Explanation of Quasi-Regular Semigroups Characterized in Terms of Neutrosophic Bipolar Valued Fuzzy Ideals in Semigroups

Pannawit Khamrot, Aiyared Iampan, Anothai Phukhaengsi, Thiti Gaketem

***Abstract*—Quasi-regular is a characteristic of semigroups that results in the relationship conditions of various types of ideals, and methods for finding this special type of semigroup using various types of fuzzy sets have been studied. In this research, we propose a discovery method of quasi-regular semigroups using neutrosophic bipolar valued fuzzy ideals.**

***Index Terms*—Neutrosophic sets, Bipolar fuzzy ideals, Neutrosophic bipolar-valued fuzzy ideals, Quasi-regular.**

I. Introduction

THE FUNDAMENTAL concept of a fuzzy set was first introduced by L. A. Zadeh in 1965 [1] with it is solving of the problem of uncertain information. Later in 1986, K. T. Atanassov [2] gave idea can displaying both the degree and non-degree of memberships, which helps with ambiguity with the name as an intuitionistic fuzzy set. In 1999, F. Smarandache [3] extended the concept of fuzzy sets by representing truth-membership, indeterminacy-membership, and falsity-membership of an object to a set independently with name as Neutrosophic sets. These concepts have been applied to various algebraic structures, including fields, rings, vector spaces, groups, and semigroups [4],[5],[6],[7],[8],[9],[10],[11]. In particular, fuzzy sets in semigroups were introduced and studied by Kuroki [12] in 1979, who investigated fuzzy (left, right) ideals and fuzzy bi-ideals in semigroups.

The decision-making difficulties dealt with it is solved by bipolar fuzzy sets by W. Zhang [13] in 1994, which allows for the representation of degrees of membership, degrees of non-membership, and degrees of partial membership simultaneously, and is a helpful extension of classical, fuzzy, and neutrosophic semigroups. Moreover, it has potential applications in handling uncertainties and partial knowledge in various fields. In 2021 T. Gaketem and P. Khamrot [14] proved the concepts of bipolar fuzzy weakly interior ideals of semigroups. We studied the relationship between bipolar fuzzy weakly interior ideals, bipolar fuzzy left (right) ideals, and bipolar fuzzy weakly interior ideals. Furthermore, in 2022, T. Gaketem et al. [15] introduced the concept of bipolar fuzzy implicative UP-filters in UP-algebras. Based on these notions, bipolar fuzzy set theory and its applications were developed [16],[17],[18], [19], [20].

Recently, in 2024 N. Deetae and P. Khamrot [21] studied the concepts neutrosophic on bipolar-valued fuzzy sets with positive and negative in ordered of truth-membership, indeterminacy-membership, and falsity-membership. We studied the basic properties of bipolar-valued fuzzy subsemigroups in semigroups.

This paper, we repeat the definitions of subsemigroups and genres of fuzzy sets in division 2. The next, division we presented methods for creating the neutrosophic bipolar-valued fuzzy bi-ideals and neutrosophic bipolar-valued fuzzy generalized bi-ideals, concinde and neutrosophic bipolar-valued fuzzy ideals and neutrosophic bipolar-valued fuzzy interior ideals, concinde by quasi-regular semigroups. In the last part, we prove the characterization of weakly regular semigroups in terms of neutrosophic bipolar-valued fuzzy ideals.

II. Preliminaries

In this clause, we reviews the types of subsemigroups, and types of fuzzy sets.

Let $\emptyset \neq \mathfrak{Q} \subseteq \mathfrak{R}$ of a semigroup (SG). Then we called

1) A *subsemigroup* $\mathfrak{Q}$ (SSG) of $\mathfrak{R}$ if $\mathfrak{Q}\mathfrak{Q}^2 \subseteq \mathfrak{Q}$.
2) A *left ideal* (LId) [right ideal (RId)] $\mathfrak{Q}$ of $\mathfrak{R}$ if $\mathfrak{Q}\mathfrak{R} \subseteq \mathfrak{Q}[\mathfrak{R}\mathfrak{Q} \subseteq \mathfrak{Q}]$.
3) An *ideal* (Id) $\mathfrak{Q}$ of an SG $\mathfrak{R}$ if it is an LID and a RID of $\mathfrak{R}$.
4) A *generalized bi-ideal* (GBId) $\mathfrak{Q}$ of an SG $\mathfrak{R}$ if $\mathfrak{Q}\mathfrak{R}\mathfrak{Q} \subseteq \mathfrak{Q}$.
5) A *bi-ideal* (BId) $\mathfrak{Q}$ of an SG $\mathfrak{R}$ if $\mathfrak{Q}$ is an SSG and $\mathfrak{Q}$ is a GIBd of $\mathfrak{R}$.
6) An *interior ideal* (INId) $\mathfrak{Q}$ of an SG $\mathfrak{R}$ if $\mathfrak{Q}$ is an SSG and $\mathfrak{R}\mathfrak{Q}\mathfrak{R} \subseteq \mathfrak{Q}$.
7) A *quasi-ideal* (QId) $\mathfrak{Q}$ of an SG $\mathfrak{R}$ if $\mathfrak{R}\mathfrak{Q} \cap \mathfrak{Q}\mathfrak{R} \subseteq \mathfrak{Q}$.

A *fuzzy set* (FS) Φ of a non-empty set $\mathfrak{Z}$ is a function from $\mathfrak{Z}$ into the closed interval $[0, 1]$, i.e., $\Phi : \mathfrak{Z} \to [0, 1]$.

Definition 2.1. *[2] An **intuitionsic fuzzy set** (IF set) $\mathfrak{W} \neq \emptyset$ in set $\mathfrak{Z}$ is an object having the form*

$$\mathfrak{W} := \{(\mathfrak{w}, \zeta_{\mathfrak{W}}(\mathfrak{w}), \phi_{\mathfrak{W}}(\mathfrak{w})) \mid \mathfrak{w} \in \mathfrak{W}\},$$

where $\zeta_{\mathfrak{W}} : \mathfrak{Z} \to [0, 1]$ is the grade of membership and $\phi_{\mathfrak{W}} : \mathfrak{Z} \to [0, 1]$ is the grade of non-membership such that $0 \leq \zeta_{\mathfrak{W}}(\mathfrak{w}) + \phi_{\mathfrak{W}}(\mathfrak{w}) \leq 1$ for all $\mathfrak{z} \in \mathfrak{Z}$.

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P. Khamrot is a lecturer at the Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University Technology Lanna Phitsanulok, Phitsanulok, Thailand. (e-mail: pk_g@rmutl.ac.th).

A. Iampan is a lecturer at the Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand. (e-mail: aiyared.ia@up.ac.th).

A. Phukhaengsi is an undergraduate student at the Department of Mathematics, School of Education, University of Phayao, Phayao, Thailand. (e-mail: 64202576@up.ac.th).

T. Gaketem is a lecturer at the Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand. (corresponding author to provide email: thiti.ga@up.ac.th).

Definition 2.2. *[13] A* bipolar fuzzy set *(shortly, BF set)* Φ *on* $\mathfrak{Z}$ *is an object having the form*

$$\Phi := \{(\mathfrak{w}, \Phi^+(\mathfrak{w}), \Phi^-(\mathfrak{w})) \mid \mathfrak{w} \in \mathfrak{Z}\},$$

where $\Phi^+ : \mathfrak{Z} \to [0,1]$ *and* $\Phi^- : \mathfrak{Z} \to [-1,0]$.

Definition 2.3. *[3] Let* $\mathfrak{Z} \neq \emptyset$*. A neutrosophic sets (NS)* $\mathfrak{W}$ *in* $\mathfrak{Z}$ *is the structure*

$$\mathfrak{W} = \{\langle \mathfrak{z}, \mathfrak{T}_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{I}_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{F}_{\mathfrak{W}}(\mathfrak{z})\rangle : \mathfrak{z} \in \mathfrak{Z}\},$$

where $\mathfrak{T}_{\mathfrak{W}} : \mathfrak{Z} \to [0,1]$ *is a truth membership function,* $\mathfrak{I}_{\mathfrak{W}} : \mathfrak{Z} \to [0,1]$ *is an indeterminate membership function, and* $\mathfrak{F}_{\mathfrak{W}} : \mathfrak{Z} \to [0,1]$ *is a false membership function.*

Next, we shall introduce the fundamental operations that can be carried out on neutrosophic bipolar-valued fuzzy sets of the SG. For brevity, we will employ the abbreviated term NSBF instead of repeatedly using the full term "neutrosophic bipolar-valued fuzzy set."

Definition 2.4. *[3] Let* $\mathfrak{Z} \neq \emptyset$*. A neutrosophic bipolar-valued fuzzy set (NSBF)* $\mathfrak{W}$ *in* $\mathfrak{Z}$ *is an object of the form* $\mathfrak{W} = \{\langle \mathfrak{z}, \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{v}), \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z})\rangle : \mathfrak{z} \in \mathfrak{V}\}$*, where* $\mathfrak{T}^+_{\mathfrak{W}}, \mathfrak{I}^+_{\mathfrak{W}}, \mathfrak{F}^+_{\mathfrak{W}} : \mathfrak{Z} \to [0,1]$ *and* $\mathfrak{T}^-_{\mathfrak{W}}, \mathfrak{I}^-_{\mathfrak{W}}, \mathfrak{F}^-_{\mathfrak{W}} : \mathfrak{Z} \to [-1,0]$.

For simplicity, we use the symbol $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ for the NSBF $\mathfrak{W} = \{\langle \mathfrak{z}, \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}), \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}, \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}))\rangle : \mathfrak{z} \in \mathfrak{Z}\}$.

Definition 2.5. *[21] An NSBF set* $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ *in an SG* $\mathfrak{R}$ *is called an NSBF subsemigroup (NSBF SSG) if it satisfies:*

$$(\forall \mathfrak{z}, \mathfrak{Q} \in \mathfrak{S}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{Q}), \\ \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{Q}), \\ \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{Q}), \\ \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{Q}), \\ \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{Q}), \\ \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{Q}) \end{pmatrix}.$$

Example 2.6. *Consider an SG* $\mathfrak{S} = \{\check{\delta}_1, \check{\delta}_2, \check{\delta}_3\}$ *with the following Cayley table:*

►	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$
$\check{\delta}_1$	$\check{\delta}_3$	$\check{\delta}_3$	$\check{\delta}_3$
$\check{\delta}_2$	$\check{\delta}_3$	$\check{\delta}_3$	$\check{\mathfrak{z}}_1$
$\check{\delta}_3$	$\check{\delta}_3$	$\check{\delta}_2$	$\check{\delta}_3$

Define an NSBF $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ in $\mathfrak{R}$ as follows:

$\mathfrak{S}$	$\mathfrak{T}^+_{\mathfrak{W}}$	$\mathfrak{I}^+_{\mathfrak{W}}$	$\mathfrak{F}^+_{\mathfrak{W}}$	$\mathfrak{T}^-_{\mathfrak{W}}$	$\mathfrak{I}^-_{\mathfrak{W}}$	$\mathfrak{F}^-_{\mathfrak{W}}$
$\check{\delta}_1$	0.3	0.5	0.6	−0.4	−0.6	−0.8
$\check{\delta}_2$	0.2	0.3	0.8	−0.6	−0.7	−0.6
$\check{\delta}_3$	0.7	0.8	0.5	−0.2	−0.3	−0.9

Then $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ is an NSBF SSG of $\mathfrak{R}$.

Definition 2.7. *[21] An NSBF set* $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ *in an SG* $\mathfrak{R}$ *is called an NSBF right ideal (NSBF RID) if it satisfies:*

$$(\forall \mathfrak{z}, \mathfrak{k} \in \mathfrak{R}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z})), \\ \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z})), \\ \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z})), \\ \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z})), \\ \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z})), \\ \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}))\} \end{pmatrix}.$$

Definition 2.8. *[21] An NSBF set* $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ *in an SG* $\mathfrak{R}$ *is called an NSBF left ideal (NSBF LID) if it satisfies:*

$$(\forall \mathfrak{z}, \mathfrak{k} \in \mathfrak{R}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{k})), \\ \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{k})), \\ \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{Q})), \\ \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{k})), \\ \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{k})), \\ \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z})(\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{k}))\} \end{pmatrix}.$$

Definition 2.9. *An NSBF set* $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ *in an SG* $\mathfrak{R}$ *is called an NSBF ideal (NSBF Id) if it satisfies Definition 2.8 and 2.7.*

Example 2.10. *Consider an SG* $S = \{\mathfrak{W}_1, \check{\delta}_2, \check{\delta}_3\}$ *with the following Cayley table:*

►	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$
$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_2$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_3$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_3$

Define an NSBF $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ in $\mathfrak{R}$ as follows:

$\mathfrak{S}$	$\mathfrak{T}^+_{\mathfrak{W}}$	$\mathfrak{I}^+_{\mathfrak{W}}$	$\mathfrak{F}^+_{\mathfrak{W}}$	$\mathfrak{T}^-_{\mathfrak{W}}$	$\mathfrak{I}^-_{\mathfrak{W}}$	$\mathfrak{F}^-_{\mathfrak{W}}$
$\check{\delta}_1$	0.7	0.8	0.1	−0.2	−0.3	−0.9
$\check{\delta}_2$	0.2	0.3	0.2	−0.6	−0.7	−0.7
$\check{\delta}_3$	0.1	0.5	0.2	−0.7	−0.5	−0.8

It is easy to verify that $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ is an NSBF ID of $\mathfrak{R}$. Every NSBF RID (resp. NSBF LID) is an NSBF SSG. But the converse may not be true, as seen in the following example.

Example 2.11. *Consider an SG* $\mathfrak{S} = \{\check{\delta}_1, \check{\mathfrak{z}}_2, \check{\delta}_3, z_4\}$ *with the following Cayley table:*

►	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$	$\check{\delta}_4$
$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_2$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_3$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_2$
$\check{\delta}_4$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$

Define an NSBF $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ in $\mathfrak{R}$ as follows:

$\mathfrak{S}$	$\mathfrak{T}^+_{\mathfrak{W}}$	$\mathfrak{I}^+_{\mathfrak{W}}$	$\mathfrak{F}^+_{\mathfrak{W}}$	$\mathfrak{T}^-_{\mathfrak{W}}$	$\mathfrak{I}^-_{\mathfrak{W}}$	$\mathfrak{F}^-_{\mathfrak{W}}$
$\check{\delta}_1$	0.5	0.7	0.1	−0.2	−0.1	−0.3
$\check{\delta}_2$	0.3	0.4	0.3	−0.6	−0.7	−0.4
$\check{\delta}_3$	0.5	0.5	0.2	−0.4	−0.5	−0.4
$\check{\delta}_4$	0.2	0.2	0.5	−0.6	−0.8	−0.5

It is easy to verify that $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ is an NSBF SSG of $\mathfrak{R}$, but it is not a left NSBF ideal of $\mathfrak{R}$, since $\mathfrak{T}^+_{\mathfrak{W}}(\check{\delta}_4\check{\delta}_3) = \mathfrak{T}^+_{\mathfrak{W}}(\check{\delta}_2) = 0.3 < 0.5 = \mathfrak{T}^+_{\mathfrak{W}}(\check{\delta}_3)$.

Definition 2.12. *[21] Let* $\mathfrak{R}$ *be an SG. An NSBF SSG* $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ *in* $\mathfrak{R}$ *is an NSBF interior ideal (NSBF IN Id) in* $\mathfrak{R}$ *if the below assertions are valid:*

$$(\forall \mathfrak{x}, \mathfrak{z}, \mathfrak{k} \in \mathfrak{S}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}), \\ \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}), \\ \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}), \\ \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}), \\ \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}), \\ \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}) \end{pmatrix}.$$

Remark 2.13. *Every NSBF Ids of an SG* $\mathfrak{R}$ *is an NSBF In Ids of* $\mathfrak{R}$.

Definition 2.14. *[21] Let NSBF $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ in an SG $\mathfrak{R}$ and $\check{\mu}_1, \check{\mu}_2, \check{\mu}_3 \in [0,1], \check{\delta}_1, \check{\delta}_2, \check{\delta}_3 \in [-1, 0]$, the sets*

$$(\mathfrak{T}^+_{\mathfrak{W}})^{\check{\mu}_1} = \{\mathfrak{z} \in \mathfrak{S} | \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}) \geq \check{\mu}_1\},$$
$$(\mathfrak{I}^+_{\mathfrak{W}})^{\check{\mu}_2} = \{\mathfrak{z} \in \mathfrak{S} | \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}) \leq \check{\mu}_2\},$$
$$(\mathfrak{F}^+_{\mathfrak{W}})^{\check{\mu}_3} = \{\mathfrak{z} \in \mathfrak{S} | \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}) \geq \check{\mu}_3\}.$$

The set $\mathfrak{W}^+(\check{\mu}_1, \check{\mu}_2, \check{\mu}_3) := \{\mathfrak{z} \in \mathfrak{S} | \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}) \geq \check{\mu}_1, \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{Q}) \leq \check{\mu}_2, \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}) \geq \check{\mu}_3\}$ is called a positive $(\check{\mu}_1, \check{\mu}_2, \check{\mu}_3)$-level of $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$. It is evident that $P^+_{\mathfrak{W}}(\check{\mu}_1, \check{\mu}_2, \check{\mu}_3) = (\mathfrak{T}^+_{\mathfrak{W}})^{\check{\mu}_1} \cap (\mathfrak{I}^+_{\mathfrak{W}})^{\check{\mu}_2} \cap (\mathfrak{F}^+_{\mathfrak{W}})^{\check{\mu}_3}$, and

$$(\mathfrak{T}^-_{\mathfrak{W}})^{\check{\delta}_1} = \{\mathfrak{z} \in \mathfrak{R} | \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}) \leq \check{\delta}_1\},$$
$$(\mathfrak{I}^-_{\mathfrak{W}})^{\check{\delta}_2} = \{\mathfrak{z} \in \mathfrak{R} | \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}) \geq \check{\delta}_2\},$$
$$(\mathfrak{F}^-_{\mathfrak{W}})^{\check{\delta}_3} = \{\mathfrak{z} \in \mathfrak{R} | \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}) \leq \check{\delta}_3\}.$$

The set $N^-_{\mathfrak{W}}(\check{\delta}_1, \check{\delta}_2, \check{\delta}_3) := \{\mathfrak{z} \in \mathfrak{R} | \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}) \leq \check{\delta}_1, \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}) \geq \check{\delta}_2, \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}) \leq \check{\delta}_3\}$ is called a negative $(\check{\delta}_1, \check{\delta}_2, \check{\delta}_3)$-level of $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$. It is evident that $N^-_{\mathfrak{W}}(\check{\delta}_1, \check{\delta}_2, \check{\delta}_3) = (\mathfrak{T}^-_{\mathfrak{W}})^{\check{\delta}_1} \cap (\mathfrak{I}^-_{\mathfrak{W}})^{\check{\delta}_2} \cap (\mathfrak{F}^-_{\mathfrak{W}})^{\check{\delta}_3}$.

The set $C^{\pm}_{\mathfrak{W}}(\check{\mu}_1, \check{\mu}_2, \check{\mu}_3, \check{\delta}_1, \check{\delta}_2, \check{\delta}_3) = P\mathfrak{W}^+(\check{\mu}_1, \check{\mu}_2, \check{\mu}_3) \cap N\mathfrak{W}^-(\check{\delta}_1, \check{\delta}_2, \check{\delta}_3)$ is called the bipolar $(\check{\mu}_1, \check{\mu}_2, \check{\mu}_3, \check{\delta}_1, \check{\delta}_2, \check{\delta}_3)$-level of $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$.

Definition 2.15. *[21] For any non-empty subset $\mathfrak{Q}$ of set $\mathfrak{X}$, the characteristic NSBF function of $\mathfrak{Q}$ in $\mathfrak{V}$ is defined to be a structure $\chi_{\mathfrak{Q}} = \{\langle x, \mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{z}), \mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{z}), \mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{z}), \mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{z}), \mathfrak{I}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{z}), \mathfrak{F}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{z})\rangle : \mathfrak{z} \in \mathfrak{z}\}$, where*

$$\mathfrak{T}^+_{\chi_{\mathfrak{Q}}} : \mathfrak{V} \to [0,1]; \mathfrak{z} \mapsto \mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{z}) := \begin{cases} 1 & if\ \mathfrak{z} \in \mathfrak{Q} \\ 0 & if\ \mathfrak{z} \notin \mathfrak{Q}, \end{cases}$$

$$\mathfrak{I}^+_{\chi_{\mathfrak{Q}}} : \mathfrak{V} \to [0,1]; \mathfrak{z} \mapsto \mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{z}) := \begin{cases} 0 & if\ \mathfrak{z} \in \mathfrak{Q} \\ 1 & if\ \mathfrak{z} \notin \mathfrak{Q}, \end{cases}$$

$$\mathfrak{F}^+_{\chi_{\mathfrak{Q}}} : \mathfrak{V} \to [0,1]; \mathfrak{z} \mapsto \mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{z}) := \begin{cases} 1 & if\ \mathfrak{z} \in \mathfrak{Q} \\ 0 & if\ \mathfrak{z} \notin \mathfrak{Q}, \end{cases}$$

$$\mathfrak{T}^-_{\chi_{\mathfrak{Q}}} : \mathfrak{V} \to [-1,0]; \mathfrak{z} \mapsto \mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{z}) := \begin{cases} -1 & if\ \mathfrak{z} \in \mathfrak{Q} \\ 0 & if\ \mathfrak{z} \notin \mathfrak{Q}, \end{cases}$$

$$\mathfrak{I}^-_{\chi_{\mathfrak{Q}}} : \mathfrak{V} \to [-1,0]; \mathfrak{z} \mapsto \mathfrak{I}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{z}) := \begin{cases} 0 & if\ \mathfrak{z} \in \mathfrak{Q} \\ -1 & if\ \mathfrak{z} \notin \mathfrak{Q}, \end{cases}$$

$$\mathfrak{F}^-_{\chi_{\mathfrak{Q}}} : \mathfrak{V} \to [-1,0]; \mathfrak{z} \mapsto \mathfrak{F}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{z}) := \begin{cases} -1 & if\ \mathfrak{z} \in \mathfrak{Q} \\ 0 & if\ \mathfrak{z} \notin \mathfrak{Q}. \end{cases}$$

For simplicity, we use the symbol $\chi_{\mathfrak{Q}} = (\chi^+_{\mathfrak{Q}}, \chi^-_{\mathfrak{Q}})$ for the characteristic NSBF (shortly, CNSBF) function $\chi_{\mathfrak{Q}} = \{\langle x, \mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{r}), \mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{r}), \mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{r}), \mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{r}), \mathfrak{I}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{r}), \mathfrak{F}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{r})\rangle : \mathfrak{r} \in \mathfrak{V}\}$. The SG $\mathfrak{R}$ can be considered a fuzzy subset of itself, i.e., $\chi_{\mathfrak{R}}(\mathfrak{r}) = \langle 1, 0, 1, -1, 0, -1\rangle$ for all $\mathfrak{r} \in \mathfrak{R}$.

Definition 2.16. *Let $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ and $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ be an NSBF in an SG $\mathfrak{S}$, Then*

1) *$\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ is called an NSBF in $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$, denoted by $\mathfrak{W} \sqsubseteq \mathfrak{B} = (\mathfrak{W}^+ \sqsubseteq \mathfrak{B}^+, \mathfrak{W}^- \sqsubseteq \mathfrak{B}^-)$ if $\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{r}) \leq \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{r}), \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{r}) \geq \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{r}), \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{r}) \leq \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{r}), \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{r}) \geq \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{r}), \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{r}) \leq \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{r}), \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{r}) \geq \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{r})$, for all $\mathfrak{r} \in \mathfrak{S}$. If $\mathfrak{W} \sqsubseteq \mathfrak{B}$ and $\mathfrak{B} \sqsubseteq \mathfrak{W}$, then we say that $\mathfrak{W} = \mathfrak{B}$.*
2) *The union of two NSBF $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ and $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ is defined as*
   $\mathfrak{W} \sqcup \mathfrak{B} = (\mathfrak{W}^+ \sqcup B^+, \mathfrak{W}^- \sqcup B^-) = \{\langle x, (\mathfrak{T}^+_{\mathfrak{W}} \cup \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{I}^+_{\mathfrak{W}} \cup \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{F}^+_{\mathfrak{W}} \cup \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{T}^-_{\mathfrak{W}} \cup \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{I}^-_{\mathfrak{W}} \cup \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{F}^-_{\mathfrak{W}} \cup \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{r})\rangle : \mathfrak{r} \in \mathfrak{r}\}$, *where* $\forall x \in S$,
   $(\mathfrak{T}^+_{\mathfrak{W}} \cup \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{r}) \vee \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{r})$, $(\mathfrak{I}^+_{\mathfrak{W}} \cup \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{r}) \wedge \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{r})$, $(\mathfrak{F}^+_{\mathfrak{W}} \cup \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{r}) \vee \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{r})$,
   $(\mathfrak{T}^-_{\mathfrak{W}} \cup \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{r}) \wedge \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{r})$, $(\mathfrak{I}^-_{\mathfrak{W}} \cup \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{r}) \vee \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{r})$, $(\mathfrak{F}^-_{\mathfrak{W}} \cup \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{r}) \wedge \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{r})$. *where* $\forall \mathfrak{r} \in \mathfrak{S}$,
3) *The intersection of two NSBF $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ and $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ is defined as*
   $\mathfrak{W} \sqcap \mathfrak{B} = (\mathfrak{W}^+ \sqcap \mathfrak{B}^+, \mathfrak{W}^- \sqcap \mathfrak{B}^-) = \{\langle \mathfrak{r}, (\mathfrak{T}^+_{\mathfrak{W}} \cap \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{I}^+_{\mathfrak{W}} \cap \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{F}^+_{\mathfrak{W}} \cap \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{r}),$
   $(\mathfrak{T}^-_{\mathfrak{W}} \cap \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{I}^-_{\mathfrak{W}} \cap \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{F}^-_{\mathfrak{W}} \cap \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{r})\rangle : \mathfrak{r} \in \mathfrak{r}\}$, *where* $\forall \mathfrak{r} \in \mathfrak{S}$,
   $(\mathfrak{T}^+_{\mathfrak{W}} \cap \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{r}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{r})$, $(\mathfrak{I}^+_{\mathfrak{W}} \cap \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{r}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{r})$, $(\mathfrak{F}^+_{\mathfrak{W}} \cap \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{r}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{r})$,
   $(\mathfrak{T}^-_{\mathfrak{W}} \cap \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{r}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{r})$, $(\mathfrak{I}^-_{\mathfrak{W}} \cap \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{r}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{r})$, $(\mathfrak{F}^-_{\mathfrak{W}} \cap \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{r}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{r})$.
4) $\mathfrak{W} \bar{\circ} \mathfrak{B} = (\mathfrak{W}^+ \circ \mathfrak{B}^+, \mathfrak{W}^- \circ \mathfrak{B}^-)$

$$(\mathfrak{T}^+_{\mathfrak{W}} \circ T^+_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \bigvee_{(\mathfrak{h},\mathfrak{r}) \in F_{\mathfrak{u}}} \{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{h}) \wedge \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{r})\} & if F_{\mathfrak{u}} \neq \emptyset \\ 0 & if F_{\mathfrak{u}} = \emptyset, \end{cases}$$

$$(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \bigwedge_{(\mathfrak{h},\mathfrak{r}) \in F_{\mathfrak{u}}} \{\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{h}) \vee \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{r})\} & if F_{\mathfrak{u}} \neq \emptyset \\ 0 & if F_{\mathfrak{u}} = \emptyset, \end{cases}$$

$$(\mathfrak{F}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \bigvee_{(\mathfrak{h},\mathfrak{r}) \in F_{\mathfrak{u}}} \{\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{h}) \wedge \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{r})\} & if F_{\mathfrak{u}} \neq \emptyset \\ 0 & if F_{\mathfrak{u}} = \emptyset, \end{cases}$$

and

$$(\mathfrak{T}^-_{\mathfrak{W}} \circ T^-_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \bigwedge_{(\mathfrak{h},\mathfrak{r}) \in F_{\mathfrak{u}}} \{\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{h}) \vee \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{r})\} & if F_{\mathfrak{u}} \neq \emptyset \\ 0 & if F_{\mathfrak{u}} = \emptyset, \end{cases}$$

$$(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \bigvee_{(\mathfrak{h},\mathfrak{r}) \in F_{\mathfrak{u}}} \{\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{h}) \wedge \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{r})\} & if F_{\mathfrak{u}} \neq \emptyset \\ 0 & if F_{\mathfrak{u}} = \emptyset, \end{cases}$$

$$(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \bigwedge_{(\mathfrak{h},\mathfrak{r}) \in F_{\mathfrak{u}}} \{\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{h}) \vee \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{r})\} & if F_{\mathfrak{u}} \neq \emptyset \\ 0 & if F_{\mathfrak{u}} = \emptyset. \end{cases}$$

where $\forall \mathfrak{r} \in \mathfrak{S}$ and $F_{\mathfrak{u}} = \{(\mathfrak{h}, \mathfrak{r}) \in \mathfrak{S} \times \mathfrak{S} \mid \mathfrak{u} = \mathfrak{h}\mathfrak{r}\}$.

Theorem 2.17. *Let $\emptyset \neq \mathfrak{K}$ of an SG $\mathfrak{S}$. Then $\mathfrak{K}$ is an SSG (LID, RID, INId) of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{Q}} = (\chi^+_{\mathfrak{Q}}, \chi^-_{\mathfrak{Q}})$ is an NSBF SSG (NSBF LID, NSBF RID, NSBF IN Id) of $\mathfrak{S}$.*

Theorem 2.18. *Let $\mathfrak{R}$ be an SG. Then the arbitrary intersection (resp., union) of NSBF SSGs (NSBF LIDs, NSBF RIDs, NSBF IN Ids) in $\mathfrak{S}$ is an NSBF SSG (NSBF LID, NSBF RID, NSBF IN Id) of $\mathfrak{S}$.*

III. Main Results

In this clause, we give definitions of NSBF bi-ideal of an SG and we prove the properties of NSBF bi-ideal and NSBF generalized bi-ideal.

Definition 3.1. *Let $\mathfrak{R}$ be an SG. An NSBF SSG $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ in $\mathfrak{R}$ is an NSBF bi-ideal (NSBF B Id) in $\mathfrak{R}$*

if the below assertions are valid:

$$(\forall \mathfrak{x},\mathfrak{z},\mathfrak{k}\in\mathfrak{S})\begin{pmatrix}\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{x})\wedge \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{k}),\\ \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{x})\vee \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{k}),\\ \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{x})\wedge \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{k})\\ \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{x})\vee \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{k})\\ \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{x})\wedge \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{k})\\ \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{x})\vee \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{k})\end{pmatrix}.$$

Example 3.2. *Consider an SG $S=\{\check{\delta}_1,\check{\mathfrak{z}}_2,\check{\delta}_3,\check{\delta}_4\}$ with the following Cayley table:*

►	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$	$\check{\delta}_4$
$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1\check{\delta}_1$	$\check{\delta}_1$	
$\check{\delta}_2$	$\check{\delta}_1$	$\check{\delta}_1\check{\delta}_1$	$\check{\delta}_1$	
$\check{\delta}_3$	$\check{\delta}_1$	$\check{\delta}_1\check{\delta}_1$	$\check{\delta}_2$	
$\check{\delta}_4$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$

Define an NSBF $\mathfrak{W}=(\mathfrak{W}^-,\mathfrak{W}^+)$ in $\mathfrak{R}$ as follows:

$\mathfrak{S}$	$\mathfrak{T}^+_{\mathfrak{W}}$	$\mathfrak{I}^+_{\mathfrak{W}}$	$\mathfrak{F}^+_{\mathfrak{W}}$	$\mathfrak{T}^-_{\mathfrak{W}}$	$\mathfrak{I}^-_{\mathfrak{W}}$	$\mathfrak{F}^-_{\mathfrak{W}}$
$\check{\delta}_1$	0.6	0.1	0.5	−0.1	−0.6	−0.1
$\check{\delta}_2$	0.3	0.3	0.2	−0.4	−0.3	−0.3
$\check{\delta}_3$	0.4	0.2	0.1	−0.3	−0.4	−0.2
$\check{\delta}_4$	0.1	0.4	0.1	−0.5	−0.1	−0.4

It is easy to verify that $\mathfrak{W}=(\mathfrak{W}^-,\mathfrak{W}^+)$ is an NSBF B Id of $\mathfrak{R}$, but it is not a left NSBF ideal of $\mathfrak{R}$, since $\mathfrak{T}^+_{\mathfrak{W}}(\check{\delta}_4\check{\delta}_3)=\mathfrak{T}^+_{\mathfrak{W}}(\check{\delta}_2)=0.3<0.4=\mathfrak{T}^+_{\mathfrak{W}}(\check{\delta}_3)$.

Remark 3.3. *Every NSBF Ids of an SG $\mathfrak{R}$ is an NSBF B Ids of $\mathfrak{R}$.*

Definition 3.4. *Let $\mathfrak{R}$ be an SG. An NSBF SG $\mathfrak{W}=(\mathfrak{W}^-,\mathfrak{W}^+)$ in $\mathfrak{R}$ is an NSBF generalized bi-ideal (NSBF GB Id) in $\mathfrak{R}$ if the below assertions are valid:*

$$(\forall \mathfrak{x},\mathfrak{z},\mathfrak{k}\in\mathfrak{R})\begin{pmatrix}\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{x})\wedge \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{k}),\\ \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{x})\vee \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{k}),\\ \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{x})\wedge \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{k})\\ \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{x})\vee \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{k})\\ \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{x})\wedge \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{k})\\ \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{x}\mathfrak{z}\mathfrak{k})\leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{x})\vee \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{k})\end{pmatrix}.$$

Remark 3.5. *Every NSBF BId of an SG $\mathfrak{R}$ is an NSBF GB Id of an SG $\mathfrak{R}$.*

In order to consider the converse of Rermak 3.3 and 3.5, we need to strengthen the condition of $\mathcal{G}$.

Definition 3.6. *[22] A semigroup $\mathfrak{R}$ called a* quasi-regular *if every left ideal and right ideal of $\mathfrak{R}$ are idempotent.*

It is easy to prove that $\mathfrak{R}$ left (right) quasi-regular if and only if $\mathfrak{R}\in\mathfrak{R}\mathfrak{R}\mathfrak{R}\mathfrak{R}\mathfrak{R}(\mathfrak{R}\in\mathfrak{R}\mathfrak{R}\mathfrak{R}\mathfrak{R})$, this implies that there exist $\mathfrak{x},\mathfrak{y}\in\mathfrak{R}$ such that $\mathfrak{r}=\mathfrak{x}\mathfrak{r}\mathfrak{y}\mathfrak{r}(\mathfrak{r}=\mathfrak{r}\mathfrak{x}\mathfrak{r}\mathfrak{y})$.

Theorem 3.7. *Let $\mathfrak{R}$ be a quasi-regular semigroup. Then the every NSBF IN Ids and the NSBF Ids coincide.*

Proof:

Suppose that $\mathfrak{W}=(\mathfrak{W}^-,\mathfrak{W}^+)$ is a NSBF IN Id of $\mathfrak{R}$ and let $\mathfrak{z},\mathfrak{y}\in\mathfrak{S}$. Since $\mathfrak{R}$ is quasi-regular, there exists $\mathfrak{t},\mathfrak{n}\in\mathfrak{R}$ such that $\mathfrak{z}=\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{t}$. Thus, $\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y})=\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{t}\mathfrak{y})=\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{t}\mathfrak{y})\geq\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z})$, $\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y})=\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{t}\mathfrak{y})=\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{t}\mathfrak{y})\leq\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z})$ and $\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y})=\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{t}\mathfrak{y})=\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{t}\mathfrak{y})\geq\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z})$. And $\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y})=\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{t}\mathfrak{y})=\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{t}\mathfrak{y})\leq\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z})$, $\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y})=\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{t}\mathfrak{y})=\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{t}\mathfrak{y})\geq\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z})$ and $\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y})=\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{t}\mathfrak{y})=\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{t}\mathfrak{y})\leq\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z})$. Hence $\mathfrak{W}=(\mathfrak{W}^-,\mathfrak{W}^+)$ is a NSBF RId of $\mathfrak{R}$. Similarly, we can show that $\mathfrak{W}=(\mathfrak{W}^-,\mathfrak{W}^+)$ is a NSBF LId of $\mathfrak{R}$. Thus, $\mathfrak{W}=(\mathfrak{W}^-,\mathfrak{W}^+)$ is a NSBF Id of $\mathfrak{R}$. ■

Theorem 3.8. *Let $\mathfrak{R}$ be an SG. Then, for any $\emptyset\neq\mathfrak{Q}\subseteq\mathfrak{S}$, the given assertions are equivalent:*

(1) $\mathfrak{Q}$ is a BId (GB Id, QId),
(2) $\chi_{\mathfrak{Q}}=(\chi^+_{\mathfrak{Q}},\chi^-_{\mathfrak{Q}})$ is an NSBF B Id (NSBF GB Id, NSBF QId).

Proof: (1⇒2) Suppose that $\mathfrak{Q}$ is a BId of $\mathfrak{R}$ and $\mathfrak{z},\mathfrak{y},\mathfrak{W}\in\mathfrak{S}$. If $\mathfrak{z},\mathfrak{y}\in\mathfrak{Q}$, then $\mathfrak{x}\mathfrak{a}\mathfrak{y}\in\mathfrak{Q}$. Thus, $\mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})=\mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x})=\mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})=1,\mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})=\mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x})=\mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})=0,\mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})=\mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x})=\mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})=1$ and $\mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})=\mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x})=\mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{y})=0,\mathfrak{I}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})=\mathfrak{I}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x})=\mathfrak{I}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{y})=1,\mathfrak{F}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})=\mathfrak{F}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x})=\mathfrak{F}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{y})=0$. Hence,
$\mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})\geq\mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x})\wedge\mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})$,
$\mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})\leq\mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x})\vee\mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})$,
$\mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})\geq\mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{x})\wedge\mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})$
$\mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})\leq\mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x})\vee\mathfrak{T}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{y})$
$\mathfrak{I}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})\geq\mathfrak{I}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x})\wedge\mathfrak{I}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{y})$
$\mathfrak{F}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y})\leq\mathfrak{F}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{x})\vee\mathfrak{F}^-_{\chi_{\mathfrak{Q}}}(\mathfrak{y})$. Therefore, $\chi_{\mathfrak{Q}}=(\chi^+_{\mathfrak{Q}},\chi^-_{\mathfrak{Q}})$ is an NSBF SSG. By Definition 2.15, $\chi_{\mathfrak{Q}}=(\chi^+_{\mathfrak{Q}},\chi^-_{\mathfrak{Q}})$ is an NSBF B Id.

(2⇒1) Assume that $\chi_{\mathfrak{Q}}=(\chi^+_{\mathfrak{Q}},\chi^-_{\mathfrak{Q}})$ is an NSBF B Id. Then $\chi_{\mathfrak{Q}}=(\chi^+_{\mathfrak{Q}},\chi^-_{\mathfrak{Q}})$ is an NSBF SSG. Thus, $\mathfrak{Q}$ is an SSG. Let $\mathfrak{z},\mathfrak{y}\in\mathfrak{Q}$ and $\mathfrak{a}\in\mathfrak{S}$. Then $\mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{z})=\mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})=1,\mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{z})=\mathfrak{I}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})=0,\mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{z})=\mathfrak{F}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})=1$. By assumptions, which imply $\mathfrak{x}\mathfrak{a}\mathfrak{y}\in\mathfrak{Q}$. Hence, by Definition 2.15, $\mathfrak{Q}$ is a BId. ■

Definition 3.9. *Let $\mathfrak{R}$ be an SG. An NSBF SG $\mathfrak{W}=(\mathfrak{W}^-,\mathfrak{W}^+)$ in $\mathfrak{R}$ is an NSBF quasi-ideal (NSBF QId) in $\mathfrak{R}$ if the below assertions are valid:*

$$(\forall\mathfrak{z}\in\mathfrak{R})\begin{pmatrix}(\mathfrak{T}^+_{\chi_{\mathfrak{R}}}\circ\mathfrak{T}^+_{\mathfrak{W}})(\mathfrak{z})\wedge(\mathfrak{T}^+_{\mathfrak{W}}\circ\mathfrak{T}^+_{\chi_{\mathfrak{Q}}})(\mathfrak{z})\geq\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}),\\ (\mathfrak{I}^+_{\chi_{\mathfrak{R}}}\circ\mathfrak{I}^+_{\mathfrak{W}})(\mathfrak{z})\vee(\mathfrak{I}^+_{\mathfrak{W}}\circ\mathfrak{I}^+_{\chi_{\mathfrak{Q}}})(\mathfrak{z})\leq\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}),\\ (\mathfrak{F}^+_{\chi_{\mathfrak{R}}}\circ\mathfrak{F}^+_{\mathfrak{W}})(\mathfrak{z})\wedge(\mathfrak{F}^+_{\mathfrak{W}}\circ\mathfrak{F}^+_{\chi_{\mathfrak{Q}}})(\mathfrak{z})\geq\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z})\\ (\mathfrak{T}^-_{\chi_{\mathfrak{R}}}\circ\mathfrak{T}^-_{\mathfrak{W}})(\mathfrak{z})\vee(\mathfrak{T}^-_{\mathfrak{W}}\circ\mathfrak{T}^-_{\chi_{\mathfrak{Q}}})(\mathfrak{z})\leq\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}),\\ (\mathfrak{I}^-_{\chi_{\mathfrak{R}}}\circ\mathfrak{I}^-_{\mathfrak{W}})(\mathfrak{z})\wedge(\mathfrak{I}^-_{\mathfrak{W}}\circ\mathfrak{I}^-_{\chi_{\mathfrak{Q}}})(\mathfrak{z})\geq\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}),\\ (\mathfrak{F}^-_{\chi_{\mathfrak{R}}}\circ\mathfrak{F}^-_{\mathfrak{W}})(\mathfrak{z})\vee(\mathfrak{F}^-_{\mathfrak{W}}\circ\mathfrak{F}^-_{\chi_{\mathfrak{Q}}})(\mathfrak{z})\leq\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z})\end{pmatrix}.$$

Lemma 3.10. *Every NSBF QId of an SG $\mathfrak{R}$ is a NSBF SSG of $\mathfrak{R}$.*

Proof: Assume that $\mathfrak{W}=(\mathfrak{W}^-,\mathfrak{W}^+)$ is a NSBF QId of $\mathfrak{R}$ and $\mathfrak{z},\mathfrak{y}\in\mathfrak{R}$. Then

$$\begin{aligned}\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) &\geq (\mathfrak{T}^+_{\chi_{\mathfrak{R}}}\circ\mathfrak{T}^+_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y})\wedge(\mathfrak{T}^+_{\mathfrak{W}}\circ\mathfrak{T}^+_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y})\\ &= \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}}}\{\mathfrak{T}^+_{\chi_{\mathfrak{R}}}(\mathfrak{i})\wedge\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{j})\}\wedge\\ &\quad \bigvee_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}}}\{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{m})\wedge\mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\}\\ &\geq \mathfrak{T}^+_{\chi_{\mathfrak{R}}}(\mathfrak{z})\wedge\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y})\wedge\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z})\wedge\mathfrak{T}^+_{\chi_{\mathfrak{Q}}}(\mathfrak{y})\\ &= 1\wedge\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y})\wedge\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z})\wedge 1\\ &= \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z})\wedge\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y}),\end{aligned}$$

$$
\begin{aligned}
\mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) &\leq (\mathfrak{I}^{+}_{\chi_{\mathfrak{R}}} \circ \mathfrak{I}^{+}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}) \vee (\mathfrak{I}^{+}_{\mathfrak{W}} \circ \mathfrak{I}^{+}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}) \\
&= \bigwedge_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{I}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{j})\} \vee \\
&\quad \bigwedge_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{m}) \vee \mathfrak{I}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\leq \mathfrak{I}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{z}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}) \\
&= 0 \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}) \vee 0 \\
&= \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{y}),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) &\geq (\mathfrak{F}^{+}_{\chi_{\mathfrak{R}}} \circ \mathfrak{F}^{+}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}) \wedge (\mathfrak{F}^{+}_{\mathfrak{W}} \circ \mathfrak{F}^{+}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{F}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{j})\} \wedge \\
&\quad \bigvee_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{m}) \wedge \mathfrak{F}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\geq \mathfrak{F}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{z}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{F}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}) \\
&= 1 \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge 1 \\
&= \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{y}),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) &\leq (\mathfrak{T}^{-}_{\chi_{\mathfrak{R}}} \circ \mathfrak{T}^{-}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}) \vee (\mathfrak{T}^{-}_{\mathfrak{W}} \circ \mathfrak{T}^{-}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}) \\
&= \bigwedge_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{T}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{j})\} \vee \\
&\quad \bigwedge_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{m}) \vee \mathfrak{T}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\leq \mathfrak{T}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{z}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}) \\
&= -1 \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee -1 \\
&= \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{y}),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) &\geq (\mathfrak{I}^{-}_{\chi_{\mathfrak{R}}} \circ \mathfrak{I}^{-}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}) \wedge (\mathfrak{I}^{-}_{\mathfrak{W}} \circ \mathfrak{I}^{-}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{I}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{j})\} \wedge \\
&\quad \bigvee_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{m}) \wedge \mathfrak{I}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\geq \mathfrak{I}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{z}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}) \\
&= 0 \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}) \wedge 0 \\
&= \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{y}),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) &\leq (\mathfrak{F}^{-}_{\chi_{\mathfrak{R}}} \circ \mathfrak{F}^{-}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}) \vee (\mathfrak{F}^{-}_{\mathfrak{W}} \circ \mathfrak{I}^{+}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}) \\
&= \bigwedge_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{F}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{j})\} \vee \\
&\quad \bigwedge_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}}} \{\mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{m}) \vee \mathfrak{F}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\leq \mathfrak{F}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{z}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}) \\
&= -1 \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee -1 \\
&= \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{y}),
\end{aligned}
$$

Thus, $\mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) \geq \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{y}), \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) \leq \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{y})$, $\mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) \geq \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{y})$ and $\mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) \leq \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{y}), \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) \geq \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{y})$, $\mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) \leq \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{y})$.
Hence $\mathfrak{W} = (\mathfrak{W}^{-}, \mathfrak{W}^{+})$ is a NSBF SSG of $\mathfrak{R}$. ■

Lemma 3.11. *Every NSBF QId of an SG $\mathfrak{R}$ is a NSBF GB Id of $\mathfrak{R}$.*

Proof: Assume that $\mathfrak{W} = (\mathfrak{W}^{-}, \mathfrak{W}^{+})$ is a NSBF QId of $\mathfrak{R}$ and let $\mathfrak{x}, \mathfrak{y}, \mathfrak{v} \in \mathfrak{R}$ we get that

$$
\begin{aligned}
\mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) &\geq (\mathfrak{T}^{+}_{\chi_{\mathfrak{R}}} \circ \mathfrak{T}^{+}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \wedge (\mathfrak{T}^{+}_{\mathfrak{W}} \circ \mathfrak{T}^{+}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{T}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \wedge \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{j})\} \wedge \\
&\quad \bigvee_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{m}) \wedge \mathfrak{T}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\geq \mathfrak{T}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{z}\mathfrak{y}) \wedge \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{v}) \wedge \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{T}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}\mathfrak{v}) \\
&= 1 \wedge \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{v}) \wedge \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge 1 \\
&= \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{v}),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) &\leq (\mathfrak{I}^{+}_{\chi_{\mathfrak{R}}} \circ \mathfrak{I}^{+}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \vee (\mathfrak{I}^{+}_{\mathfrak{W}} \circ \mathfrak{I}^{+}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}) \\
&= \bigwedge_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{I}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{j})\} \vee \\
&\quad \bigwedge_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{m}) \vee \mathfrak{I}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\leq \mathfrak{I}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{z}\mathfrak{y}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{v}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}\mathfrak{v}) \\
&= 0 \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{v}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}) \vee 0 \\
&= \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{v}),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) &\geq (\mathfrak{F}^{+}_{\chi_{\mathfrak{R}}} \circ \mathfrak{F}^{+}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \wedge (\mathfrak{F}^{+}_{\mathfrak{W}} \circ \mathfrak{F}^{+}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{F}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{j})\} \wedge \\
&\quad \bigvee_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{m}) \wedge \mathfrak{F}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\geq \mathfrak{F}^{+}_{\chi_{\mathfrak{R}}}(\mathfrak{z}\mathfrak{y}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{v}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) \wedge \mathfrak{F}^{+}_{\chi_{\mathfrak{Q}}}(\mathfrak{v}) \\
&= 1 \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{v}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge 1 \\
&= \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{y}),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) &\leq (\mathfrak{T}^{-}_{\chi_{\mathfrak{R}}} \circ \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \vee \mathfrak{T}^{-}_{\mathfrak{W}} \circ \mathfrak{T}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{z}\mathfrak{y}\mathfrak{v})) \\
&= \bigwedge_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{T}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{j})\} \vee \\
&\quad \bigwedge_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{m}) \vee \mathfrak{T}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\leq \mathfrak{T}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{z}\mathfrak{y}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{v}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}\mathfrak{v}) \\
&= -1 \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{v}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee -1 \\
&= \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{y}),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) &\geq (\mathfrak{I}^{-}_{\chi_{\mathfrak{R}}} \circ \mathfrak{I}^{-}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \wedge (\mathfrak{I}^{-}_{\mathfrak{W}} \circ \mathfrak{I}^{-}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \\
&= \bigvee_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{I}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{j})\} \wedge \\
&\quad \bigvee_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{m}) \wedge \mathfrak{I}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\geq \mathfrak{I}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{z}\mathfrak{y}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{v}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}\mathfrak{v}) \\
&= 0 \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{v}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}) \wedge 0 \\
&= \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{v}),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}) &\leq (\mathfrak{F}^{-}_{\chi_{\mathfrak{R}}} \circ \mathfrak{F}^{-}_{\mathfrak{W}})(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \vee (\mathfrak{F}^{-}_{\mathfrak{W}} \circ \mathfrak{I}^{+}_{\chi_{\mathfrak{Q}}})(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \\
&= \bigwedge_{(\mathfrak{i},\mathfrak{j})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{F}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{i}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{j})\} \vee \\
&\quad \bigwedge_{(\mathfrak{m},\mathfrak{n})\in F_{\mathfrak{z}\mathfrak{y}\mathfrak{v}}} \{\mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{m}) \vee \mathfrak{F}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{n})\} \\
&\leq \mathfrak{F}^{-}_{\chi_{\mathfrak{R}}}(\mathfrak{z}\mathfrak{y}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{v}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^{-}_{\chi_{\mathfrak{Q}}}(\mathfrak{y}\mathfrak{v}) \\
&= -1 \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{v}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee -1 \\
&= \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{y}),
\end{aligned}
$$

Thus, $\mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \geq \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{T}^{+}_{\mathfrak{W}}(\mathfrak{v}), \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \leq \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^{+}_{\mathfrak{W}}(\mathfrak{v})$, $\mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \geq \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{F}^{+}_{\mathfrak{W}}(\mathfrak{v})$ and $\mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \leq \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^{-}_{\mathfrak{W}}(\mathfrak{v}), \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \geq \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^{-}_{\mathfrak{W}}(\mathfrak{v})$, $\mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}\mathfrak{y}\mathfrak{v}) \leq \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^{-}_{\mathfrak{W}}(\mathfrak{v})$.
Hence, $\mathfrak{W} = (\mathfrak{W}^{-}, \mathfrak{W}^{+})$ is a NSBF GB Id of $\mathfrak{R}$. ■

Theorem 3.12. *Every NSBF QId of an SG $\mathfrak{R}$ is a NSBF BId of $\mathfrak{R}$.*

Proof: By Lemma 3.10 and 3.11. ■

IV. Characterize quasi-regular semigroups in terms of generalized neutrosophic bipolar-valued fuzzy ideals.

In this clause, we will characterize weakly regular in terms of types NSBF Ids.

Lemma 4.1. *Let $\mathfrak{Q}$ and $\mathfrak{L}$ be non-empty subsets of an SG $\mathfrak{R}$. Then the following statements are true*

(1) $(\chi_{\mathfrak{Q}}) \wedge (\chi_{\mathfrak{Q}}) = (\chi_{\mathfrak{Q} \cap \mathfrak{L}})$.

(2) $(\chi_{\mathfrak{Q}})\bar{\circ}(\chi_{\mathfrak{Q}}) = (\chi_{\mathfrak{K}\mathfrak{L}})$.

The following definition and lemma will be used to prove in Theorem 4.2

Lemma 4.2. *[22] A semigroup $\mathfrak{R}$ is a left quasi-regular if and only if $\mathfrak{L}\mathfrak{L} = \mathfrak{L}$, for every left ideal $\mathfrak{L}$ of $\mathfrak{R}$.*

Theorem 4.3. *A semigroup $\mathfrak{R}$ is a left quasi-regular if and only if $\mathfrak{W}\bar{\circ}\mathfrak{W} = \mathfrak{W}$, for every NSBF LId $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ of $\mathfrak{R}$.*

Proof: Assume that $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ is a NSBF LId of $\mathfrak{R}$. Then $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ is a NSBF SSG of $\mathfrak{R}$. Let $\mathfrak{z} \in \mathfrak{R}$.
If $A_{\mathfrak{z}} = \emptyset$, then it is easy to verify that,
$(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{T}^+_{\mathfrak{W}})(\mathfrak{z}) \leq \mathfrak{T}^+_{\mathfrak{W}}$, $(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{W}})(\mathfrak{z}) \geq \mathfrak{I}^+_{\mathfrak{W}}$, $(\mathfrak{F}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{W}})(\mathfrak{z}) \leq \mathfrak{F}^+_{\mathfrak{W}}$ and $(\mathfrak{T}^-_{\mathfrak{W}} \circ \mathfrak{T}^-_{\mathfrak{W}})(\mathfrak{z}) \geq \mathfrak{T}^-_{\mathfrak{W}}$, $(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{W}})(\mathfrak{z}) \leq \mathfrak{I}^-_{\mathfrak{W}}$, , $(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{W}})(\mathfrak{z}) \geq \mathfrak{F}^-_{\mathfrak{W}}$.
If $A_{\mathfrak{R}} \neq \emptyset$, then

$$\begin{aligned}(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{T}^+_{\mathfrak{W}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in A_{\mathfrak{z}}} \{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{n})\} \\ &\leq \bigvee_{(\mathfrak{y},\mathfrak{z})\in A_{\mathfrak{R}}} \{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y}\mathfrak{z})\} = \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{W}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{n})\} \\ &\leq \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{\mathfrak{z}}} \{\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{y}\mathfrak{z})\} = \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{F}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{W}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{n})\} \\ &\leq \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{\mathfrak{z}}} \{\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{y}\mathfrak{z})\} = \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{T}^-_{\mathfrak{W}} \circ \mathfrak{T}^-_{\mathfrak{W}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{n})\} \\ &\leq \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{\mathfrak{z}}} \{\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{y}\mathfrak{z})\} = \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{W}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{n})\} \\ &\leq \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{\mathfrak{z}}} \{\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{y}\mathfrak{z})\} = \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{W}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{n})\} \\ &\leq \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{\mathfrak{z}}} \{\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{y}\mathfrak{z})\} = \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

Thus, $(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{T}^+_{\mathfrak{W}})(\mathfrak{z}) \leq \mathfrak{T}^+_{\mathfrak{W}}$, $(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{W}})(\mathfrak{z}) \geq \mathfrak{I}^+_{\mathfrak{W}}$, $(\mathfrak{F}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{W}})(\mathfrak{z}) \leq \mathfrak{F}^+_{\mathfrak{W}}$ and $(\mathfrak{T}^-_{\mathfrak{W}} \circ \mathfrak{T}^-_{\mathfrak{W}})(\mathfrak{z}) \geq \mathfrak{T}^-_{\mathfrak{W}}$, $(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{W}})(\mathfrak{z}) \leq \mathfrak{I}^-_{\mathfrak{W}}$, , $(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{W}})(\mathfrak{z}) \geq \mathfrak{F}^-_{\mathfrak{W}}$. Hence, $\mathfrak{W}\bar{\circ}\mathfrak{W} \sqsubseteq \mathfrak{W}$. On other hand since $\mathfrak{R}$ is left quasi-regular, there exist $\mathfrak{k}, \mathfrak{t} \in \mathfrak{R}$ such that $\mathfrak{z} = \mathfrak{k}\mathfrak{z}\mathfrak{t}\mathfrak{z}$. Thus,

$$\begin{aligned}(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{T}^+_{\mathfrak{W}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{n})\} \\ &= \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{k}\mathfrak{z})(\mathfrak{t}\mathfrak{z})}} \{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{n})\} \\ &\geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{k}\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}) \\ &\geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}) = \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{W}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{n})\} \\ &= \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{k}\mathfrak{z})(\mathfrak{t}\mathfrak{z})}} \{\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{n})\} \\ &\leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{k}\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}) \\ &\leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}) = \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{F}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{W}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{n})\} \\ &= \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{k}\mathfrak{z})(\mathfrak{t}\mathfrak{z})}} \{\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{n})\} \\ &\geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{k}\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}) \\ &\geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}) = \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{T}^-_{\mathfrak{W}} \circ \mathfrak{T}^-_{\mathfrak{W}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{n})\} \\ &= \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{k}\mathfrak{z})(\mathfrak{t}\mathfrak{z})}} \{\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{n})\} \\ &\leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{k}\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}) \\ &\leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}) = \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{W}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{n})\} \\ &= \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{k}\mathfrak{z})(\mathfrak{t}\mathfrak{z})}} \{\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{n})\} \\ &\geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{k}\mathfrak{z}) \wedge \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}) \\ &\geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}) = \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{W}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{n})\} \\ &= \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{k}\mathfrak{z})(\mathfrak{t}\mathfrak{z})}} \{\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{n})\} \\ &\leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{k}\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{t}\mathfrak{z}) \\ &\leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}) = \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}),\end{aligned}$$

Hence,$(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{T}^+_{\mathfrak{W}})(\mathfrak{z}) \geq \mathfrak{T}^+_{\mathfrak{W}}$, $(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{W}})(\mathfrak{z}) \leq \mathfrak{I}^+_{\mathfrak{W}}$, $(\mathfrak{F}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{W}})(\mathfrak{z}) \geq \mathfrak{F}^+_{\mathfrak{W}}$ and $(\mathfrak{T}^-_{\mathfrak{W}} \circ \mathfrak{T}^-_{\mathfrak{W}})(\mathfrak{z}) \leq \mathfrak{T}^-_{\mathfrak{W}}$, $(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{W}})(\mathfrak{z}) \geq \mathfrak{I}^-_{\mathfrak{W}}$, , $(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{W}})(\mathfrak{z}) \leq \mathfrak{F}^-_{\mathfrak{W}}$. Therefore, $\mathfrak{W} \sqsubseteq \mathfrak{W}\bar{\circ}\mathfrak{W}$. Thus, $\mathfrak{W} = \mathfrak{W}\bar{\circ}\mathfrak{W}$.

Conversely, Let $\mathfrak{L}$ be a left ideal of $\mathfrak{R}$. Then by Theorem 2.17, $\chi_{\mathfrak{L}} = (\chi^+_{\mathfrak{L}}, \chi^-_{\mathfrak{L}})$ is a NSBF LId of $\mathfrak{R}$. By supposition and Lemma 4.1, we have

$$\begin{aligned}1 &= (\mathfrak{T}^+_{\chi_{\mathfrak{L}^2}})(\mathfrak{z}) = (\mathfrak{T}^+_{\chi_{\mathfrak{L}}} \bar{\circ} \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &= \mathfrak{T}^+_{\chi_{\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}0 &= ((\mathfrak{I}^+_{\chi_{\mathfrak{L}^2}})(\mathfrak{z}) = (\mathfrak{I}^+_{\chi_{\mathfrak{L}}} \bar{\circ} \mathfrak{I}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &= \mathfrak{I}^+_{\chi_{\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}1 &= (\mathfrak{F}^+_{\chi_{\mathfrak{L}^2}})(\mathfrak{z}) = (\mathfrak{F}^+_{\chi_{\mathfrak{L}}} \bar{\circ} \mathfrak{F}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &= \mathfrak{F}^+_{\chi_{\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

and

$$\begin{aligned}-1 &= (\mathfrak{T}^-_{\chi_{\mathfrak{L}^2}})(\mathfrak{z}) = (\mathfrak{T}^-_{\chi_{\mathfrak{L}}} \bar{\circ} \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &= \mathfrak{T}^-_{\chi_{\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}0 &= ((\mathfrak{I}^-_{\chi_{\mathfrak{L}^2}})(\mathfrak{z}) = (\mathfrak{I}^-_{\chi_{\mathfrak{L}}} \bar{\circ} \mathfrak{I}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &= \mathfrak{I}^-_{\chi_{\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}-1 &= (\mathfrak{F}^-_{\chi_{\mathfrak{L}^2}})(\mathfrak{z}) = (\mathfrak{F}^-_{\chi_{\mathfrak{L}}} \bar{\circ} \mathfrak{F}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &= \mathfrak{F}^-_{\chi_{\mathfrak{L}}}(\mathfrak{z}).\end{aligned}$$

Thus $\mathfrak{z} \in \mathfrak{L}^2$. Hence $\mathfrak{L}^2 = \mathfrak{L}$. By Lemma 4.2, we have $\mathfrak{R}$ is left quasi regular. ■

Lemma 4.4. *[22] A semigroup $\mathfrak{R}$ is a left quasi-regular semigroup if and only if $\mathfrak{K} \cap \mathfrak{L} \subseteq \mathfrak{K}\mathfrak{L}$, for every ideal $\mathfrak{K}$ and left ideal $\mathfrak{L}$ of $\mathfrak{R}$.*

Theorem 4.5. *A semigroup $\mathfrak{R}$ is a left quasi-regular semigroup if and only if $\mathfrak{W} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{W}\bar{\circ}\mathfrak{B}$, for every NSBF Id $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ and every NSBF LId $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ of $\mathfrak{R}$.*

Proof: Assume that $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ is a NSBF Id and $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ is a NSBF LId of $\mathfrak{R}$. Let $\mathfrak{z} \in \mathfrak{R}$. Since $\mathfrak{S}$ is left quasi regular, there exist $\mathfrak{p}, \mathfrak{q} \in \mathfrak{R}$ such that $\mathfrak{z} = \mathfrak{p}\mathfrak{z}\mathfrak{q}\mathfrak{z}$. Thus,

$$\begin{aligned}(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &\geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{q}\mathfrak{z}) \\ &\geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{T}^+_{\mathfrak{W}} \cap \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &\leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{q}\mathfrak{z}) \\ &\leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{I}^+_{\mathfrak{W}} \cap \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{F}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &\geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{q}\mathfrak{z}) \\ &\geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{F}^+_{\mathfrak{W}} \cap \mathfrak{F}^+_{\mathfrak{W}})(\mathfrak{z}),\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{T}^-_{\mathfrak{W}} \circ \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &\leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{q}\mathfrak{z}) \\ &\leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{T}^-_{\mathfrak{W}} \cap \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &\geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{q}\mathfrak{z}) \\ &\geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{I}^-_{\mathfrak{W}} \cap \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &\leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{q}\mathfrak{z}) \\ &\leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{F}^-_{\mathfrak{W}} \cap \mathfrak{F}^-_{\mathfrak{W}})(\mathfrak{z}),\end{aligned}$$

Hence, $(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}) \geq (\mathfrak{T}^+_{\mathfrak{W}} \cap \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z})$ $(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}) \leq (\mathfrak{I}^+_{\mathfrak{W}} \cap \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z})$ $(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z}) \geq (\mathfrak{F}^+_{\mathfrak{W}} \cap \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z})$ and $(\mathfrak{T}^-_{\mathfrak{W}} \circ \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}) \leq (\mathfrak{T}^-_{\mathfrak{W}} \cap \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z})$ $(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}) \geq (\mathfrak{I}^-_{\mathfrak{W}} \cap \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z})$ $(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{z}) \leq (\mathfrak{F}^-_{\mathfrak{W}} \cap \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{z})$ Therefore, $\mathfrak{W} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{W} \bar{\circ} \mathfrak{B}$.

Conversely, let $\mathfrak{K}$ and $\mathfrak{L}$ be a Id and a LId of $\mathfrak{S}$. Then by Theorem 2.17, $\chi_{\mathfrak{K}} = (\chi^+_{\mathfrak{K}}, \chi^-_{\mathfrak{K}})$ and $\chi_{\mathfrak{L}} = (\chi^+_{\mathfrak{L}}, \chi^-_{\mathfrak{L}})$ is a NSBF Id and a NSBF LId of $\mathfrak{R}$. By supposition and Lemma 4.1, we have

$$\begin{aligned}1 &= \mathfrak{T}^+_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{T}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &\sqsubseteq (\mathfrak{T}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{T}^+_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}0 &= \mathfrak{I}^+_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{I}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &\sqsubseteq (\mathfrak{I}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{I}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{I}^+_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}1 &= \mathfrak{F}^+_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &\sqsubseteq (\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{F}^+_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

and

$$\begin{aligned}-1 &= \mathfrak{T}^-_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{T}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &\sqsubseteq (\mathfrak{T}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{T}^-_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}0 &= \mathfrak{I}^-_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{I}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &\sqsubseteq (\mathfrak{I}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{I}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{I}^-_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}-1 &= \mathfrak{F}^-_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{F}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\ &\sqsubseteq (\mathfrak{F}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{F}^-_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),\end{aligned}$$

Thus, $\mathfrak{z} \in \mathfrak{K}\mathfrak{L}$. Hence, $\mathfrak{K} \cap \mathfrak{L} \subseteq \mathfrak{K}\mathfrak{L}$. Therefore, by Lemma 4.4, $\mathfrak{R}$ is left quasi regular. ■

Lemma 4.6. *[22] A semigroup $\mathfrak{S}$ is a left quasi-regular semigroup if and only if $\mathfrak{K} \cap \mathfrak{L} \subseteq \mathfrak{K}\mathfrak{L}$, for every ideal $\mathfrak{K}$ and bi-ideal $\mathfrak{B}$ of $\mathfrak{S}$.*

Theorem 4.7. *Let S be a semigroup. Then the following are equivalent:*

(1) *$\mathfrak{R}$ is a left quasi-regular semigroup,*
(2) *$\mathfrak{W} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{W} \bar{\circ} \mathfrak{B}$, for every NSBF Id $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ and every NSBF QId $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ of $\mathfrak{R}$,*
(3) *$\mathfrak{W} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{W} \bar{\circ} \mathfrak{B}$, for every NSBF Id $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ and every NSBF BId $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ of $\mathfrak{R}$.*

Proof: Assume that $\mathfrak{W} = (\mathfrak{W}^-, \mathfrak{W}^+)$ is a NSBF Id and $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ is a NSBF QId of $\mathfrak{R}$. Let $\mathfrak{z} \in \mathfrak{R}$. Since $\mathfrak{S}$ is left quasi regular, there exist $\mathfrak{p}, \mathfrak{q} \in \mathfrak{R}$ such that $\mathfrak{z} = \mathfrak{p}\mathfrak{z}\mathfrak{q}\mathfrak{z}$. Thus,

$$\begin{aligned}(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &\geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}\mathfrak{q}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{z}) \\ &\geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{q}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{z}) \\ &\geq \mathfrak{T}^+_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{T}^+_{\mathfrak{W}} \cap \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &\leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}\mathfrak{q}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}) \\ &\leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{q}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}) \\ &\leq \mathfrak{I}^+_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{I}^+_{\mathfrak{W}} \cap \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{F}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{n})\} \\ &\geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}\mathfrak{q}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{z}) \\ &\geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}\mathfrak{q}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{z}) \\ &\geq \mathfrak{F}^+_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{F}^+_{\mathfrak{W}} \cap \mathfrak{F}^+_{\mathfrak{W}})(\mathfrak{z}),\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{T}^-_{\mathfrak{W}} \circ \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &\leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}\mathfrak{q}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{z}) \\ &\leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{q}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{z}) \\ &\leq \mathfrak{T}^-_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{T}^-_{\mathfrak{W}} \cap \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}),\end{aligned}$$

$$\begin{aligned}(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &= \bigvee_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{y}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{n})\} \\ &\geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}\mathfrak{q}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{z}) \\ &\geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{q}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{z}) \\ &\geq \mathfrak{I}^-_{\mathfrak{W}}(\mathfrak{z}) \wedge \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{I}^-_{\mathfrak{W}} \cap \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}),\end{aligned}$$

$$
\begin{aligned}
(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{(\mathfrak{y},\mathfrak{n})\in F_{\mathfrak{z}}} \{\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{n})\} \\
&= \bigwedge_{(\mathfrak{y},\mathfrak{z})\in F_{(\mathfrak{p}\mathfrak{z})(\mathfrak{q}\mathfrak{z})}} \{\mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{y}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{n})\} \\
&\leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{p}\mathfrak{z}\mathfrak{q}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{z}) \\
&\leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}\mathfrak{q}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{z}) \\
&\leq \mathfrak{F}^-_{\mathfrak{W}}(\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{z}) = (\mathfrak{F}^-_{\mathfrak{W}} \cap \mathfrak{F}^-_{\mathfrak{W}})(\mathfrak{z}),
\end{aligned}
$$

Hence, $(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}) \geq (\mathfrak{T}^+_{\mathfrak{W}} \cap \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z})$ $(\mathfrak{I}^+_{\mathfrak{W}} \circ \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}) \leq (\mathfrak{I}^+_{\mathfrak{W}} \cap \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z})$ $(\mathfrak{T}^+_{\mathfrak{W}} \circ \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z}) \geq (\mathfrak{F}^+_{\mathfrak{W}} \cap \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z})$ and $(\mathfrak{T}^-_{\mathfrak{W}} \circ \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}) \leq (\mathfrak{T}^-_{\mathfrak{W}} \cap \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z})$ $(\mathfrak{I}^-_{\mathfrak{W}} \circ \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}) \geq (\mathfrak{I}^-_{\mathfrak{W}} \cap \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z})$ $(\mathfrak{F}^-_{\mathfrak{W}} \circ \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{z}) \leq (\mathfrak{F}^-_{\mathfrak{W}} \cap \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{z})$ Therefore, $\mathfrak{W} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{W} \bar{\circ} \mathfrak{B}$.

$(2) \Rightarrow (3)$ This is obvious because every NSBF QId is a NSBF BId of $\mathfrak{R}$.

$(3) \Rightarrow (1)$ Let $\mathfrak{K}$ and $\mathfrak{L}$ be a Id and a BId of $\mathfrak{R}$. Then by Theorem 2.17, $\chi_{\mathfrak{K}} = (\chi^+_{\mathfrak{K}}, \chi^-_{\mathfrak{K}})$ and $\chi_{\mathfrak{L}} = (\chi^+_{\mathfrak{L}}, \chi^-_{\mathfrak{L}})$ is a NSBF Id and a NSBF QId of $\mathfrak{R}$. By supposition and Lemma 4.1, we have

$$
\begin{aligned}
1 &= \mathfrak{T}^+_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{T}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\
&\sqsubseteq (\mathfrak{T}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{T}^+_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathfrak{I}^+_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{I}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\
&\sqsubseteq (\mathfrak{I}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{I}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{I}^+_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),
\end{aligned}
$$

$$
\begin{aligned}
1 &= \mathfrak{F}^+_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\
&\sqsubseteq (\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{F}^+_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),
\end{aligned}
$$

and

$$
\begin{aligned}
-1 &= \mathfrak{T}^-_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{T}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\
&\sqsubseteq (\mathfrak{T}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{T}^-_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathfrak{I}^-_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{I}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\
&\sqsubseteq (\mathfrak{I}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{I}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{I}^-_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),
\end{aligned}
$$

$$
\begin{aligned}
-1 &= \mathfrak{F}^-_{\chi_{\mathfrak{K}\mathfrak{L}}}(\mathfrak{z}) = (\mathfrak{F}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \\
&\sqsubseteq (\mathfrak{F}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = \mathfrak{F}^-_{\chi_{\mathfrak{K}\cap\mathfrak{L}}}(\mathfrak{z}),
\end{aligned}
$$

Thus, $\mathfrak{z} \in \mathfrak{K}\mathfrak{L}$. Hence, $\mathfrak{K} \cap \mathfrak{L} \subseteq \mathfrak{K}\mathfrak{L}$. Therefore, by Lemma 4.6, $\mathfrak{G}$ is left quasi regular. ■

V. Conclusion

This paper has presented the concept of an NSBF B Id and NSBF GB Id, NSBF IN Id and NSBF Ids that has been discussed and shown to coincide with quasi regular. Furthermore, we characterize qausi regular semigroups in terms NSBF Id. Further, we extend to NSBF bi-interior ideal, NSBF Qausi-bi-ideals, fuzzy A-ideals, and algebraic systems. The study of NSBF set in semigroup theory opens up a new area of research and paves the way for further investigation in this field.

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