Fuzzy Generalization of ZZ Transform

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Abstract—**This study introduces a novel fuzzy transform, developed as a generalization of the ZZ transform, to effectively solve first-order fuzzy differential equations. We provide a thorough theoretical foundation, including detailed discussions of essential properties and associated theorems, to clarify the principles underlying this fuzzy transform. The efficacy and versatility of the proposed fuzzy ZZ transform method are demonstrated through illustrative examples, highlighting its ability to simplify fuzzy differential equations into solvable algebraic forms. Our findings suggest that this generalized fuzzy transform significantly enhances analytical precision, offering reliable solutions for practical problems within fuzzy environments. This work contributes meaningfully to the existing literature, opening avenues for broader applications and deeper investigation into fuzzy integral transforms and their utilization in real-world scenarios.**

Index Terms—**Fuzzy number, differential equation, integral transform, ZZ transform.**

I. Introduction

In recent years, there has been growing interest in studying the stability, existence of solutions, and analytical approaches to various integral and differential equations, especially those involving random kernels, fractional derivatives, and fuzzy environments. For instance, in their work on Volterra integral equations with random kernels, Qazza, Hatamleh, and Alodat examined crucial aspects related to solution stability [1]. Qazza and Hatamleh further investigated the existence of solutions in semi-linear abstract differential equations with infinite B-chains of the characteristic sheaf [2]. Altaie et al. applied the Homotopy Analysis Method in a fuzzy setting to tackle partial differential equations [3], while Saadeh et al. derived analytical solutions for coupled Hirota–Satsuma and KdV equations [4]. Several authors, including Hazaymeh et al. and Qawasmeh et al., have made notable contributions to numerical radius inequalities [5], [6], and Alzahrani et al. explored effective methods for analyzing chaotic circuit models using novel fractional derivatives [7]. More recently, Hazaymeh et al. established a perturbed Milne's Quadrature Rule with Lp-error estimates [8]. Additionally, the work of Gharib and Saadeh has provided valuable insights into the reduction of self-dual Yang–Mills equations to the Sinh–Poisson form, offering exact solutions [9].

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In recent decades, fuzzy differential equations have found extensive applications across diverse scientific and engineering disciplines due to their flexibility in modeling uncertainty in complex real-world problems. Motivated by the broad utility of these equations, this research introduces an innovative fuzzy transform aimed at effectively solving first-order fuzzy differential equations. The foundational concepts of fuzzy differential equations date back to the pioneering work of Chang and Zadeh [10], who initially defined the fuzzy derivative. Subsequent advancements were made by Kandel and Byatt [11], who formally established fuzzy differential equations. Abbasbandy and Allahviranloo [12] developed numerical approaches to tackle fuzzy differential equations dynamically, while Seikkala [13] further generalized fuzzy derivatives by extending the classical Hukuhara derivative. Notably, comprehensive discussions on generalized fuzzy derivatives are found in works such as those by Bede and Gal [14], which Bede et al. [15] study. Building upon these foundational contributions, several scholars have proposed fuzzy analogs of classical integral transforms commonly used in crisp contexts, such as the Fuzzy Emad-Falih and fuzzy Aboodh transforms [16], [17]. Additionally, practical applications, such as cost estimation using two-dimensional fuzzy systems, have also been explored by researchers including Samer et al.[18]. These prior contributions collectively provide the theoretical foundation upon which our new fuzzy transform is developed, extending the analytical tools available for practical and theoretical exploration of fuzzy differential equations.

II. Fundamental Preliminaries

To ensure clarity and completeness, we present the essential concepts and theorems relevant to our research in this area. These foundational elements are crucial for understanding the theoretical framework upon which our study is built.

Definition 2.1: [19] By $\mathbb{R}$, the set of all real numbers is represented as, the mapping $D : \mathbb{R} \to [0, 1]$ is fuzzy number if it fulfills

1) D is upper semi-continuous.
2) D is fuzzy convex, i.e., $D(\sigma X + (1-\sigma) Y) \geq \min\{D(X), D(Y)\}$,for all $X, Y \in \mathbb{R}$ and $\sigma \in [0, 1]$.
3) D is normal i.e., $\exists\, X_0 \in \mathbb{R}$ for which $D(X) = 1$.
4) $supp(D) = X \in \mathbb{R}; D(X) > 0$, and $cl(Supp\ (D))$ is compact.

Let ϖ be the set of all fuzzy number on $\mathbb{R}$. The σ -level set of a fuzzy number $D \in \varpi, 0 \leq \sigma \leq 1$ denoted by $[D]_\sigma$ is defined as

$$[D]_\sigma = \begin{cases} \{X \in \mathbb{R}, D(X) \geq \sigma\}, & 0 \leq \sigma \leq 1 \\ cl\,(Supp\ \ (D)), & \sigma = 0 \end{cases}$$

Done $[D]_\sigma = [\overline{D}_\sigma, \underline{D}_\sigma]$, so the σ –level set$[D]_\sigma$ is a bounded and closed interval for all $\sigma \in [0,1]$.

To compute the addition of two fuzzy numbers D and Ω defined on the fuzzy space ϖ, Zadeh's extension principle specifies the following relation:

$$(D \oplus \Omega)(X) = \sup_{Y \in \mathbb{R}} \left(\min\{D(Y), \Omega(X-Y)\}\right), \quad X \in \mathbb{R}.$$

Moreover, scalar multiplication for a fuzzy number is defined by:

$$(\rho \odot D)(X) = \begin{cases} D\left(\frac{X}{\rho}\right), & \rho \neq 0, \\ \hat{0}, & \rho = 0, \end{cases} \quad \text{with} \quad \hat{0} \in \varpi.$$

The following properties are universally recognized and hold true across all levels:

$$[D \oplus \Omega]_\sigma = [\ D]_\sigma + [\Omega]_\sigma\,,\ [\rho\ \odot D]_\sigma = \rho\,[\ D]_\sigma\,.$$

Definition 2.2: [20] A parametrically defined pair $(\underline{D}, \overline{D})$ represents a fuzzy number, where $\underline{D}(\sigma)$ and $\overline{D}(\sigma)$ are functions for $\sigma \in [0,1]$, satisfying the following conditions:

1) $\underline{D}(\sigma)$ is a continuous function that is non-decreasing, with a right limit at 0 and a left limit over $(0,1]$.
2) $\overline{D}(\sigma)$ is a bounded, non-increasing function, continuous from the right at 0 and from the left over $(0,1]$.
3) $\underline{D}(\sigma) \leq \overline{D}(\sigma)$ for all $\sigma \in [0,1]$.

For arbitrary $D = (\underline{D}\,(\sigma)\,,\ \overline{D}\,(\sigma))$, $D = (\underline{D}\ \ (\sigma)\,, \overline{D}\ \ (\sigma))$, $0 \leq \sigma \leq 1$ and $\beta > 0$ we define:

1) Addition $D \oplus\ \Omega = (\underline{D}\,(\sigma) + \underline{\Omega}\,(\sigma)\,,\ \overline{D}\,(\sigma) + \overline{\Omega}\ \ (\sigma))$.
2) Subtraction $D \ominus_h \Omega = (\underline{D}\,(\sigma) - \overline{\Omega}\,(\sigma)\,,\ \overline{D}\ (\sigma) - \underline{\Omega}\,(\sigma))$.
3) Multiplication

$$\begin{aligned} D \odot D = \Big(&\min\Big\{\underline{D}\,(\sigma)\,\overline{\Omega}\ \ (\sigma)\,, \underline{D}\,(\sigma)\,\underline{D}\,(\sigma)\,, \\ &\qquad \overline{D}\ \ (\sigma)\,\overline{\Omega}\ \ (\sigma)\,, \overline{D}\ \ \ (\sigma)\,\underline{D\Omega}\ \ (\sigma)\Big\}, \\ &\max\Big\{\underline{D}\,(\sigma)\,\overline{\Omega}\ \ (\sigma)\,, \underline{D}\ \ (\sigma)\,\underline{\Omega}\,(\sigma)\,, \\ &\qquad \overline{D}\ \ (\sigma)\,\overline{\Omega}\ \ (\sigma)\,, \overline{D}\ \ (\sigma)\,\underline{\Omega}\ \ (\sigma)\Big\}\Big). \end{aligned}$$

4) Scalar multiplication

$$\beta \odot D = \begin{cases} (\beta\,\underline{D}, \beta\overline{D}\,), & \beta \geq 0, \\ \left(\beta\ \overline{D}, \beta\ \underline{D}\right), & \beta < 0. \end{cases}$$

If $\beta = 1$ then $\beta \odot D = -D$.

Definition 2.3: [15] Let D and Ω be fuzzy numbers. The Hausdorff distance between these fuzzy numbers is defined as:

$$\Xi :\ \varpi \times \varpi \to [0, +\infty],$$

$$\Xi\,(D, \Omega) = \sup_{\sigma \in [0,1]} \max\left\{\left|\underline{D}(\sigma) - \underline{\Omega}(\sigma)\right|, \left|\overline{D}(\sigma) - \overline{\Omega}(\sigma)\right|\right\}.$$

where $D = \left(\underline{D}\,(\sigma)\,,\ \overline{D}\ \ (\sigma)\right),\ D = \left(\ \underline{\Omega}\ \ (\sigma)\,, \overline{\Omega}\ \ (\sigma)\ \right) \subset \mathbb{R}$ and following properties are well known:

1) $\Xi\,(D \oplus \pi, \Omega \oplus D) = \Xi\,(D\ ,\ \Omega)\,,\quad \forall\ D, \Omega, \pi \in \varpi$.
2) $\Xi\,(\beta \odot D, \beta \odot \Omega) = |\beta|\ \ \Xi\,(D, \Omega)\,, \forall D, \Omega \in \varpi, \beta \in \mathbb{R}$.
3) $\Xi\,(D \oplus \Omega, \pi \oplus h) \leq \Xi\,(D, \Omega) + \Xi\,(\pi, h)\,,$ $\forall\ D, \Omega, \pi, h \in \varpi$.
4) (Ξ, ϖ) is a complete metric space.

Definition 2.4: [20] Let $\psi : \mathbb{R} \to \varpi$ be a fuzzy-valued function. For a fixed point $X_0 \in \mathbb{R}$ and any $\epsilon > 0$, there exists $\delta > 0$ such that if $|X - X_0| < \delta$, then $\Xi\left(\psi(X), \psi(X_0)\right) < \epsilon$. In this case, D is referred to as a continuous fuzzy-valued function.

Definition 2.5: [21] A mapping $\psi : \mathbb{R} \times \varpi \longrightarrow \varpi$ is said to be continuous at a point $(\tau_0, X_0) \in \mathbb{R} \times \varpi$ if, for any fixed $\sigma_0 \in [0,1]$ and arbitrary $\varepsilon > 0$, there exists a $\delta(\varepsilon, \sigma)$ such that

$$\Xi\left(\left[\psi\left(\tau, X\right)\right]_\sigma, \left[\psi\left(\tau_0, X_0\right)\right]_\sigma\right) < \varepsilon$$

whenever

$$|\tau - \tau_0| < \delta(\varepsilon, \sigma) \quad \text{and} \quad \Xi\left([X]_\sigma, [X_0]_\sigma\right) < \delta(\varepsilon, \sigma)$$

for all $\tau \in \mathbb{R}$ and $X \in \varpi$.

Theorem 2.1: [22] Let $\Psi(\chi)$ be a fuzzy-valued function defined on $[e, \infty)$, represented as $\left(\underline{\Psi}(\chi, \sigma), \overline{\Psi}(\chi, \sigma)\right)$. For any fixed $\sigma \in [0,1]$, assume that $\underline{\Psi}(\chi, \sigma)$ and $\overline{\Psi}(\chi, \sigma)$ are Riemann-integrable on $[p, q]$. If, for every $q \geq p$, there exist two positive functions $\underline{\theta}(\sigma)$ and $\overline{\theta}(\sigma)$ such that

$$\int_p^q |\underline{\Psi}(\chi, \sigma)|\, d\chi \leq \underline{\theta}(\sigma) \quad \text{and} \quad \int_p^q \left|\overline{\Psi}(\chi, \sigma)\right| d\chi \leq \overline{\theta}(\sigma),$$

then the fuzzy number is improperly fuzzy Riemann-integrable, and $\Psi(\chi)$ is said to be improperly fuzzy Riemann-integrable on $[p, \infty]$. That is,

$$\int_p^\infty \Psi(\chi)\, d\chi = \left[\int_p^\infty \underline{\Psi}(\chi, \sigma)\, d\chi, \int_p^\infty \overline{\Psi}(\chi, \sigma)\, d\chi\right].$$

Definition 2.6: [21] Let $D, D \in \varpi$. If there exists $\pi \in \varpi$ such that $D = D \oplus \pi$, then π is referred to as the H-differential of D and is denoted by $D \ominus D$. In this context, the symbol "$\ominus$" consistently represents the H-difference. It is important to note that

$$\ominus \neq \ominus_h \quad \text{and} \quad D \ominus D \neq D + (-1)D.$$

Definition 2.7: [23] A function $\Psi : (p, q) \to \varpi$ with $\chi_0 \in (p, q)$ is said to be strongly generalized differentiable at χ_0 if there exists an element $\Psi'(\chi_0) \in \varpi$ such that one of the following conditions holds:

i. $\wp > 0$ that is adequately little, there are $\Psi\left(\chi_0\ + \wp\right) \ominus \Psi\left(\chi_0\right)$, $\Psi\left(\chi_0\ \right) \ominus \Psi\left(\chi_0 - \wp\right)$, where

$$\begin{aligned} &\lim_{\wp \to 0} \frac{\Psi\left(\chi_0\ + \wp\right) \ominus \Psi\left(\chi_0\ \right)}{\wp} \wp \\ &\quad = \lim_{\wp\ \to 0} \frac{\Psi\left(\chi_0\ \right) \ominus \Psi\left(\chi_0\ - \wp\right)}{\wp} = \Psi'(\chi_0). \end{aligned}$$

or

ii. $\forall \wp > 0$ that is adequately little, there are $\Psi\left(\chi_0\right) \ominus \Psi\left(\chi_0\ + \wp\right), \Psi\left(\chi_0\ - \wp\right) \ominus \Psi\left(\chi_0\right)$ where

$$\begin{aligned} &\lim_{\wp \to 0} \frac{\Psi\left(\chi_0\ \right) \ominus \Psi\left(\chi_0\ + \wp\right)}{-\wp} \\ &\quad = \lim_{\wp\ \to 0} \frac{\Psi\left(\chi_0\ - \wp\right) \ominus \Psi\left(\chi_0\right)}{-\wp} = \Psi'(\chi_0). \end{aligned}$$

or

iii. $\forall \wp > 0$ that is adequately little, there are $\Psi\left(\chi_0 + \wp\right) \ominus \Psi\left(\chi_0\right)$, $\Psi\left(\chi_0 - \wp\right) \ominus \Psi\left(\chi_0\right)$ where

$$\begin{aligned} &\lim_{\wp \to 0} \frac{\Psi\left(\chi_0 + \wp\right) \ominus \Psi\left(\chi_0\right)}{\wp} \\ &\quad = \lim_{\tau \to 0} \frac{\Psi\left(\chi_0 - \wp\right) \ominus \Psi\left(\chi_0\right)}{-\wp} = \Psi'(\chi_0). \end{aligned}$$

or

iv. $\forall \wp > 0$ that is adequately little, there are $\Psi(\chi_0) \ominus \Psi(\chi_0 + \wp), \Psi(\chi_0) \ominus \Psi(\chi_0 - \wp)$ where

$$\lim_{\wp \to 0} \frac{\Psi(\chi_0) \ominus \Psi(\chi_0 + \wp)}{-\wp} = \lim_{\wp \to 0} \frac{\Psi(\chi_0) \ominus \Psi(\chi_0 - \wp)}{\wp} = \Psi'(\chi_0).$$

Theorem 2.2: [24] Let $\Psi(\chi) : \mathbb{R} \to \overline{\omega}$ be a function represented as $\Psi(\chi) = \left(\underline{\Psi}(\chi,\sigma), \overline{\Psi}(\chi,\sigma)\right)$ for every $\sigma \in [0,1]$. Then:

1) If $\Psi(\chi)$ is differentiable in form (i), then $\underline{\Psi}(\chi,\sigma)$ and $\overline{\Psi}(\chi,\sigma)$ are differentiable functions, and
$$\Psi'(\chi) = \left(\underline{\Psi}'(\chi,\sigma), \overline{\Psi}'(\chi,\sigma)\right).$$
2) If $\Psi(\chi)$ is differentiable in form (ii), then $\underline{\Psi}(\chi,\sigma)$ and $\overline{\Psi}(\chi,\sigma)$ are differentiable functions, and
$$\Psi'(\chi) = \left(\overline{\Psi}'(\chi,\sigma), \underline{\Psi}'(\chi,\sigma)\right).$$

III. Fuzzy Generalization of ZZ Transform

The fuzzy version of the ZZ transform provides a powerful tool for addressing fuzzy initial and boundary value problems linked to fuzzy differential equations. By extending the ZZ transform to the fuzzy domain, it simplifies fuzzy differential equations into algebraic problems, significantly easing their solution. This transition from calculus-based operations to algebraic manipulations through transformations is referred to as operational calculus, a critical and practical branch of mathematics.

A. Generalization of ZZ Transform

Let $\Psi(\chi)$ be function defined $\forall \chi \geq 0$. The general ZZ transform of $\Psi(\chi)$ is defined as:

$$\mathbb{H}_{\mathfrak{d}}\left[\Psi(\chi)\right] = q(s,v) \int_0^\infty \Psi(\chi)\ e^{-p(s,v)\chi}\ d\chi,$$

where q and p are functions of a parameters s and v [25].

Definition 3.1: Let $\Psi(\chi)$ be a fuzzy-valued continuous function. Assume that $q(s,v)\Psi(\chi) \odot e^{-p(s,v)\chi}$ is improperly fuzzy integrable in the Riemann sense on $[0,\infty)$. Then,

$$q(s,v) \int_0^\infty \Psi(\chi) \odot e^{-p(s,v)\chi}\, d\chi$$

is referred to as the fuzzy generalization of the ZZ transform and is defined as

$$\widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right] = q(s,v) \int_0^\infty \Psi(\chi) \odot e^{-p(s,v)\chi}\, d\chi,$$

$$q(s,v) \int_0^\infty \Psi(\chi) \odot e^{-p(s,v)\chi}\, d\chi = \left(q(s,v) \int_0^\infty \underline{\Psi}(\chi,\sigma) \odot e^{-p(s,v)\chi}\, d\chi,\ q(s,v) \int_0^\infty \overline{\Psi}(\chi,\sigma) \odot e^{-p(s,v)\chi}\, d\chi \right).$$

Using the definition of the classical generalization of the ZZ transform, we obtain:

$$\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\right] = q(s,v) \int_0^\infty \underline{\Psi}(\chi,\sigma) \odot e^{-p(s,v)\chi}\, d\chi,$$

$$\mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\right] = q(s,v) \int_0^\infty \overline{\Psi}(\chi,\sigma) \odot e^{-p(s,v)\chi}\, d\chi.$$

Thus,

$$\widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right] = \left(\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\right], \mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\right]\right).$$

Theorem 3.1: Let $\Psi(\chi)$, $Y(\chi)$ be continuous fuzzy-valued functions d_1 and d_2 are constants, then

1) $\widehat{\mathbb{H}_{\mathfrak{d}}}\left[d_1 \odot \Psi(\chi)\right] = d_1 \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right]$.
2) $\widehat{\mathbb{H}_{\mathfrak{d}}}\left[(d_1 \odot \Psi(\chi)) \oplus (d_2 \odot Y(\chi))\right] = \left(d_1 \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right]\right) \oplus \left(d_2 \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[Y(\chi)\right]\right)$

Proof:

$$\begin{aligned}
\widehat{\mathbb{H}_{\mathfrak{d}}}\left[d_1 \odot \Psi(\chi)\right] &= \left(\mathbb{H}_{\mathfrak{d}}\left[d_1\underline{\Psi}(\chi,\sigma)\right], \mathbb{H}_{\mathfrak{d}}\left[d_1\overline{\Psi}(\chi,\sigma)\right]\right) \\
&= \left(q(s,v) \int_0^\infty d_1\underline{\Psi}(\chi,\sigma)\, e^{-p(s,v)\chi}\ d\chi,\ q(s,v) \int_0^\infty d_1\overline{\Psi}(\chi,\sigma)\, e^{-p(s,v)\chi} d\chi \right) \\
&= \left(d_1 q(s,v) \int_0^\infty \underline{\Psi}(\chi,\sigma)\, e^{-p(s,v)\chi}\ d\chi,\ d_1 q(s,v) \int_0^\infty \overline{\Psi}(\chi,\sigma)\, e^{-p(s,v)\chi}\ d\chi \right) \\
&= d_1 \left(q(s,v) \int_0^\infty \underline{\Psi}(\chi,\sigma)\, e^{-p(s,v)\chi}\ d\chi,\ q(s,v) \int_0^\infty \overline{\Psi}(\chi,\sigma)\, e^{-p(s,v)\chi}\ d\chi \right) \\
&= d_1 \left(\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\right], \mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\right]\right) = d_1 \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right]
\end{aligned}$$

■

(2) Suppose $\Psi(\chi) = (\underline{\Psi}(\chi,\sigma), \overline{\Psi}(\chi,\sigma)$ and $Y(\chi) = (\underline{Y}(\chi,\sigma), \overline{Y}(\chi,\sigma)$

$$\widehat{\mathbb{H}_{\mathfrak{d}}}\ \left[\Psi(\chi)\right] = q(s,v) \int_0^\infty \Psi(\chi) \odot\ e^{-p(s,v)\chi}\ d\chi,$$

$$\begin{aligned}
&\widehat{\mathbb{H}_{\mathfrak{d}}}\left[(d_1 \odot \Psi(\chi)) \oplus (d_2 \odot Y(\chi))\right] \\
&= \Big(\mathbb{H}_{\mathfrak{d}}\ \left[\ d_1\ \underline{\Psi}(\chi,\sigma) + d_2\underline{Y}(\chi,\sigma)\right], \\
&\quad \mathbb{H}_{\mathfrak{d}}\left[d_1\ \overline{\Psi}(\chi,\sigma) + d_2\overline{Y}(\chi,\sigma)\right]\Big) \\
&= \Big(q(s,v)\int_0^\infty e^{-p(s,v)\chi}\left(d_1\underline{\Psi}(\chi,\sigma) + d_2\underline{Y}(\chi,\sigma)\right)d\chi, \\
&\quad q(s,v)\int_0^\infty e^{-p(s,v)\chi}\left(d_1\overline{\Psi}(\chi,\sigma) + d_2\overline{Y}(\chi,\sigma)\right)d\chi\Big) \\
&= \Big(q(s,v)\int_0^\infty e^{-p(s,v)\chi}d_1\ \underline{\Psi}(\chi,\sigma)\,d\chi, \\
&\quad q(s,v)\int_0^\infty d_1\overline{\Psi}(\chi,\sigma)\ \ e^{-p(s,v)\chi}d\chi)\Big) \\
&+ \Big(q(s,v)\int_0^\infty e^{-p(s,v)\chi}d_2\ \underline{Y}(\chi,\sigma)\,d\chi, \\
&\quad q(s,v)\int_0^\infty d_2\overline{Y}(\chi,\sigma)\ \ e^{-p(s,v)\chi}d\chi)\Big) \\
&= d_1\Big(q(s,v)\int_0^\infty e^{-p(s,v)\chi}\ \underline{\Psi}(\chi,\sigma)\,d\chi, \\
&\quad q(s,v)\int_0^\infty \overline{\Psi}(\chi,\sigma)\ \ e^{-p(s,v)\chi}d\chi)\Big) \\
&+ d_2\Big(q(s,v)\int_0^\infty e^{-p(s,v)\chi}\ \underline{Y}(\chi,\sigma)\,d\chi, \\
&\quad q(s,v)\int_0^\infty \overline{Y}(\chi,\sigma)\ \ e^{-p(s,v)\chi}d\chi)\Big) \\
&= d_1\Big(\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\right], \mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\right]\Big) \\
&\quad + d_2\left(\mathbb{H}_{\mathfrak{d}}\left[\underline{Y}(\chi,\sigma)\right], \mathbb{H}_{\mathfrak{d}}\left[\overline{Y}(\chi,\sigma)\right]\right) \\
&= \Big(d_1 \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right]\Big) \oplus \Big(d_2 \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[Y(\chi)\right]\Big)
\end{aligned}$$

IV. Fuzzy Generalization of ZZ Transform for First -Order Fuzzy Differential Equation

Solving high-order fuzzy differential equations necessitates an analysis of the fuzzy generalization of the ZZ transform applied to the first-order derivative under the framework of generalized H-differentiability. This approach allows for the transformation of fuzzy differential equations into a more manageable algebraic form, facilitating their solution.

Theorem 4.1: Let $\Psi(\chi)$ be the primitive of $\Psi'(\chi)$ on $[0,\infty)$, and assume $\Psi(\chi)$ is an integrable fuzzy-valued function. Then:

(a) If $\Psi(\chi)$ is (i)-differentiable, then

$$\widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi'(\chi)\right] = p(s,v) \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right] \ominus q(s,v) \odot \Psi(0).$$

(b) If $\Psi(\chi)$ is (ii)-differentiable, then

$$\begin{aligned}
\widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi'(\chi)\right] &= (-q(s,v) \odot \Psi(0)) \\
&\quad \ominus \Big(-p(s,v) \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right]\Big).
\end{aligned}$$

Proof: (a) For a fixed, arbitrary $0 \le \sigma \le 1$,

$$\begin{aligned}
&p(s,v) \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right] \ominus q(s,v) \odot \Psi(0) \\
&= \Big(p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\right] \\
&- q(s,v)\underline{\Psi}(0,\sigma), p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\right] \\
&- q(s,v)\overline{\Psi}(0,\sigma)\Big).
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}'(\chi,\sigma)\right] &= p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\right] \\
&-\ q(s,v)\underline{\Psi}(0,\sigma), \mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}'(\chi,\sigma)\right] \\
&= p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\right] -\ q(s,v)\overline{\Psi}(0,\sigma).
\end{aligned}$$

Since $\Psi(\chi)$ is differentiable in the form (i) according to Theorem 2.2, the following holds:

$$\begin{aligned}
&\underline{\Psi'}(\chi,\sigma) = \underline{\Psi}'(\chi,\sigma), \overline{\Psi'}(\chi,\sigma) = \overline{\Psi}'(\chi,\sigma), \\
&\mathbb{H}_{\mathfrak{d}}\left[\ \underline{\Psi'}(\chi,\sigma)\right] = \mathbb{H}_{\mathfrak{d}}\left[\ \underline{\Psi}'(\chi,\sigma)\right] \\
&\quad = p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\ \right] - q(s,v)\underline{\Psi}(0,\sigma), \\
&\mathbb{H}_{\mathfrak{d}}\left[\ \overline{\Psi'}(\chi,\sigma)\right] = \mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}'(\chi,\sigma)\right] \\
&\quad = p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\ \right] -\ q(s,v)\overline{\Psi}(0,\sigma), \\
&p(s,v)\widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right] \ominus q(s,v)\Psi(0) \\
&\quad = \left(\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi'}(\chi,\sigma)\right], \mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi'}(\chi,\sigma)\right]\right) = \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi'(\chi)\right].
\end{aligned}$$

■

(b)

$$\begin{aligned}
&\left(-q(s,v) \odot \Psi(0)\right) \ominus \Big(-p(s,v) \odot \widehat{\mathbb{EF}}\left[\Psi(\chi)\right]\Big) \\
&= \Big(-q(s,v)\overline{\Psi}(0,\sigma) + p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\right], \\
&\quad -q(s,v)\underline{\Psi}(0,\sigma) + p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\right]\Big).
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{EF}\left[\underline{\Psi}'(\chi,\sigma)\right] &= p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\right] - q(s,v)\underline{\Psi}(0,\sigma), \\
\mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}'(\chi,\sigma)\right] &= p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\right] -\ q(s,v)\overline{\Psi}(0,\sigma).
\end{aligned}$$

Since $\Psi(\chi)$ is differentiable in the form (i) according to Theorem 2.2, the following holds:

$$\begin{aligned}
&\underline{\Psi'}(\chi,\sigma) = \overline{\Psi}'(\chi,\sigma), \overline{\Psi'}(\chi,\sigma) = \underline{\Psi}'(\chi,\sigma) \\
&\mathbb{H}_{\mathfrak{d}}\left[\ \underline{\Psi'}(\chi,\sigma)\right] = \mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}'(\chi,\sigma)\right] \\
&\quad = p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi}(\chi,\sigma)\right] -\ q(s,v)\overline{\Psi}(0,\sigma), \\
&\mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi'}(\chi,\sigma)\right] = \mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}'(\chi,\sigma)\right] \\
&\quad = p(s,v)\mathbb{H}_{\mathfrak{d}}\left[\underline{\Psi}(\chi,\sigma)\right] -\ q(s,v)\underline{\Psi}(0,\sigma),
\end{aligned}$$

$$\begin{aligned}
&(-q(s,v) \odot \Psi(0)) \ominus \Big(-p(s,v) \odot \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right]\Big) \\
&= \left(\mathbb{H}_{\mathfrak{d}}\left[\ \underline{\Psi'}(\chi,\sigma)\right], \mathbb{H}_{\mathfrak{d}}\left[\overline{\Psi'}(\chi,\sigma)\right]\right) \\
&= \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi'(\chi)\right].
\end{aligned}$$

Example 4.1: Consider a fuzzy initial value problem:

$$\Psi'(\chi) = \Psi(\chi), \quad \Psi(0,\ \sigma) = (\sigma - 1, 1 - \sigma),\ 0 \le \sigma \le 1.$$

Solution. Apply both sides' fuzzy generalization of ZZ transforms to get

$$\widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi'(\chi)\right] = \widehat{\mathbb{H}_{\mathfrak{d}}}\left[\Psi(\chi)\right].$$

Case (1) $\Psi(\chi)$ be (i)-differentiable,

$$q(s,v)\odot\widehat{\mathbb{H}_{\mathfrak{F}}}[\Psi(\chi)]\ominus q(s,v)\odot\Psi(0)=\widehat{\mathbb{H}_{\mathfrak{F}}}[\Psi'(\chi)].$$

Using upper and lower functions to have

$$\begin{aligned} q(s,v)\mathbb{H}_{\mathfrak{F}}[\underline{\Psi}(\chi,\sigma)]- \quad & q(s,v)\underline{\Psi}(0,\sigma) \\ &=\mathbb{H}_{\mathfrak{F}}[\underline{\Psi}(\chi,\sigma)], q(s,v)\mathbb{H}_{\mathfrak{F}}\left[\overline{\Psi}(\chi,\sigma)\right] \\ &\quad - q(s,v)\overline{\Psi}(0,\sigma)=\left[\overline{\Psi}(\chi,\sigma)\right], \end{aligned}$$

$$(q(s,v)-1)\,\mathbb{H}_{\mathfrak{F}}[\underline{\Psi}(\chi,\sigma)]=q(s,v)\,(\sigma-1),$$
$$(q(s,v)-1)\;\mathbb{H}_{\mathfrak{F}}\left[\overline{\Psi}(\chi,\sigma)\right]=q(s,v)(1-\sigma),$$

$$\mathbb{H}_{\mathfrak{F}}[\underline{\Psi}(\chi,\sigma)]=\frac{1}{(q(s,v)-1)}\;q(s,v)\,(\sigma-1),$$
$$\mathbb{H}_{\mathfrak{F}}\left[\overline{\Psi}(\chi,\sigma)\right]=\frac{1}{(q(s,v)-1)}q(s,v)(1-\sigma),$$

$$\underline{\Psi}(\chi,\sigma)=(\mathbb{H}_{\mathfrak{F}})^{-1}\left(\frac{1}{(q(s,v)-1)}\;q(s,v)\,(\sigma-1)\right),$$
$$\overline{\Psi}(\chi,\sigma)=(\mathbb{H}_{\mathfrak{F}})^{-1}\left(\frac{1}{(q(s,v)-1)}q(s,v)\,(1-\sigma)\right).$$

Using inverse generalization of ZZ transform

$$\underline{\Psi}(\chi,\sigma)=(\sigma-1)\;e^{\chi},\;\overline{\Psi}(\chi,\sigma)=(1-\sigma)\;e^{\chi}.$$

Case (2) $\Psi(\chi)$ be (ii)-differentiable,

$$\widehat{\mathbb{H}_{\mathfrak{F}}}[\Psi'(\chi)]=(-q(s,v)\odot\Psi(0))\ominus\left(-q(s,v)\odot\widehat{\mathbb{H}_{\mathfrak{F}}}[\Psi(\chi)]\right).$$

Using upper and lower functions, to have

$$q(s,v)\mathbb{H}_{\mathfrak{F}}[\underline{\Psi}(\chi,\sigma)]-\;q(s,v)\underline{\Psi}(0,\sigma)=\mathbb{H}_{\mathfrak{F}}\left[\overline{\Psi}(\chi,\sigma)\right],$$
$$q(s,v)\mathbb{H}_{\mathfrak{F}}\left[\overline{\Psi}(\chi,\sigma)\right]-\;q(s,v)\overline{\Psi}(0,\sigma)=\mathbb{EF}[\underline{\Psi}(\chi,\sigma)],$$

$$q(s,v)\mathbb{EF}[\underline{\Psi}(\chi,\sigma)]=q(s,v)\,(\sigma-1)+\;\mathbb{H}_{\mathfrak{F}}\left[\overline{\Psi}(\chi,\sigma)\right],$$
$$q(s,v)\mathbb{H}_{\mathfrak{F}}\left[\overline{\Psi}(\chi,\sigma)\right]=q(s,v)\,(1-\sigma)+\mathbb{H}_{\mathfrak{F}}[\underline{\Psi}(\chi,\sigma)].$$

With simple calculation and Using inverse generalization of ZZ transform, we obtained the solution of case (2)

$$\underline{\Psi}(\chi,\sigma)=(\sigma-1)\;e^{-\chi},\quad\overline{\Psi}(\chi,\sigma)=(1-\sigma)\;e^{-\chi}.$$

V. Conclusion

Using the extremely extended differentiability notion, we have developed the fuzzy generalization of ZZ transform to solve fuzzy initial-value problems for first-order linear fuzzy differential equations. This might lead to solutions whose support fluctuates over time.

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