m) = \{u\}$ and $V(H_2) \cap V(T_m) = \{v\}$. Deleting the edges of T_m, we obtain $H_1uS_mvH_2$ from $H_1uT_mvH_2$. For a positive integer k, when $2 \leq k \leq n-1$. Then

$$SW_k(H_1uT_mvH_2) \geq SW_k(H_1uS_mvH_2)$$

if and only if the equality $H_1uT_mvH_2 \cong H_1uS_mvH_2$ holds.

Proof. If $S \subseteq V(H_1uT_mvH_2) = V(H_1uS_mvH_2)$ where $|S| = k$, let $H_1uT_mvH_2 = A$, $H_1uS_mvH_2 = B$, we discuss the following seven cases.

Case 1. If $S \subseteq V(H_1)$, then $d_A(S) = d_B(S)$.

Case 2. If $S \subseteq V(H_2)$, then $d_A(S) = d_B(S)$.

Case 3. If $V(H_1 \backslash \{u\}) \cap S \neq \emptyset$, $V(H_2 \backslash \{v\}) \cap S \neq \emptyset$, regardless of the inclusion of $u(v)$ in S, $u(v)$ must beincluded in the Steiner tree connecting H_1 and H_2. Then, $d_A(S) = d_B(S)$.

Case 4. If $S \subseteq V(T_m)$, $k > m$, no such set S exists. The impact of the deletion on $SW_k(T_m)$ or $SW_k(S_m)$ and $SW_k(A)$ or $SW_k(B)$ is identical. Since $SW_k(T_m) \geq SW_k(S_m)$, with the equality holding if and only if $T_m \cong S_m$, then $SW_k(A) \geq SW_k(B)$.

Case 5. If $V(T_m \setminus \{u(v)\}) \cap S \neq \emptyset$, $V(H_1 \setminus \{u\}) \cap S \neq \emptyset$, then regardless of whether $u(v)$ is included in S, it must be included in the Steiner tree that connects H_1uT_m and H_2uS_m. We partition the Steiner tree into two subtrees T_{H_1} and T_{T_m}, with$V(T_{T_m}) \subseteq V(T_m)$, $V(T_{H_1}) \subseteq V(H_1)$, $V(T_{H_1}) \cap V(T_m) = \{u\}$, $|V(T_{H_1uT_m}(S))| = |V(T_{H_1})| + |V(T_{T_m})| - 1$, $d_{H_1uT_m}(S) = d_{H_1uT_m}(V(T_{T_m})) + d_{H_1uT_m}(V(T_{H_1}))$. The Steiner tree $T_{H_1uS_m}$ is similarly partitioned into two subtrees T_{H_1} and T_{S_m}. Since $d_{H_1uT_m}(V(T_{T_m})) \geq d_{H_1uS_m}(V(T_{Sm}))$, and $d_{H_1uT_m}(V(T_{H_1})) = d_{H_1uS_m}(V(T_{H_1}))$, then $d_{H_1uT_m}(S) \geq d_{H_1uS_m}(S)$.

Case 6. If $V(H_2 \setminus \{v\}) \cap S \neq \emptyset$, $V(T_m \setminus \{u(v)\}) \cap S \neq \emptyset$, then similarly, it can be proven that $d_{H_2uT}(S) \geq d_{H_2uS}(S)$.

Case 7. If $V(H_1 \setminus \{u\}) \cap S \neq \emptyset$, $V(H_2 \setminus \{v\}) \cap S \neq \emptyset$, and $V(T_m \backslash \{u(v)\}) \cap S \neq \emptyset$, regardless of the inclusion of $u(v)$ in S, $u(v)$ must be included in the Steiner tree connecting A and B. We partition the Steiner tree T_A into three subtrees T_{H_1}, T_{H_2}, and T_{T_m}, with$V(T_{H_2}) \subseteq V(H_2)$, $V(T_{H_1}) \subseteq V(H_1)$, $V(T_{T_m}) \subseteq V(T_m)$, $V(T_{T_m}) \cap V(T_{H_1}) \cap V(T_{H_2}) = \{u(v)\}$, $|V(T_A)(S)| = |V(T_{H_1})| + |V(T_{T_m})| + |V(T_{H_2})| - 2$, since $d_A(S) = d_A(V(T_{T_m})) + d_A(V(T_{H_1})) + d_A(V(T_{H_2}))$. Similarly, we partition the Steiner tree T_B into three subtrees T_{H_1}, T_{H_2} and T_{S_m}. Since $d_A(V(T_{T_m})) \geq d_A(V(T_{S_m}))$, $d_A(V(T_{H_1})) + d_A(V(T_{H_2})) = d_B(V(T_{H_1})) + d_B(T_{H_2})$, then $d_A(S) \geq d_B(S)$.

From the above discussion, the following conclusion can be drawn.

Given a connected graph G, the vertices u and v satisfy the following conditions: vertex u has p pendant vertices $(u_1, u_2, \ldots, u_p)$, and vertex v has q pendant vertices $(v_1, v_2, \ldots, v_q)$, with $u_i \neq v_j$ $(i = 1, \ldots, p, j = 1, \ldots, q)$. For the graph G, moving all pendant vertices from v to u results in a new graph denoted as G', while moving all pendant vertices from u to v results in a new graph denoted as G'' (see Figure 2). Then

$$\begin{aligned} G' =& G - \{vv_1, vv_2, ..., vv_q\} + \{uv_1 + uv_2, ..., uv_q\} \\ G'' =& G - \{uu_1, uu_2, ..., uu_p\} + \{vu_1 + vu_2, ..., vu_p\} \end{aligned}$$

Theorem 3.2: Let G is a connected graph and G' and G'' are graphs obtained from transformations of G, for $2 \leq k \leq n-1$. Then

$$SW_k(G) \geq SW_k(G') or SW_k(G) \geq SW_k(G'')$$

Proof. $X = \{u_1, ..., u_p\}$, $Y = \{v_1, ..., v_q\}$ and $V = V(G) \setminus (X \cup Y)$. For $S \subseteq V(G)$, $S \subseteq V(X) \setminus \{u\}$ or $S \subseteq V(Y) \setminus \{v\}$, we have $d_G(S) = d_{G'}(S)$ and $d_G(S) = d_{G''}(S)$. Then,

$$\begin{aligned} & SW_k(G) - SW_k(G') \\ =& \sum_{\substack{S \cap V(G) = \emptyset \\ S \cap V(X) \setminus \{u\} \neq \emptyset \\ S \cap V(Y) \setminus \{v\} \neq \emptyset}} (d_G(S) - d_{G'}(S)) \\ &+ \sum_{\substack{S \cap V(G) \neq \emptyset \\ S \cap V(X) \setminus \{u\} \neq \emptyset \\ S \cap V(Y) \setminus \{v\} = \emptyset}} (d_G(S) - d_{G'}(S)) \\ &+ \sum_{\substack{S \cap V(G) \neq \emptyset \\ S \cap V(X) \setminus \{u\} = \emptyset \\ S \cap V(Y) \setminus \{v\} \neq \emptyset}} (d_G(S) - d_{G'}(S)) \\ &+ \sum_{\substack{S \cap V(G) \neq \emptyset \\ S \cap V(X) \setminus \{u\} \neq \emptyset \\ S \cap V(Y) \setminus \{v\} \neq \emptyset}} (d_G(S) - d_{G'}(S)) \\ >& \sum_{\substack{S \cap V(G) \neq \emptyset \\ S \cap V(X) \setminus \{u\} = \emptyset \\ S \cap V(Y) \setminus \{v\} \neq \emptyset}} (d_G(S) - d_{G'}(S)) \\ =& \sum_{S' = S \cap V} (d_V(S' \cup \{u\}) - d_V(S' \cup \{v\})) \end{aligned}$$

$$
\begin{aligned}
&SW_k(G) - SW_k(G'') \\
=& \sum_{\substack{S\cap V(G)=\emptyset \\ S\cap V(X)\setminus\{u\}\neq\emptyset \\ S\cap V(Y)\setminus\{v\}\neq\emptyset}} (d_G(S) - d_{G''}(S)) \\
&+ \sum_{\substack{S\cap V(G)\neq\emptyset \\ S\cap V(X)\setminus\{u\}=\emptyset \\ S\cap V(Y)\setminus\{v\}\neq\emptyset}} (d_G(S) - d_{G''}(S)) \\
&+ \sum_{\substack{S\cap V(G)\neq\emptyset \\ S\cap V(X)\setminus\{u\}\neq\emptyset \\ S\cap V(Y)\setminus\{v\}\neq\emptyset}} (d_G(S) - d_{G''}(S)) \\
&+ \sum_{\substack{S\cap V(G)\neq\emptyset \\ S\cap V(X)\setminus\{u\}\neq\emptyset \\ S\cap V(Y)\setminus\{v\}=\emptyset}} (d_G(S) - d_{G''}(S)) \\
>& \sum_{\substack{S\cap V(G)\neq\emptyset \\ S\cap V(X)\setminus\{u\}=\emptyset \\ S\cap V(Y)\setminus\{v\}\neq\emptyset}} (d_G(S) - d_{G''}(S)) \\
=& \sum_{S'=S\cap V} (d_V(S'\cup\{u\}) - d_V(S'\cup\{v\}))
\end{aligned}
$$

If $SW_k(G) - SW_k(G') \leq 0$, then $\sum_{S\subseteq V}(d_{G'}(S\cup\{v\}) - d_{G'}(S\cup\{u\})) < 0$. In this case, $SW_k(G) - SW_k(G'') > 0$. That is, $SW_k(G) \geq SW_k(G')$ or $SW_k(G) \geq SW_k(G'')$.

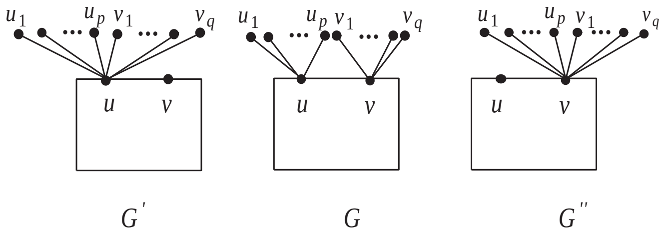

Fig. 2: The graphs G , G' and G''.

Let G_0 be a connected graph with $V(G_0) = q$, and $C_p = u_1, u_2, ..., u_p, u_1$ be a unicyclic graph of cycle length p. The connected components $S_1, S_2, ..., S_c$ ($0 \leq c \leq p$) of $G - E(G_0, C_p)$ represent the pendant edges on the cycle graph. When $|V(S_i)| = l_i$, $i = 1, 2, ..., c$, the vertex u_1 is a common vertex of the connected graph G_0 and the unicyclic graph C_p. Such connected graphs are denoted as $C_{G_0}^{C_p}(S_1, S_2, ..., S_c)$, and when $c = 0$, they are denoted as $G = C_{G_0}^{C_p}$. If the pendant edge S_i on the unicyclic graph C_p is moved to the common vertex u_1, a new graph denoted as $G' = C_{G_0}^{C_p}(S_{n-p-q+1})$ is obtained. Let G' be the graph obtained from $G = C_{G_0}^{C_p}(S_1, S_2, ..., S_c)$ by transformation, as shown in Figure 3.

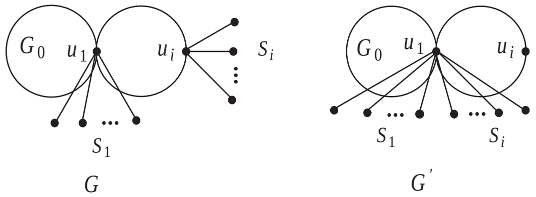

Fig. 3: The transformation of graph G to graph G'.

Repeating the operation in Theorem 3.2, we have the following corollary.

Corollary 3.3: Let $k \in \mathbb{Z}$, $2 \leq k \leq n-1$, if $G = C_{G_0}^{C_p}(S_1, S_2, ..., S_c)$ be a connected graph. Then

$$SW_k(G) \geq SW_k(C_{G_0}^{C_p}(S_{n-p-q+1}))$$

with the equality holding if and only if $G \cong C_{G_0}^{C_p}(S_{n-p-q+1})$.

Theorem 3.4: Let $k \in \mathbb{Z}$, $2 \leq k \leq n-1$, if $G = C_{G_0}^{C_p}(S_{n-p-q+1})$ is a connected graph. Then

$$SW_k(G) \geq SW_k(C_{G_0}^{C_3}(S_{n-2-q}))$$

with the equality holding if and only if $G \cong C_{G_0}^{C_3}(S_{n-2-q})$ (see Figure 4).

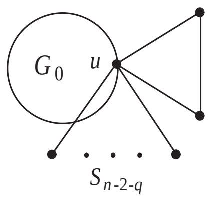

Fig. 4: The graphs $C_{G_0}^{C_3}(S_{n-2-q})$.

Proof. Let $G' = C_{G_0}^{C_3}(S_{n-2-q})$. According to the definition of $SW_k(G)$, the calculation is discussed based on the vertex set S being in different subsets of $V(G)$ and $V(G')$, as follows.

For any set $S \subseteq V(G)$ where $|S| = k$, we examine the following three scenarios.

Case 1. If $S \subseteq V(G_0)$, then $d_G(S) = d_{G_0}(S)$.

Case 2. If $S \subseteq V(C_p(S_{n-p-q+1}))$. Then

$$d_G(S) = d_{C_p(S_{n-p-q+1})}(S)$$

Case 3. If $V(G_0) \setminus \{u\} \cap S \neq \emptyset$ and $V(C_p(S_{n-p-q+1})) \setminus \{u\} \cap S \neq \emptyset$, let $S_1 = S \cap V(G_0)\setminus\{\text{u}\} = S \cap V(G_0 - u)$, $S_2 = S \cap V(C_p(S_{n-p-q-1})\setminus\{\text{u}\} = S \cap V(C_p - u)$.

Case 3.1. When $u \notin S$, let $|S_1| = j$ and $|S_2| = k - j$. Then,

$$
\begin{aligned}
&\sum_{\substack{u\notin S \\ S\cap V(G_0-u)\neq\emptyset \\ S\cap V(C_p(S_{n-p-q+1})-u)\neq\emptyset}} d_G(S) \\
=& \sum_{j=1}^{k-1} \sum_{\substack{|S_1|=j \\ |S_2|=k-j \\ S_1\subset V(G_0-u) \\ S_2\subset V(C_p(S_{n-p-q+1})-u)}} d_G(S_1\cup S_2\cup\{u\}) \\
=& \sum_{j=1}^{k-1} \sum_{\substack{|S_1|=j \\ |S_2|=k-j \\ S_1\subset V(G_0-u) \\ S_2\subset V(C_p(S_{n-p-q+1})-u)}} d_{G_0}(S_1\cup\{u\}) \\
&+ \sum_{j=1}^{k-1} \sum_{\substack{|S_1|=j \\ |S_2|=k-j \\ S_1\subset V(G_0-u) \\ S_2\subset V(C_p(S_{n-p-q+1})-u)}} d_{C_p(S_{n-p-q+1})}(S_2\cup\{u\}) \\
=& \sum_{j=1}^{k-1} \binom{|V(C_p(S_{n-p-q+1}))|-1}{k-j} SW_{j+1}(G_0, u) \\
&+ \sum_{j=1}^{k-1} \binom{|V(G_0)|-1}{j} SW_{k-j+1}(C_p(S_{n-p-q+1}), u)
\end{aligned}
$$

Case 3.2. When $u \in S$, let $|S_1| = j$ and $|S_2| = k-1-j$. Then

$$\sum_{\substack{u\in S\\ S\cap V(G_0)\setminus\{u\}\neq\emptyset\\ S\cap V(C_p(S_{n-p-q+1})-u)\neq\emptyset}} d_G(S)$$
$$=\sum_{j=1}^{k-2} \sum_{\substack{|S_1|=j\\ |S_2|=k-1-j\\ S_1\subset V(G_0-u)\\ S_2\subset V(C_p(S_{n-p-q+1})-u)}} d_G(S_1\cup S_2\cup\{u\})$$
$$=\sum_{j=1}^{k-2}\binom{|V(C_p(S_{n-p-q+1})|-1}{k-1-j} SW_{j+1}(G_0,u)$$
$$+\sum_{j=1}^{k-2}\binom{|V(G_0)|-1}{j} SW_{k-j}(C_p(S_{n-p-q+1}),u)$$

Through simple calculations, we can derive the following results.

$$SW_k(G)=SW_k(G_0)+SW_k(C_p(S_{n-p-q+1}))$$
$$+\sum_{j=1}^{k-1}\binom{|V(C_p(S_{n-p-q+1})|-1}{k-j} SW_{j+1}(G_0,u)$$
$$+\sum_{j=1}^{k-1}\binom{|V(G_0)|-1}{j} SW_{k-j+1}(C_p(S_{n-p-q+1}),u)$$
$$+\sum_{j=1}^{k-2}\binom{|V(C_p(S_{n-p-q+1})|-1}{k-1-j} SW_{j+1}(G_0,u)$$
$$+\sum_{j=1}^{k-2}\binom{|V(G_0)|-1}{j} SW_{k-j}(C_p(S_{n-p-q+1}),u)$$

Similarly, if the set $S \subseteq V(G')$ and $|S| = k$, we have

$$SW_k(G')=SW_k(G_0)+SW_k(C_3(S_{n-2-q}))$$
$$+\sum_{j=1}^{k-1}\binom{|V(C_3(S_{n-2-q})|-1}{k-j} SW_{j+1}(G_0,u)$$
$$+\sum_{j=1}^{k-1}\binom{|V(G_0)|-1}{j} SW_{k-j+1}(C_3(S_{n-2-q}),u)$$
$$+\sum_{j=1}^{k-2}\binom{|V(C_3(S_{n-2-q})|-1}{k-1-j} SW_{j+1}(G_0,u)$$
$$+\sum_{j=1}^{k-2}\binom{|V(G_0)|-1}{j} SW_{k-j}(C_3(S_{n-2-q}),u)$$

In summary, we conclude that

$$SW_k(G)-SW_k(G')$$
$$=SW_k(C_p(S_{n-p-q+1}))-SW_k(C_3(S_{n-2-q}))$$
$$+\sum_{j=1}^{k-2}\binom{|V(G_0)|-1}{j}(SW_{k-j+1}(C_p(S_{n-p-q+1}),u)$$
$$-SW_{k-j+1}(C_3(S_{n-2-q}),u))$$
$$+\sum_{j=1}^{k-2}\binom{|V(G_0)|-1}{j}(SW_{k-j}(C_p(S_{n-p-q+1}),u)$$
$$-SW_{k-j}(C_3(S_{n-2-q}),u))$$

Through Lemma 2.1 and 2.3, it is known that

$$SW_k(C_p(S_{n-p-q+1})) \geq SW_k(C_3(S_{n-2-q}))$$

For $SW_{k-j+1}(C_3(S_{n-2-q}),u)$, since the Steiner distance $d(S) = k-j$ for all $k-j+1$ element subsets S containing u in $V(C_3(S_{n-2-q}))$, and the Steiner distance $d(S) \geq k-j$ for all $k-j+1$ element subsets S containing u in $V(C_p(S_{n-p-q+1}))$, we conclude that $SW_{k-j+1}(C_p(S_{n-p-q+1}),u) \geq SW_{k-j+1}(C_3(S_{n-2-q}),u)$. Then, we have that $SW_k(G) \geq SW_k(C_{G_0}^{C_3}(S_{n-2-q}))$, with the equality holding if and only if $G \cong C_{G_0}^{C_3}(S_{n-2-q})$.

A. Type I bicircular graph

Let G_0 be a connected graph, where $C_m = v_1v_2...v_mv_1$ is a cycle of length m, and $C_p = u_1u_2...u_pu_1$ is a cycle of length p, with $u_1(v_1)$ being the common vertex of C_m and C_p. If $T_i(0 \leq i \leq m)$ and $T_i'(0 \leq i \leq p)$ represent the pendant edges of v_i and u_i respectively, then such a bicircular graph is denoted as $G = C_{T_1,T_2,...,T_m}^{T_1',T_2',...,T_p'}(u_1)$.

Theorem 3.5: Let $G = C_{T_1,T_2,...,T_m}^{T_1',T_2',...,T_p'}(u_1)$be a bicircular graph of order $n(n \geq 5)$. If $k \in \mathbb{Z}$, $3 \leq k \leq n-2$. Then

$$SW_k(G) \geq SW_k(C_{S_{n-5},0,0}^{S_{n-5},0,0}(u_1))$$

with the equality holding if and only if $G \cong C_{S_{n-5},0,0}^{S_{n-5},0,0}(u_1)$ (see Figure 5).

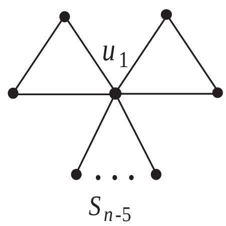

Fig. 5: Type I bicircular graph $C_{S_{n-5},0,0}^{S_{n-5},0,0}(u_1)$.

Proof. From Theorem 3.1, we obtain $SW_k(G) \geq SW_k(C_{S_1,S_2,...,S_m}^{S_1',S_2',...,S_p'}(u_1))$. Moving the pendant edges on cycles C_m and C_p to the common vertex u_1, and applying Theorem 3.2 and Corollary 3.3, we find $SW_k(G) \geq SW_k(C_{S_{n-p-m+1},0,...,0}^{S_{n-p-m+1},0,...,0}(u_1))$. As the lengths of cycles C_m and C_p gradually decrease, Theorem 3.4 indicates that when the cycle lengths $p = m = 3$, the SW_k index of graph $C_{S_{n-5,0,0}}^{S_{n-5,0,0}}(u_1)$ is minimized.

For any set $S \subseteq V(C_{S_{n-5,0,0}}^{S_{n-5,0,0}}(u_1))$ with $|S| = k \geq 3$. If $d_{C_{S_{n-5,0,0}}^{S_{n-5,0,0}}(u_1)}(S) = k-1$, the number of such subsets is $\binom{n-1}{k-1}$, with each subset contributing $\binom{n-1}{k-1}(k-1)$ to SW_k. If $d_{C_{S_{n-5,0,0}}^{S_{n-5,0,0}}(u_1)}(S) = k$, the number of such sets is $\binom{n-1}{k}$. Then

$$SW_k(C_{S_{n-5},0,0}^{S_{n-5},0,0}(u_1)) = \binom{n-1}{k}k + \binom{n-1}{k-1}(k-1)$$
$$= \binom{n-1}{k-1}(n-1)$$

B. Type II Bicircular Graph

Let $P_{l+1} = w_1 w_2 ... w_{l+1}$ be the shortest path connecting C_m and C_p, with the common vertex of P_{l+1} and C_p being v_1, and the common vertex of P_{l+1} and C_m being u_1. Each branch T_{u_i}, T_{v_i}, and T_{w_i} of $G - E(C_m) - E(C_p) - E(P_{l+1})$ is a tree. Such a bicircular graph is denoted as $G = C^{T_{v_1}, T_{v_2}, ..., T_{v_i}}_{T_{u_1}, T_{u_2}, ..., T_{u_i}}(T_{w_1}, T_{w_2}, ..., T_{w_l})$.

Theorem 3.6: Let $G = C^{T_{v_1}, T_{v_2}, ..., T_{v_i}}_{T_{u_1}, T_{u_2}, ..., T_{u_i}}(T_{w_1}, T_{w_2}, ..., T_{w_l})$ be a bicircular graph of order $n (n \geq 6)$. If $k \in \mathbb{Z}$, $3 \leq k \leq n - 2$. Then

$$SW_k(G) \geq SW_k(C_3^{S_{n-6},0,0}(u_1, v_1))$$

with the equality holding if and only if $G \cong C_3^{S_{n-6},0,0}(u_1, v_1)$.

Proof. Moving the pendant edges of cycles C_m and C_p to the connecting points u_1 and v_1, and the pendant edges T_{w_i} on P_{l+1} to v_1, from Theorems 3.1 and 3.2, we obtain $SW_k(G) > SW_k(C_m^{S_{n-m-p}}(u_1, v_1))$. Further reducing the lengths of cycles C_m and C_p, and applying Theorem 3.4, we find that when the cycle lengths $p = m = 3$, the SW_k index of graph $C_3^{S_{n-6}}(u_1, v_1)$ is minimized.

For any set $S \subseteq C_3^{S_{n-6},0,0}(u_1, v_1)$ with $|S| = k \geq 3$, if the selected k vertices include u_1 and v_1, then $d_{C_3^{S_{n-6},0,0}(u_1,v_1)}(S) = k - 1$, there are $\binom{n-2}{k-2}$ such sets contributing to SW_k by $\binom{n-2}{k-2}(k-1)$. If the selected k vertices do not include u_1 and v_1, then $d_{C_3^{S_{n-6},0,0}(u_1,v_1)}(S) \geq k$ and there are $\binom{n-2}{k}$ such sets. If the selected k vertices include v_1 but do not include u_1, then $d_{C_3^{S_{n-6},0,0}(u_1,v_1)}(S) \geq k-1$, and there are $\binom{n-2}{k-1}$ such sets. If the selected k vertices include u_1 but do not include v_1, then $d_{C_3^{S_{n-6},0,0}(u_1,v_1)}(S) = k$, and there are $\binom{n-2}{k-1}$ such sets.

It follows that

$$\begin{aligned}
&SW_k(C_3^{S_{n-6},0,0}(u_1, v_1)) \\
\geq &\binom{n-2}{k-2}(k-1) + \binom{n-2}{k}k \\
&+ \binom{n-2}{k-1}(k-1) + \binom{n-2}{k-1}k \\
= &\left[\binom{n-2}{k-2} + \binom{n-2}{k-1}\right](k-1) \\
&+ \left[\binom{n-2}{k} + \binom{n-2}{k-1}\right]k \\
= &\binom{n-1}{k-1}(n-1)
\end{aligned}$$

C. Type III bicircular graph

For two distinct vertices u and v in G, let p_r, p_s, and p_t be three internally disjoint paths connecting u and v. Here, T_j denotes the pendant edges on p_r $(1 \leq r)$, T'_j denotes the pendant edges on p_s $(1 \leq s)$, and T''_j denotes the pendant edges on p_t $(1 \leq t)$. Such a bicircular graph is denoted as $G = C^{T'_1, T'_2, ..., T'_s}_{T_1, T_2, ..., T_r}(T''_1, T''_2, ..., T''_t)$.

Theorem 3.7: Let $G = C^{T'_1, T'_2, ..., T'_s}_{T_1, T_2, ..., T_r}(T''_1, T''_2, ..., T''_t)$ be a bicircular graph of order $n (n \geq 4)$, If $k \in \mathbb{Z}$, $3 \leq k \leq n-2$. Then

$$SW_k(G) \geq SW_k(C_4^u(S_{n-4}))$$

with the equality holding if and only if $G \cong C_4^u(S_{n-4})$ (see Figure 6).

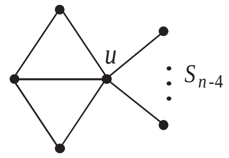

Fig. 6: Type III bicyclic graphs $C_4^u(S_{n-4})$.

Proof. Moving all the pendant edges in graph G to vertex u, and applying Theorem 3.1-3.4 and Corollary 3.3, it is evident the SW_k index of the graph $C_4^u(S_{n-4})$ is minimized.

If $S \subseteq V(C_4^u(S_{n-4})$ and $|S| = k \geq 3$, we examine the following two scenarios.

Case 1. When $k = 3$, if $d_{C_4^u(S_{n-4})} = 2$, there are $\binom{n-1}{2}$ such sets. If the set does not include vertex u, there is only 1 such subset. If $d_{C_4^u(S_{n-4})} = 3$, there are $\binom{n-1}{2} - 1$ such sets.

$$\begin{aligned}
SW_3(C_4^u(S_{n-4})) &= 3\left(\binom{n-1}{3} - 1\right) + 2\left(1 + \binom{n-1}{2}\right) \\
&= \binom{n-1}{2}(n-1) - 1
\end{aligned}$$

Case 2. When $k > 3$, if $d_{C_4^u}(S_{n-4}) = k-1$, there are $\binom{n-1}{k-1}$ such sets contributing to SW_k by $\binom{n-1}{k-1}(k-1)$. If $d_{C_4^u}(S_{n-4}) = k$, the count of such S would be $\binom{n-1}{k}$.

$$\begin{aligned}
SW_k(C_4^u(S_{n-4})) &= \binom{n-1}{k}k + \binom{n-1}{k-1}(k-1) \\
&= \binom{n-1}{k-1}(n-1)
\end{aligned}$$

It follows that

$$SW_k(C_4^u(S_{n-4})) = \begin{cases} \frac{(n-1)^2(n-2)}{2} - 1 & \text{if } k = 3 \\ \binom{n-1}{k-1}(n-1) & \text{if } k > 3 \end{cases}$$

IV. Conclusion

This study primarily investigates the Steiner k-Wiener index of bicircular graphs. Based on the structural characteristics of bicircular graphs, they are classified into three types, and through graph transformations, the graphs with the minimum SW_k index for each type of bicircular graph are obtained. it is found that the graph with the smallest SW_k index among Type I bicircular graphs is denoted as $C^{S_{n-5},0,0}_{S_{n-5},0,0}(u_1)$; among Type II bicircular graphs, it is denoted as $C_3^{S_{n-6},0,0}(u_1, v_1)$; and among Type III bicircular graphs, it is denoted as $C_4^u(S_{n-4})$. Comparative analysis reveals that the graph with the smallest SW_k index among all bicircular graphs is $C_4^u(S_{n-4})$.

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