Diagnosability of Lexicographic Product of Wheels and Paths under the PMC Model

Bu Chen, Feng Li

***Abstract*— Diagnosability is a critical metric for evaluating the fault diagnosis capabilities of interconnection networks in multiprocessor systems. Accurate assessment of diagnosability requires system-level fault diagnosis models, which play a key role in the design of new interconnection networks. In this paper, we introduce a novel network, denoted as $P_m \circ W_n$, which represents the lexicographic product of a wheel and a path. Under the PMC model, we prove that the diagnosability of $P_m \circ W_n$ is $3 + n$ and its h-edge tolerable diagnosability is $3 + n - h$ for $0 \leq h < 3 + n$, $m \geq 4$, and $n \geq 7$. These results reveal that $P_m \circ W_n$ exhibits strong fault diagnosis capabilities. Furthermore, the lexicographic product offers a promising approach to designing interconnection network architectures for large-scale multiprocessor systems.**

***Index Terms*—Lexicographic product, Diagnosability, PMC model, Multiprocessor system, Wheels, Paths.**

I. Introduction

A multiprocessor system is a computing architecture composed of multiple processors that share memory and operate collaboratively. By leveraging parallel computing, such systems enable multitasking, enhanced data processing throughput, and improved scalability. This offers computational power that far surpasses traditional single-processor systems. These capabilities make multiprocessor systems particularly effective in handling complex tasks, including advanced data analysis, machine learning, and real-time applications. Consequently, they serve as the fundamental technological support for high-performance computing domains such as scientific simulations, big data processing, and engineering computations. In a multiprocessor system, all processors are interconnected via dedicated interconnection networks. These networks play a critical role in determining the system's fault diagnosis capability and fault tolerance. Additionally, they shape communication efficiency and reliability. A well-designed interconnection network can greatly enhance the system's diagnosability, allowing for efficient fault detection, isolation, and recovery. This ensures the system's sustained computational performance and reliability, thereby maximizing its potential in high-demand computing environments.

In the early days of multiprocessor systems, the number of processors was relatively small, and these processors were typically interconnected using simple networks such as bus networks. However, the inherent limitations of bus networks significantly constrained the parallel processing capabilities of such systems, restricting them to relatively basic tasks. Moreover, processor failures often caused disruptions to the entire system's operation, since fault diagnosis was typically conducted manually. The concept of fault diagnosis capability refers to a system's ability to detect and precisely locate internal faults. An interconnection network with robust fault diagnosis capability enables the system to quickly and accurately identify faults, thereby streamlining system maintenance and ensuring uninterrupted operation. The diagnosability of a system provides a quantitative metric for evaluating its fault diagnosis capability. Specifically, if a system can accommodate up to t faulty processors while accurately diagnosing all of them without requiring component replacements, the system's diagnosability is defined as t.

To accommodate the increasing data volume, multiprocessor systems have adopted more advanced interconnection networks through various approaches, such as graph products. Some of them such as mesh networks and hypercubes are constructed by Cartesian products. These interconnection networks not only increase the number of processors but also significantly enhance the system's parallel processing and fault diagnosis capabilities. However, as the number of processors continues to grow, diagnosing faulty processors becomes increasingly difficult. The emergence of system-level diagnostic models has significantly improved this issue. Among these models, the most commonly used are the PMC model proposed by Preparata, Metze, and Chien [1]. In the PMC model, topological graphs are commonly used to visually represent the connections and relationships between processors and links in a system. For instance, assuming a graph $G = (V(G), E(G))$ represents a multiprocessor system, $V(G)$ represents all the processors in the system, while $E(G)$ denotes all links connecting the processors. Each edge in G represents a testing process. In the following context, 'vertices' can represent processors, and 'edges' can represent links. Test results can be represented by 1 or 0. If $(a, b) = 1 (= 0)$, it indicates that vertex a considers vertex b to be faulty (fault-free). In [2], Chang et al. used graph theory to transform complex networks into graphs and analyzed the diagnosability of regular networks under the PMC model and MM* model. Following this, the diagnosability of more networks has been determined, such as hypercubes and enhanced hypercubes [3], Möbius hypercubes [4], crossed hypercubes [5], and star graphs [6].

As systems have grown increasingly complex, the nature of faults has expanded beyond simple single-processor failures to more intricate scenarios, including hybrid faults, which refer to the simultaneous failures of both links and

Manuscript received November 26, 2024; revised April 12, 2025. This work was supported in part by the National Natural Science Foundation of China (No.11551002), Natural Science Foundation of Qinghai Province(No.2019 ZJ-7093), students' innovation and entrepreneurship training program of Qinghai Normal University under $qhnucxcy$2024024.

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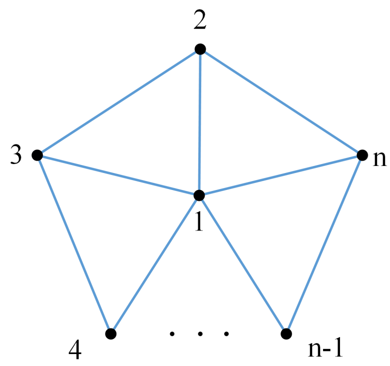

Fig. 1. An n-order wheel W_n

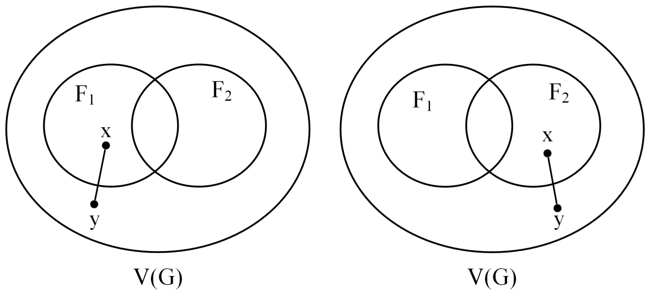

Fig. 2. Two distinguishable vertex sets F_1 and F_2

processors. To address such hybrid fault scenarios, Zhu et al. [7] introduced the concept of h-edge tolerable diagnosability. This concept is particularly well-suited for analyzing and managing hybrid faults. Since its introduction, the h-edge tolerable diagnosability of various specialized networks, such as regular networks, has been extensively explored and studied [8], [9], and [10]. Another notable advancement is conditional diagnosability, first proposed by Lai et al. [11]. This concept has garnered significant attention in recent years, becoming a popular topic of research [12], [13], [14], [15], [16], [17], [18], and [19].

Nowadays, graph theory is commonly used to study most properties of specific graphs or networks, as detailed in [26], [27] and [28]. To determine the diagnosability of a system using graph theory, it is essential to first understand some fundamental concepts. In a graph G, the number of vertices and edges are denoted by $|V(G)|$ and $|E(G)|$, respectively, where $|V(G)|$ is also called the order of G. Let a and b be any two vertices in G. If $(a,b) \in E(G)$, then a and b are said to be adjacent, and the edge (a,b) is considered incident with both a and b.

The set of all vertices adjacent to a vertex a in graph G is denoted by $N_G(a)$ and is referred to as the neighborhood of a. The number of edges incident with a is called the degree of vertex a, denoted by $d_G(a)$. In graph G, the vertex with the smallest degree is known as the minimum degree vertex, while the vertex with the largest degree is referred to as the maximum degree vertex. These are represented by $\delta(G) = \min\{d_G(x)|x \in V(G)\}$ and $\Delta(G) = \max\{d_G(x)|x \in V(G)\}$, respectively.

Paths and wheels are special types of graphs. A path is composed of distinct vertices connected by edges, where both the edges and vertices are unique. If a path consists of m vertices, denoted as $p_1, p_2, \cdots, p_{m-1}, p_m$, it is represented as P_m. In P_m, the vertices p_i and p_j are connected by an edge if and only if $i-j = \pm 1$. A wheel, denoted as W_n, consists of a cycle of length $n-1$ $(n \geq 4)$ and an additional central vertex that is directly connected to every vertex in the cycle (see Fig. 1). Given an edge set $E \subset E(G)$, the graph obtained by removing all edges in E from G is denoted as $G - E$. Similarly, given a vertex set $Q \subset V(G)$, the graph obtained by removing the vertices in Q along with all edges incident to these vertices is denoted as $G - Q$. For any two vertex subsets $F_1, F_2 \subset V(G)$, the symmetric difference between F_1 and F_2 is denoted as $F_1 \triangle F_2$ and is defined as $F_1 \triangle F_2 = (F_1 - F_2) \cup (F_2 - F_1) = (F_1 \cup F_2) - (F_1 \cap F_2)$. The set of all test results produced by a multiprocessor system is referred to as syndromes. Let $W \subset V(G)$ represent a set of faulty vertices. The syndromes produced by W are denoted as $\sigma(W)$. Similarly, let $F \subset V(G)$ be another set of faulty vertices. If $\sigma(W) \cap \sigma(F) = \emptyset$, then W and F are said to be distinguishable under the PMC model; otherwise, they are considered indistinguishable.

Similar to the Cartesian product, the lexicographic product is another significant method for constructing large networks from smaller ones while preserving certain properties of the smaller networks. Let two undirected finite graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be considered as factors. The lexicographic product of G_1 and G_2 is denoted as $G_1 \circ G_2$, with $V(G_1 \circ G_2) = V(G_1) \times V(G_2)$. In $G_1 \circ G_2$, two distinct vertices (x_1, y_1) and (x_2, y_2), where $x_1, x_2 \in V(G_1)$ and $y_1, y_2 \in V(G_2)$, are adjacent if and only if one of the following conditions is satisfied: $x_1 = x_2$ and $(y_1, y_2) \in E(G_2)$, or $(x_1, x_2) \in E(G_1)$.

In [20], [21], and [22], various properties of the lexicographic product have been studied in greater detail. Interconnection networks generated by the Cartesian product of simple graphs, such as hypercubes and grid networks, are widely employed in multiprocessor systems. However, based on recent research on lexicographic products, it has become evident that, in certain respects, the lexicographic product outperforms the Cartesian product. Consequently, it is reasonable to hypothesize that interconnection networks constructed using the lexicographic product could contribute to improving and optimizing multiprocessor systems. To explore the application of the lexicographic product in multiprocessor systems, this paper investigates the lexicographic product of a path and a wheel, $P_m \circ W_n$. It is determined that, under the PMC model, the diagnosability of $P_m \circ W_n$ is $3+n$, and its h-edge tolerable diagnosability is $3+n-h$ when $0 \leq h < 3+n$. MATLAB simulations were conducted to compare these properties with those of the hypercube. The experimental results demonstrate that the fault diagnosis capability of $P_m \circ W_n$ is superior to that of the hypercube.

II. Main Results

The previous section has outlined the essential concepts required for the subsequent proofs. For additional details

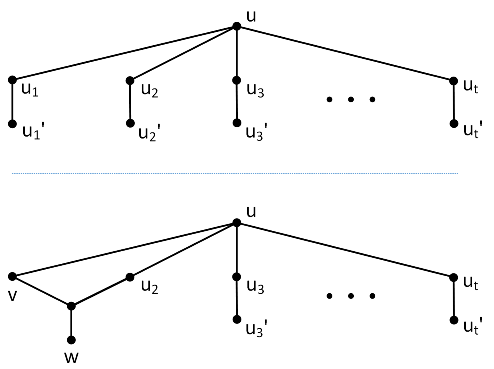

Fig. 3. Two types of t-order extending star

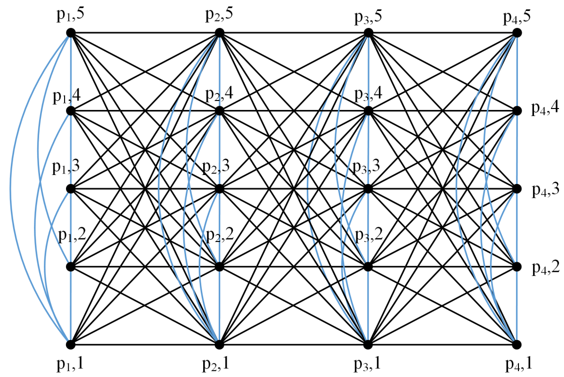

Fig. 4. The structure of $P_5 \circ W_5$

on these concepts, refer to [23]. Lemma 1 and Definition 2 play a fundamental role in determining the diagnosability of interconnection networks.

Lemma 1. *[1] Given a multiprocessor system $G = (V, E)$, let $F_1, F_2 \subset V(G)$ be any two distinct sets of faulty vertices in G. Under the PMC model, F_1 and F_2 are distinguishable if and only if there exists a vertex $y \in V - (F_1 \cup F_2)$ such that $N_G(y) \cap (F_1 \triangle F_2) \neq \emptyset$ (see Fig. 2).*

Definition 2. *[2] In a multiprocessor system $G = (V, E)$, consider two distinct sets of faulty vertices F_1 and F_2. If $|F_1| \leq t$, $|F_2| \leq t$, and F_1 and F_2 are distinguishable, then the system is t-diagnosable. The maximum integer t for which the system G remains t-diagnosable under the PMC model is called the diagnosability of the system and is denoted by $t(G)$.*

Lemma 1 and Definition 2 transform the problem of determining a network's diagnosability into the task of representing the system as a graph and identifying two distinguishable faulty sets within the graph that include the maximum number of vertices.

Definition 3. *[7] Given a multiprocessor system $G = (V, E)$ and two non-negative integers h and t, let $A \subseteq E(G)$ represent the set of faulty edges in G, where $|A| \leq h$. For any two distinct faulty sets F_1 and F_2 with $|F_1| \leq t$ and $|F_2| \leq t$, if F_1 and F_2 are distinguishable under the PMC model in $G - A$, then G is said to be h-edge tolerable t-diagnosable under the PMC model. The maximum integer t such that G is h-edge tolerable t-diagnosable is defined as the h-edge tolerable diagnosability of G, denoted by $t_h(G)$.*

In [25], Hsu and Tan proposed two innovative structures, referred to as the extending star (illustrated in Fig. 3). These structures establish an elegant connection between the diagnosability of the network and its vertices.

Lemma 4. *[25] For a multiprocessor system $G = (V, E)$, if an extending star of order t is rooted at any vertex in G, then the diagnosability of G is t.*

Before determining the diagnosability of $P_m \circ W_n$, it is essential to first examine its structure. To facilitate this, we will perform an in-depth examination using a specific example. Let $V(P_5) = \{p_1, p_2, p_3, p_4, p_5\}$ and $V(W_5) = \{1, 2, 3, 4, 5\}$, where the central vertex of W_5 is 1. The structure of $P_5 \circ W_5$ is illustrated in Fig. 4. By examining Fig. 4 and combining it with the construction method of the lexicographic product, we can obtain $P_m \circ W_n$ by replacing m vertices in P_m with m copies of W_n and then connecting each vertex in every W_n to every vertex in the neighboring W_n. According to the definitions of wheels and paths, we have $\delta(W_n) = 3$ and $\Delta(W_n) = n - 1$, while $\delta(P_m) = 1$ and $\Delta(P_m) = 2$. Thus,

By examining Fig. 4 and considering the construction method of the lexicographic product, we can obtain $P_m \circ W_n$ by replacing the m vertices of P_m with m copies of W_n, and then connecting each vertex in every W_n to every vertex in the neighboring W_n. According to the definitions of W_n and P_m, we have $\delta(W_n) = 3$ and $\Delta(W_n) = n - 1$, while $\delta(P_m) = 1$ and $\Delta(P_m) = 2$. Thus,

$$\delta(P_5 \circ W_5) = \delta(W_5) + \delta(P_5) \times |V(W_5)| = 8. \quad (1)$$

and,

$$\Delta(P_5 \circ W_5) = \Delta(W_5) + \Delta(P_5) \times |V(W_5)| = 14. \quad (2)$$

From this, we can derive Corollary 5.

Corollary 5. *If $G = P_m \circ W_n$, then $\delta(G) = 3 + n$ and $\Delta(G) = 3n - 1$.*

Next, we will prove the diagnosability and h-edge tolerable diagnosability of $P_m \circ W_n$.

Corollary 6. *Given a multiprocessor system $G = P_m \circ W_n$ under the PMC model, $t(P_m \circ W_n) \leq 3 + n$.*

Proof: Let x be a vertex with the minimum degree in $P_m \circ W_n$. According to Corollary 5, we have $\delta(P_m \circ W_n) = d_{P_m \circ W_n}(x) = 3 + n$. By utilizing the neighborhood of x and x itself, we construct two vertex sets, F_1 and F_2. Specifically, we define $F_1 = \{x\} \cup N_{P_m \circ W_n}(x)$ and $F_2 = N_{P_m \circ W_n}(x)$.

It is evident that $|F_1| = 3 + n + 1$ and $|F_2| = 3 + n$. Then $F_1 \triangle F_2 = \{x\}$ and $F_1 \cup F_2 = \{x\} \cup N_{P_m \circ W_n}(x)$. We can conclude that $N_{P_m \circ W_n}(F_1 \triangle F_2) \cap (P_m \circ W_n - F_1 \cup F_2) = \emptyset$. Obviously F_1 and F_2 are indistinguishable vertex sets according to Lemma 1, and by Definition 2, we have $t(P_m \circ W_n) \leq 3 + n$.

It is evident that $|F_1| = 3 + n + 1$ and $|F_2| = 3 + n$. Furthermore, we have $F_1 \triangle F_2 = \{x\}$ and $F_1 \cup F_2 = \{x\} \cup N_{P_m \circ W_n}(x)$. Consequently, we can conclude that $N_{P_m \circ W_n}(F_1 \triangle F_2) \cap (P_m \circ W_n - F_1 \cup F_2) = \emptyset$. Clearly, F_1 and F_2 are indistinguishable according to Lemma 1. Therefore, by Definition 2, we obtain $t(P_m \circ W_n) \leq 3 + n$. ■

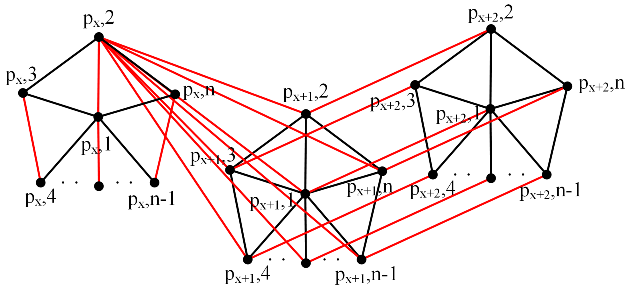

Fig. 5. An $n+3$-order extending star of Case 1

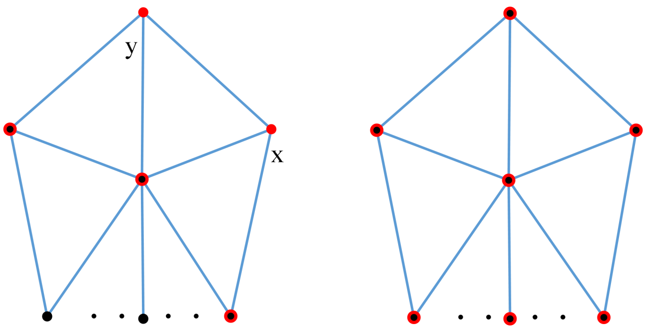

Fig. 7. The minimal neighborhood set($xy \in E\left(P_m \circ W_n - A\right)$)

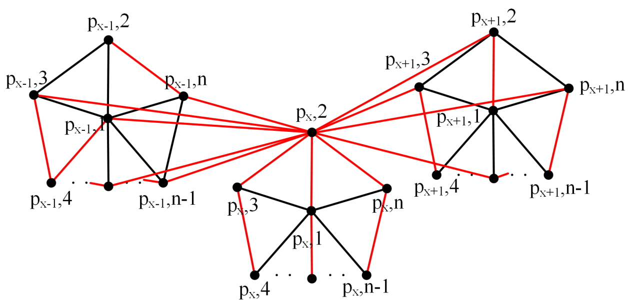

Fig. 6. An $n+3$-order extending star of Case 2

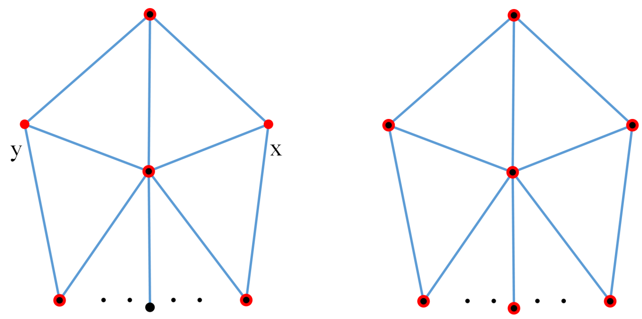

Fig. 8. The minimal neighborhood set($xy \notin E\left(P_m \circ W_n - A\right)$)

Corollary 6 provides an upper bound for $t\left(P_m \circ W_n\right)$. Next, we aim to establish its lower bound to determine the exact value of $t\left(P_m \circ W_n\right)$.

Theorem 7. *Given a multiprocessor system $G = P_m \circ W_n$, when $m \geq 4$ and $n \geq 7$, under the PMC model, we have $t\left(P_m \circ W_n\right) = 3 + n$.*

Proof: By Lemma 4, to determine the lower bound of $t\left(P_m \circ W_n\right)$, it is sufficient to identify the minimum order of the extending star that can be formed at each vertex in $P_m \circ W_n$. As shown in Fig. 5, when $m \geq 4$ and $n \geq 7$, based on the properties of the lexicographic product, $P_m \circ W_n$ can be divided into m segments of W_n, denoted as xW_n, where $1 \leq x \leq m$.

There are two distinct cases suitable for constructing an extending star of order $n+3$.

Case 1: In any subgraph xW_n within $P_m \circ W_n$, we can select a vertex of minimum degree, denoted as u. By the properties of W_n, the vertex u can form a 3-order extending star rooted at itself within the subgraph xW_n. Furthermore, according to the properties of the lexicographic product, the corresponding vertices in $(x+1)W_n$ and $(x+2)W_n$ are adjacent, thereby forming an n-order extending star rooted at u. Consequently, the vertex u can construct an $n+3$-order extending star, as illustrated in Fig. 5.

Case 2: If n is even, an n-order extending star rooted at u can be formed by connecting to the adjacent vertices in subgraphs $(x-1)W_n$ and $(x+1)W_n$. Conversely, if n is odd, an extending star of order $\lceil n/2 \rceil$ can be constructed through the adjacent subgraph $(x-1)W_n$, while an extending star of order $\lfloor n/2 \rfloor$ can be constructed through $(x+1)W_n$. In summary, the vertex u can construct an $n+3$-order extending star, as illustrated in Fig. 6. In summary, an extending star of at least order $n+3$ can be constructed at any vertex in $P_m \circ W_n$. According to Lemma 4, we conclude that $t\left(P_m \circ W_n\right) \geq 3+n$. Combined with the result of Corollary 6, Theorem 7 is thus proved. ■

In practical applications, multiprocessor systems face extremely large, complex, and difficult tasks, and the scale of the systems themselves is immense. Thus, system failures are inevitable. If only the processors experience failures, the diagnosability can effectively measure the system's fault diagnosis capability at that time. However, in real-world scenarios, link failures are also unavoidable, and the h-edge tolerable diagnosability proposed by Zhu et al. [7] can better reflect the fault diagnosis capability of multiprocessor systems under hybrid faults of links failures and processors failures. In practical applications, multiprocessor systems are required to handle extremely large, complex, and challenging tasks, and the systems themselves often operate on a massive scale. As a result, system failures are inevitable. When only processor failures occur, diagnosability serves as an effective metric to evaluate the system's fault diagnosis capability. However, in real-world scenarios, link failures are also unavoidable. The h-edge tolerable diagnosability, proposed by Zhu et al. [7], provides a more comprehensive measure of a multiprocessor system's fault diagnosis capability under hybrid faults, encompassing both link and processor failures simultaneously.

In $P_m \circ W_n\,(m \geq 4, n \geq 7)$, if all the links of a minimum-degree vertex fail, that vertex becomes isolated. Since an isolated processor cannot be diagnosed by the

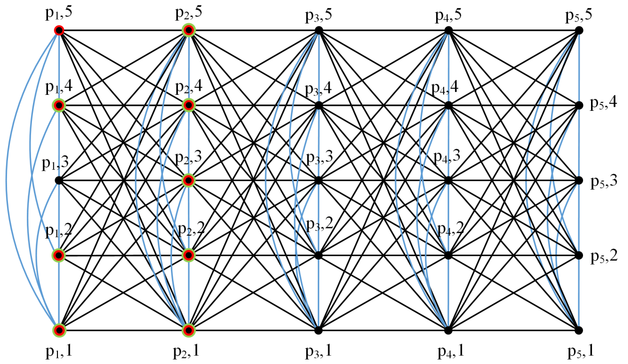

Fig. 9. Indistinguishable faulty vertex sets A and B in $P_5 \circ W_5$

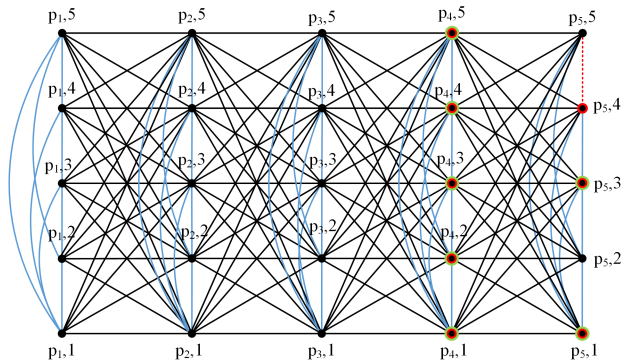

Fig. 10. Two indistinguishable faulty vertex sets in $P_5 \circ W_5 - (p_5, 4, p_5, 5)$

system, such link failures have the greatest impact on system performance. To address this issue, we define a worst-case scenario where link failures in a multiprocessor system preferentially occur at the processor with the minimum degree. In the following discussion, we assume that the number of faulty edges h is less than $3 + n$. The h-edge tolerable diagnosability in this worst-case scenario effectively measures the system's minimum fault diagnosis capability when link failures occur anywhere in the system. Next, we present the h-edge tolerable diagnosability for $P_m \circ W_n\,(m \geq 4, n \geq 7)$.

Corollary 8. *Given a multiprocessor system $P_m \circ W_n$, when $0 \leq h < 3+n$ and $m \geq 4$, $n \geq 7$, in the worst-case scenario, we have $t_h\left(P_m \circ W_n\right) \leq 3 + n - h$.*

Proof: We will analyze two cases based on the value of h:

Case 1: When $h = 0$, no link failures occur in the system. In this case, we have $t_h\left(P_m \circ W_n\right) = t\left(P_m \circ W_n\right)$. Based on the proof above, it follows that when $h = 0$, $t_h\left(P_m \circ W_n\right) \leq 3 + n - h = 3 + n$.

Case 2: When $0 < h < 3 + n$, let $A \subset E\left(P_m \circ W_n\right)$ represent the set of faulty links in the system, and assume $|A| \leq 3 + n$. Let x be a vertex with the minimum degree in $P_m \circ W_n$. In the worst-case scenario, the links incident with the minimum-degree vertex fail first, resulting in $d_{P_m \circ W_n - A}\left(x\right) = 3 + n - h$. Define $B = N_{P_m \circ W_n - A}\left(x\right)$ and $C = N_{P_m \circ W_n - A}\left(x\right) \cup \{x\}$. Then, we have $|B| = 3 + n - h$ and $|C| = 4 + n - h$, with $B \triangle C = \{x\}$. Based on the construction of B and C, it follows that $\left(B \triangle C\right) \cap \left(P_m \circ W_n - A - B \cup C\right) = \emptyset$. Thus, by Lemma 1, B and C are indistinguishable. Combining these results with Definition 3, we conclude that when $0 < h < 3 + n$, $t_h\left(P_m \circ W_n\right) \leq 3 + n - h$. ■

Theorem 9. *Given a multiprocessor system $P_m \circ W_n$, when $0 \leq h < 3+n$ and $m \geq 4$, $n \geq 7$, in the worst-case scenario, we have $t_h\left(P_m \circ W_n\right) = 3 + n - h$.*

Proof: By Definition 2 and the result of Corollary 8, it follows that we only need to prove that either B or C contains at least $3 + n - h + 1$ vertices. Under the PMC model, assume $A \subset E\left(P_m \circ W_n\right)$ represents the set of all faulty links in $P_m \circ W_n$, and $|A| \leq h$. Let B, $C \subset V\left(P_m \circ W_n\right)$ denote two distinct indistinguishable faulty vertex sets in $P_m \circ W_n - A$, satisfying $|B| < 3 + n - h + 1$ and $|C| < 3 + n - h + 1$. Since $|A| < 3 + n$, by the definition of lexicographic product, $P_m \circ W_n - A$ remains a connected graph. Furthermore, since B and C are indistinguishable, by Lemma 1, we have $N_{P_m \circ W_n - A}\left(B \triangle C\right) \cap \left(P_m \circ W_n - A - B \cup C\right) = \emptyset$. Clearly, $N_{P_m \circ W_n - A}\left(B \triangle C\right) \subseteq \left(B \cap C\right)$, which implies $|B \cap C| \geq |N_{P_m \circ W_n - A}\left(B \triangle C\right)|$ and $|B \triangle C| + |B \cap C| \geq |B \triangle C| + |N_{P_m \circ W_n - A}\left(B \triangle C\right)|$.

Next, we analyze the value of $|B \triangle C|$ by examining different cases.

Case 1: If $|B \triangle C| = 1$, without loss of generality, let $B \triangle C = B - C = \{x\}$, in this situation, obviously $B \cap C = C$. Therefore, we have

$$\begin{aligned} |B| &= |B - C| + |B \cap C| \\ &\geq 1 + |N_{P_m \circ W_n - A}\left(B \triangle C\right)| \\ &\geq 1 + |N_{P_m \circ W_n - A}\left(x\right)| - |A| \\ &\geq 1 + 3 + n - h. \end{aligned} \tag{3}$$

Therefore, when $|B \triangle C| = 1$, it follows that $|B| \geq 3 + n - h + 1$.

Case 2: If $|B \triangle C| = 2$, without loss of generality, let $B \triangle C = \{x, y\}$. In this case, we need to consider whether x and y are adjacent. If $xy \in E\left(P_m \circ W_n - A\right)$, based on the structural analysis of the network in the previous section, the scenario where the number of neighboring vertices of x and y is minimized is shown in Fig.7 (the neighboring vertices of x and y are highlighted with red circles), so $|N_{P_m \circ W_n - A}\left(B \triangle C\right)| \geq (3 + n) - h$. If $xy \notin E\left(P_m \circ W_n - A\right)$, then according to Fig.8, it is clear that $|N_{P_m \circ W_n - A}\left(B \triangle C\right)| > (3 + n) - h$. Since $N_{P_m \circ W_n - A}\left(B \triangle C\right) \subseteq \left(B \cap C\right)$, we can conclude:

$$\begin{aligned} |B| + |C| &= |B \cup C| + |B \cap C| \\ &= |B \triangle C| + |B \cap C| + |B \cap C| \\ &\geq 2 + 2 \times |N_{P_m \circ W_n - A}\left(B \triangle C\right)| \\ &> 2 + 2 \times \{(3 + n) - h\} \\ &> 2 + 2 \times (3 + n) - 2h. \end{aligned} \tag{4}$$

Obviously it follows that $|B| \geq 3 + n - h + 1$ and $|C| \geq 3 + n - h + 1$.

Case 3: When $|B \triangle C| \geq 3$, in this case, if $|B| \leq 3+n-h$ and $|C| \leq 3+n-h$, obviously, $|B \cap C| < 3+n-h$. Based on the structural analysis of $P_m \circ W_n$, when $0 \leq h < 3+n$, $m \geq 4$, and $n \geq 7$, $|N_{P_m \circ W_n - A}\left(B \triangle C\right)| \geq 3 + n - h$, contradict to $N_{P_m \circ W_n - A}\left(B \triangle C\right) \subseteq \left(B \cap C\right)$. So when $|B \triangle C| \geq 3$,

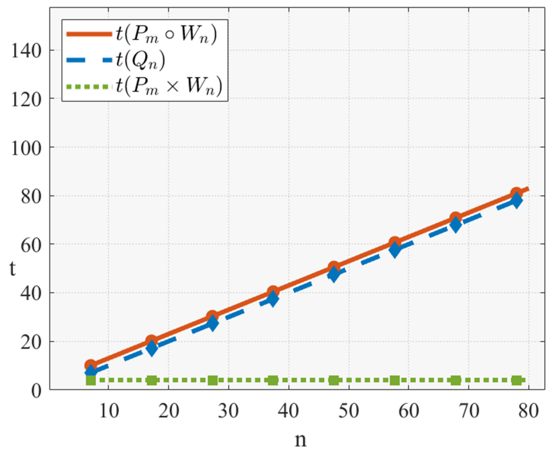

Fig. 11. Diagnosability of $P_m \circ W_n$, $P_m \times W_n$ and Q_n

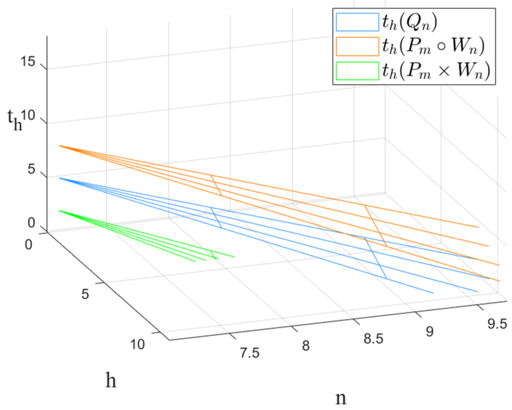

Fig. 12. The t_h of $P_m \circ W_n$, $P_m \times W_n$ and Q_n

we have $|B| \geq 3 + n - h + 1$ and $|C| \geq 3 + n - h + 1$. In summary, Theorem 9 is proven. ■

III. Application And Numerical Simulation

When a failure occurs in a multiprocessor system, it is crucial to diagnose faulty processors quickly and adjust task allocation in parallel computing to maintain normal operation. Therefore, a strong fault diagnosis capability is a fundamental requirement for a high-quality multiprocessor system. Modern high-performance multiprocessor systems incorporate complex interconnection networks to connect a large number of processors. When certain processors or links within the system fail, manual handling significantly reduces the system's operational efficiency. The only effective solution is system-level fault diagnosis, autonomously performed by the system itself. Consequently, studying the diagnosability of interconnection networks is essential to ensure stable and efficient operation. Graph products serve as a key method for constructing new interconnection networks. In this paper, we focus on the interconnection network $P_m \circ W_n$, generated by the lexicographic product of a path and a wheel. Specifically, we examine its diagnosability and h-edge tolerable diagnosability. To conclude, we will illustrate our findings with a concrete example.

Example. Given a multiprocessor system $P_5 \circ W_5$, its diagnosability is 8. In the worst-case scenario, when $0 \leq h < 8$, its h-edge tolerable diagnosability is $8 - h$.

First, we provide the structure of $P_5 \circ W_5$, as shown in Fig.10. If the vertices enclosed by the red circle in Fig.10 form a faulty vertex set A, with $|A| = 9$, and the vertices enclosed by the green circle form a faulty vertex set B, with $|B| = 8$, then clearly $A \triangle B = \{p_1, 5\}$, and $N_{P_5 \circ W_5}(A \triangle B) \cap (P_5 \circ W_5 - A \cup B) = \emptyset$, making A and B indistinguishable. Obviously, in Fig. 10, any two distinct faulty vertex sets with a vertex count less than or equal to 8 are distinguishable. Therefore, $t(P_5 \circ W_5) = 8$.

Since, in the worst-case scenario, link failures occur preferentially on the edges incident with the minimum-degree vertex, let us illustrate the case when $h = 1$. Taking the minimum-degree vertex $p_5, 4$ in $P_5 \circ W_5$ as an example, suppose the edge $(p_5, 4, p_5, 5)$ fails, as shown in Fig. 9. In Fig. 9, the red dashed line $(p_5, 4, p_5, 5)$ represents the faulty link. The vertices enclosed by the red circle form a faulty set A with $|A| = 8$, while the vertices enclosed by the green circle form a faulty set B with $|B| = 7$. In this case, $A \triangle B = \{p_5, 4\}$, and $N_{P_5 \circ W_5 - (p_5,4,p_5,5)}(A \triangle B) \cap V(P_5 \circ W_5 - (p_5, 4, p_5, 5) - A \cup B) = \emptyset$.

Therefore, based on Lemma 1, A and B are indistinguishable faulty vertex sets. Furthermore, as illustrated in Fig. 9, any two distinct faulty vertex sets with a vertex count less than or equal to 7 are distinguishable. This observation leads to the conclusion that, when $h = 1$, $t_1(P_5 \circ W_5) = 7$, with the proof for other values of h following a similar approach.

In fact, graph products can generate many network structures with excellent properties, such as the hypercube Q_n generated by the Cartesian product. Q_n not only exhibits an efficient topological structure but also possesses excellent fault diagnosis capabilities. These features make it widely used in the construction of interconnection networks for large-scale parallel computing systems. In fact, the diagnosability and h-edge tolerable diagnosability of the hypercube have been extensively studied, as presented in [7] and [24], respectively. Building upon this foundation, we conducted simulations to evaluate the fault diagnosis performance of the network structure generated by the lexicographical product. Specifically, we tested $P_m \circ W_n$, $P_m \times W_n$, and Q_n using Matlab, with the results presented in Fig.11 and Fig.12. The simulation results, when $0 \leq h < 3 + n$ and $m \geq 4$, $n \geq 7$, reveal that $P_m \circ W_n$ outperforms Q_n in terms of both diagnosability and h-edge tolerable diagnosability. This demonstrates the potential of the lexicographic product as a promising method for designing interconnection networks and highlights its potential for further research.

In multiprocessor systems, if each processor is connected to every other processor via physical links, the resulting network provides a complete communication path, offering low communication latency and load balancing advantages. These features significantly enhance the system's parallel computing capabilities. However, the hardware and cost burden associated with such a network structure is exceedingly high, and it substantially increases energy consumption, making it less practical for real-world applications. In contrast, the $P_m \circ W_n$ network structure offers a more efficient solution. Each processor maintains close connections with its neighboring processors, while long-distance links improve overall communication efficiency. The layered connection design not only effectively reduces communication latency but also enhances fault diagnosis capabilities. Furthermore, the existence of multiple paths ensures system redundancy, enabling the network to remain operational even in the event of failures. This network structure is particularly well-suited for high-performance computing scenarios, such as distributed deep learning, distributed data processing, large-scale data centers, distributed neural network training, and social network analysis. However, it may not be ideal for small-scale tasks or applications that are highly sensitive to network overhead.

IV. Conclusion

For high-performance multiprocessor systems, excellent fault diagnosis capability is indispensable. Researching methods to develop interconnection networks with outstanding diagnosability is therefore of significant importance. This paper establishes that the diagnosability of the lexicographic product of a path and a wheel, $P_m \circ W_n\,(m \geq 4,\ n \geq 7)$, under the PMC model, is $3+n$. Additionally, for $0 \leq h < 3+n$, its h-edge tolerable diagnosability, which is suitable for hybrid fault scenarios, is $3+n-h$. A comparative simulation with Q_n was conducted, and the results demonstrate that, when n is the same, $P_m \circ W_n$ exhibits superior fault diagnosis capabilities. This finding suggests that lexicographic products are a promising approach for designing large-scale interconnection networks. But when implemented at scale, $P_m \circ W_n$ introduces significant hardware costs and operational complexity, which necessitates weighing these trade-offs against application-specific requirements. Future research will focus on investigating the fault diagnosability of lexicographic products involving more specialized graphs, as well as exploring the strengths and limitations of the lexicographic product as a method for generating new network structures.

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