# Degree Partition Number of Some Derived Graphs

N. Malathi, M. Bhuvaneshwari and Selvam Avadayappan

*Abstract*— **In many places, there may emerge a situation in which a certain group of individuals or components need to be partitioned into many groups in order to meet certain requirements. To investigate the characteristics and nature of a network, our mathematicians create a variety of partitioning techniques. Usage of graph theoretical method simplifies the partitioning procedure. On the vertex set of graph G, we define a partition $\pi_k$ consisting of k partition classes. $\pi_k$ is said to be a similar degree partition of G if the absolute difference of the sum of the degrees of all vertices between any two partition classes is at most 1. The largest value k across all the similar degree partition of the graph G is defined to be the degree partition number of G and is denoted by $\psi_D(G)$. This partition suits the need for partitioning any resource into groups of almost equal strength. In this paper we have established some interesting facts and theorems regarding the degree partition number of some derived graphs.**

***Index Terms***— **Degree Partition Number, Graph Partitioning, Partitions of sets.**

## I. INTRODUCTION

In this paper, we consider finite, undirected simple graphs. Graphs serve as mathematical models to analyse many concrete real-world problems. For the basic definitions and notations of graph theory, we refer the text book by Harary [5]. The vertex set and edge set of $G$ are denoted by $V(G)$ and $E(G)$ respectively. A graph G of order $p$ and size $q$ is referred to as a $(p,q)$ $graph$. (1,0) graph is called *trivial graph.*

For a vertex $v$, the number of vertices adjacent to $v$ in $G$ is called its *degree* and is denoted by $\deg_G(v)$ or simply $\deg(v)$. A vertex $v$ of G with $deg(v) = 1$ is a *pendant vertex.* If $v$ is a vertex of degree 0, then $v$ is called an *isolated vertex.* A graph G is called $r-regular$ if $deg(v) = r$ for each $v \in V(G)$. The graph is known to be $(r,k)-biregular$ if any vertex of G is of degree either $r$ or $k$. The minimum and maximum degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A graph of order $n$ is said to be *complete* if $\delta(G) = \Delta(G) = n-1$ and it is denoted by $K_n$.

The set of vertices adjacent to $v$ is called the *open neighbourhood* of $v$. It is denoted by $N(v)$. The set $N(v) \cup \{v\}$ is called the *closed neighbourhood* of $v$ which is denoted by $N[v]$. A closed path is called a *cycle.* The path and cycle with $n$ vertices are denoted by $P_n$ and $C_n$ respectively.

The *join* of two graphs $G$ and $H$, denoted by $G \vee H$ is obtained by joining each vertex of $G$ to every other vertices of $H$ by means of edges. The *wheel graph* $W_n (n \geq 4)$ is nothing but $K_1 \vee C_{n-1}$.

The cartesian product of two graphs $G$ and $H$ denoted by $G \square H$ is defined such that $V(G \square H) = V(G) \times V(H)$ and the two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if $u_1 = v_1$ and $u_2$ is adjacent to $v_2$ in $H$ or $u_1$ is adjacent to $v_1$ in $G$ and $u_2 = v_2$.

The *subdivision* of an edge $e = \{u, v\}$ in the graph $G$ is nothing but the replacement of the edge $e$ with a path of length 2. The graph obtained by subdividing all edges of $G$ is called the *subdivision graph* of $G$.

The *splitting graph* $S_p(G)$ of a graph $G$ is obtained from $G$ by adding a new vertex $v'$ for each vertex v of $G$ such that $v'$ is adjacent to all the vertices in $N(v)$.

Let $I_n$ $(n \geq 4)$ denote the irregular most graph on $n$ vertices with degree sequence $n-1, n-2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor, \ldots, 2, 1$. That is, only two vertices have same degree in $I_n$ and all the other vertices have distinct degrees.

The *Kneser graph* $K(n,k)$ is the graph whose vertices correspond to the $k-$ element subsets of a set of $n$ elements whose two vertices are adjacent if and only if the two corresponding sets are disjoint. $K(n,k)$ is a $\binom{n-k}{k}$ – regular graph of order $\binom{n}{k}$.

The $n-$ *hypercube graph* $Q_n$ is the cartesian product of $n-path$ graphs. It is a $n-regular$ graph of order $2^n$.

Graph partitioning is a process of reducing a graph to smaller graphs by partitioning its vertex set/ edge set into mutually incompatible groups. There are numerous research concepts in the literature that are based on partitioning the vertex and edge sets of a graph [2],[3],[4],[5],[7],[10],[11],[12],[13].

A *Barbell partition* of a graph G is a partition of $V(G)$ into three disjoint parts $\{R, W_1, W_2\}$ such that

1. $R$ is allowed to be an empty set, but $W_i \neq \phi$ for $i = 1,2$.
2. there are no edges between vertices in $W_1$ and $W_2$
3. for each $v \in R$, $|N_G(v) \cap W_i| \neq 1$ for $i = 1,2$.

For more details on this partitioning, one can refer [1].

This paper deals a kind of partitioning in graphs which aims at having smaller groups of vertices with almost same degree sum.

Let $\pi_k$ $(k \geq 2)$ be a partition of the vertex set $V(G)$ with the partition classes $V_1, V_2, \ldots, V_k$. The *degree sum* of the vertex class $V_i$ is defined as the sum of the degrees of the vertices in the class $V_i$ which is denoted by $D(V_i)$. $\pi_k$ is called a *similar degree partition* if the absolute difference between

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the *degree sum* of any two classes is at most 1, that is, if $|D(V_i) - D(V_j)| \leq 1$ for $1 \leq i,j \leq k$.

$\pi_k$ is called a *perfect similar degree partition* if the degree sum of the vertex classes are the same. $\pi_k$ is said to be a *maximum similar degree partition* if we cannot find a *similar degree partition* $\pi_l$ such that $l > k$ and such $k$ is defined to be the *degree partition number* of the graph G which is denoted by $\psi_D(G)$.

Here, we denote the $n$ disjoint copies of a graph $G$ by $nG$.

For interesting results on this parameter, one can refer [8],[9]. In this paper, we establish degree partition number of some subdivision graphs, spanning subgraphs and some family of derived graphs. Also, we present a family of graphs for which the maximum similar degree partition as well as the barbell partition are one and the same.

## II. Main Results

In this section, we present some interesting facts and theorems on the *degree partition number*. We first recall the following results proved in [9] which are useful in determining the *degree partition number* of various graph families.

**Theorem 1** [9] $\psi_D(G) = |V(G)|$ if and only if G is either a regular graph or a $(r, r+1)$ – biregular graph. □

**Theorem 2** [9] $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v \in V(G)} deg\ v - 1}{\Delta - 1} \right\rfloor$

**Proof** Let G be a graph with n vertices. Let $\pi_k = \{V_1, V_2, \ldots, V_k\}$ be a maximal similar degree partition of G. Then $\psi_D(G) = k$.

Clearly, there exists at least one partition say $V_1$ such that $\sum_{v_i \in V_1} deg\ v_i \geq \Delta$.

Also, $\sum_{v_i \in V_j} deg\ v_i \geq \Delta - 1$ for $j = 2,3,\ldots,k$.

Adding the above k inequalities, we get

$$\sum_{v_i \in V(G)} deg\ v_i \geq \Delta + (k-1)(\Delta - 1).$$

$$\therefore k - 1 \leq \frac{\sum_{v_i \in V_1} deg\ v_i - \Delta}{\Delta - 1} \Rightarrow k \leq \frac{\sum_{v_i \in V_1} deg\ v_i - \Delta}{\Delta - 1} + 1$$
$$\Rightarrow k \leq \frac{\sum_{v_i \in V_1} deg\ v_i - 1}{\Delta - 1}$$

Hence, $k \leq \left\lfloor \frac{\sum_{v_i \in V_1} deg\ v_i - 1}{\Delta - 1} \right\rfloor$ since k is an integer.

Always $k \geq 1$.

Thus $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} deg\ v_i - 1}{\Delta - 1} \right\rfloor$ □

**Theorem 3** [9] If $G$ has the vertices even degree only, then $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v \in V(G)} deg\ v}{\Delta} \right\rfloor$

**Proof** Let G be a graph with n vertices and degree of each vertex be even.

Let $\pi_k = \{V_1, V_2, \ldots, V_k\}$ be a maximal similar degree partition of G. Then $\psi_D(G) = k$.

Since degree of each vertex be even, $\pi_k$ should be a perfect similar degree partition of G.

Then, $\sum_{v_i \in V_j} deg\ v_i \geq \Delta$ for all $j = 1,2,3,\ldots,k$.

Adding the above k inequalities, we get

$$\sum_{v_i \in V(G)} deg\ v_i \geq k\Delta.$$

$\therefore k \leq \frac{\sum_{v_i \in V_1} deg\ v_i}{\Delta}$. Hence, $k \leq \left\lfloor \frac{\sum_{v_i \in V_1} deg\ v_i}{\Delta} \right\rfloor$ since k is an integer.

Always $k \geq 1$. Thus $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} deg\ v_i}{\Delta} \right\rfloor$. □

We also note the following facts which are vacuously true.

**Fact 4** There does not exist a graph of order $n$, $2 \leq n \leq 4$ such that $\psi_D(G) = 1$. □

**Fact 5** There does not exist a graph of order 3 such that $\psi_D(G) = 2$. □

**Fact 6** For $n \geq 3$, $\psi_D(K_1 \vee K_{n,n}) = 1$.

**Proof** The degree sequence of the graph $K_1 \vee K_{n,n}$ is given by $((n+1)^{2n}, 2n)$.

We cannot find any similar degree partition for $K_1 \vee K_{n,n}$.

Hence $\psi_D(K_1 \vee K_{n,n}) = 1$ for $n \geq 3$. □

**Fact 7** $\psi_D(W_n) = n$ for $n = 4,5$.

**Proof** As $W_4$ is $3-regular$ and $W_5$ is $(3,4)-biregular$, $\psi_D(W_n) = n$ for $n = 4,5$. □

**Theorem 8** For any wheel graph $W_n \cong K_1 \vee C_{n-1}$,

$$\psi_D(W_n) = \left.\begin{cases} 4 & if\ n \equiv 1(mod\ 3) \\ 3 & if\ n \equiv 3,8(mod\ 9) \\ 1 & if\ n \equiv 0,2,5,6(mod\ 9) \end{cases}\right\} \text{for } n \geq 6$$

**Proof** Let $V(W_n) = \{v_1, v_2, \ldots, v_n\}$ with $\deg v_1 = n-1$ and $\deg v_i = 3$ for $i = 2,3,\ldots,n$.

It can be seen that $\left\lfloor \frac{\sum_{i=1}^{n} \deg v_i - 1}{\Delta - 1} \right\rfloor = 4$.

$\therefore \psi_D(W_n) \leq 4$ for $n \geq 6$.

**Case i** Let $\psi_D(W_n) = 4$

In this case $\{v_1\}$ should be a partition class in our degree partition whose degree sum is $n-1$.

Since the degree of remaining vertices are the same, the degree sum of the other classes should be the same.

This is possible only when $n-1$ is a multiple of 3. That is $n - 1 \equiv 0(mod\ 3)$ or $n \equiv 1(mod\ 3)$.

**Case ii** Let $\psi_D(W_n) = 3$

Let $V_1 = \{v_1, v_2, \ldots, v_k\}$ be a partition class in our maximum similar degree partition.

We may see that $D(V_1) = (n-1) + 3(k-1) = n + 3k - 4$.

As we have seen above, the degree sum of the remaining two partition classes should be the same and that may be $n + 3k - 3$ or $n + 3k - 4$ or $n + 3k - 5$.

Since the sum of all the degree of vertices of $W_n$ is $4(n-1)$, the degree of the other two partition classes should be $\frac{4(n-1)-(n+3k-4)}{2} = \frac{3n-3k}{2}$.

If $\frac{3n-3k}{2} = n + 3k - 3$, then $3n - 3k = 2n + 6k - 6 \Rightarrow n - 9k + 6 = 0$

$\Rightarrow n \equiv -6(mod\ 9)$ or $n \equiv 3(mod\ 9)$

Suppose that $\frac{3n-3k}{2} = n + 3k - 4$, then $3n - 3k = 2n + 6k - 8 \Rightarrow n - 9k + 8 = 0$

$\Rightarrow n \equiv -8(mod\ 9)$ or $n \equiv 1(mod\ 9)$, which leads to Case i.

Next if $\frac{3n-3k}{2} = n + 3k - 5$, then $3n - 3k = 2n + 6k - 10 \Rightarrow n - 9k + 10 = 0$

$\Rightarrow n \equiv -1(mod\ 9)$ or $n \equiv 8(mod\ 9)$

Hence $\psi_D(W_n) = 3$ only when $n \equiv 3,8(mod\ 9)$.

**Case iii** Let $\psi_D(W_n) = 2$

Let $V_1 = \{v_1, v_2, \ldots, v_k\}$ be a partition class in our maximum similar degree partition with

$D(V_1) = n + 3k - 4$.

Again, the degree sum of the other partition class may be $n + 3k - 3$ or $n + 3k - 4$ or $n + 3k - 5$ and the degree sum of the partition classes should be $3n - 3k$.

If $3n - 3k = n + 3k - 3$, then $2n = 6k - 3$, a contradiction.

Therefore, $3n - 3k \neq n + 3k - 3$. Similarly, $3n - 3k \neq n + 3k - 5$.

Thus $3n-3k=n+3k-4$. This forces that $n\equiv 1 (mod\ 3)$ which leads to Case i.
Hence $\psi_D(W_n)\neq 2$.
For all the remaining $n$ $i.e., n\equiv 0,2,5,6 (mod\ 9)$, $\psi_D(W_n)=1$. □

**Proposition 9** Consider the family of graphs $F_n$ defined as

$$F_n=\begin{cases} K_1\vee\left(K_1\cup\left(\frac{n-2}{2}\right)K_2\right) & if\ n\ is\ even \\ K_1\vee\left(\frac{n-1}{2}\right)K_2 & if\ n\ is\ odd \end{cases}$$

Then $\psi_D(F_n)=3$ when $n\geq 4$.

**Proof** For $n=4$, a maximum similar degree partition is shown in the Fig 1.

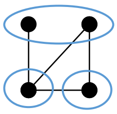


Fig 1. The graph $F_4$

For $n\geq 5$, $\psi_D(F_n)\leq 3+\frac{2}{n-2}\leq 3$ by Theorem 2.
Let $V(F_n)=\{v_1,v_2,\dots,v_{n-1},v_n\}$ with $\deg v_n=n-1$, $\deg v_i=2$ for $i=1,2,\dots,n-2$ and $\deg v_{n-1}=2\ or\ 1$ according as n is odd or even.
When n is odd,

$$\pi_3=\left\{\{v_n\},\left\{v_1,v_2,\dots,v_{\frac{n-1}{2}}\right\},\left\{v_{\frac{n-1}{2}+1},v_{\frac{n-1}{2}+2},\dots,v_{n-1}\right\}\right\}$$

forms a perfect similar degree partition with the degree sum $n-1$ and hence $\psi_D(F_n)=3$.
When n is even,

$$\pi_3=\left\{\{v_n\},\left\{v_1,v_2,\dots,v_{\frac{n-2}{2}}\right\},\left\{v_{\frac{n+1}{2}},v_{\frac{n+3}{2}},\dots,v_{n-2},v_{n-1}\right\}\right\}$$

forms a maximum similar degree partition with degree sum as $n-1,n-2,n-1$ respectively implying $\psi_D(F_n)=3$. □

**Theorem 10** For any given positive integer $n\geq 5$, there exists a graph of order $n$ whose barbell partition and the similar degree partition are the same.

**Proof** Consider the family of graphs $H_n$ defined as $H_n=$

$$\begin{cases} K_1\vee\left(2P_3\cup K_1\cup\left(\frac{n-8}{2}\right)K_2\right) & if\ n\equiv 0\ (mod\ 4) \\ K_1\vee\left(\frac{n-1}{2}\right)K_2 & if\ n\equiv 1\ (mod\ 4) \\ K_1\vee\left(K_1\cup\left(\frac{n-2}{2}\right)K_2\right) & if\ n\equiv 2\ (mod\ 4) \\ K_1\vee\left(2P_3\cup\left(\frac{n-7}{2}\right)K_2\right) & if\ n\equiv 3\ (mod\ 4) \end{cases}$$

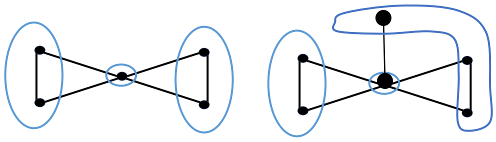
By Theorem 2, we can easily verify that $\psi_D(H_n)\leq 3$ for $n\geq 5$.
Let $V(H_n)=\{v_1,v_2,\dots,v_n\}$.

**Case i** Let $n\equiv 0\ (mod\ 4)$
In this case, $H_n$ has one vertex of degree $n-1$, one vertex of degree 1, two vertices of degree 3 and $n-4$ vertices of degree 2.
Let $\deg v_i=2$ for $i=1,2,\dots,n-4$,
$\deg v_j=3$ for $j=n-3,n-2$, $\deg v_{n-1}=1$ and $\deg v_n=n-1$.
Now, $\pi_3=\{V_1,V_2,V_3\}$ where $V_1=\{v_n\}$,
$V_2=\left\{v_1,v_2,\dots,v_{\frac{n-4}{2}},v_{n-3}\right\}$ and
$V_3=\left\{v_{\frac{n-4}{2}+1},v_{\frac{n-4}{2}+2},\dots,v_{n-4},v_{n-2},v_{n-1}\right\}$ forms a maximum similar degree partition for $H_n$ with the degree sum as $n-1,n-1,n$ respectively.

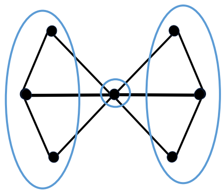
**Case ii** Let $n\equiv 1\ (mod\ 4)$
Here, $H_n$ consists of one vertex of degree $n-1$ and $n-1$ vertices of degree 2.
Let $\deg v_i=2$ for $i=1,2,\dots,n-1$ and $\deg v_n=n-1$.
Consider $\pi_3=\{V_1,V_2,V_3\}$ where $V_1=\{v_n\}$,
$V_2=\left\{v_1,v_2,\dots,v_{\frac{n-1}{2}}\right\}$ and
$V_3=\left\{v_{\frac{n-1}{2}+1},v_{\frac{n-1}{2}+2},\dots,v_{n-1}\right\}$ which forms a maximum perfect similar degree partition for $H_n$ with the degree sum as $n-1$.

**Case iii** Let $n\equiv 2\ (mod\ 4)$
It is clear that $H_n$ has one vertex of degree $n-1$, one vertex of degree 1, and $n-2$ vertices of degree 2.
Let $\deg v_i=2$ for $i=1,2,\dots,n-2$, $\deg v_{n-1}=1$ and $\deg v_n=n-1$.
Then, $\pi_3=\{V_1,V_2,V_3\}$ where $V_1=\{v_n\}$,
$V_2=\left\{v_1,v_2,\dots,v_{\frac{n-2}{2}}\right\}$ and
$V_3=\left\{v_{\frac{n-2}{2}+1},v_{\frac{n-2}{2}+2},\dots,v_{n-2},v_{n-1}\right\}$ is one of the required maximum similar degree partitions for $H_n$ with the degree sum as $n-1,n-2,n-1$ respectively.

**Case iv** Let $n\equiv 3\ (mod\ 4)$
Note that $H_n$ has one vertex of degree $n-1$, two vertices of degree 3 and $n-3$ vertices of degree 2.
Let $\deg v_i=2$ for $i=1,2,\dots,n-3$, $\deg v_j=3$ for $j=n-2,n-1$ and $\deg v_n=n-1$.
The partition $\pi_3=\{V_1,V_2,V_3\}$ with $V_1=\{v_n\}$,
$V_2=\left\{v_1,v_2,\dots,v_{\frac{n-3}{2}},v_{n-2}\right\}$ and
$V_3=\left\{v_{\frac{n-3}{2}+1},v_{\frac{n-3}{2}+2},\dots,v_{n-3},v_{n-1}\right\}$ serves as a maximum similar degree partition for $H_n$ with the degree sum as $n-1,n,n$ respectively.
Hence $\psi_D(H_n)=3$ for $n\geq 5$.
If we take $W=V_1,R_1=V_2$ and $R_2=V_3$, then in each case, the partition $\pi_3$ constitutes a barbell partition for $H_n$. □
As an illustration, the maximum similar degree partitions as well as the barbell partitions of $H_5,H_6$ and $H_7$ are given in Fig 2.



(a) *The graph* $H_5$ (b) *The graph* $H_6$



(c) *The graph* $H_7$

Fig 2. Family of Graphs having the maximum similar degree partition as same as the barbell partition

**Fact 11** The degree partition number of the subdivision graph of $K_{1,1}$ is 3 and that of $K_{1,n}$ is 5 for $n=2,3,4$. □

**Notation** If there are three conditions say $A_1, A_2, A_3$ such that
$$f(x) = \begin{cases} a_1 & if\ A_1 \\ a_2 & if\ A_2\ but\ not\ A_1 \\ a_3 & if\ A_3\ but\ not\ both\ A_1\ and\ A_2 \end{cases},$$
we use to denote
$$f(x) = \begin{cases} a_1 & if\ A_1 \\ a_2 & else\ if\ A_2 \\ a_3 & else\ if\ A_3 \end{cases}$$
□

**Theorem 12** Let $G^*$ be the graph obtained from $K_{1,n}, n \geq 5$, by subdividing k edges. Then
$$\psi_D(G^*) = \begin{cases} 2 & if\ 1 \leq k < \left\lceil \frac{n-1}{2} \right\rceil \\ 3 & else\ if\ k \leq n-2 \\ 4 & else\ if\ k \leq n \end{cases}$$

**Proof** Let $V' = \{v\}$ and $V'' = \{v_1, v_2, \dots, v_n\}$ be the bipartition of $K_{1,n}$ and $U = \{u_1, u_2, \dots, u_k\}$ be the vertices obtained by subdividing $k$ edges of $K_{1,n}$.
We may notice that $\deg_{G^*} v = n$, $\deg_{G^*} v_i = 1$ and $\deg_{G^*} u_j = 2$ for all $1 \leq i \leq n$ and $1 \leq j \leq k$.
Now, $\frac{\sum_{v \in V(G^*)} \deg v - 1}{\Delta - 1} = \frac{n+2k+n(1)-1}{n-1} = \frac{2n+2k-1}{n-1} = \frac{2(n-1)+2(k-1)+3}{n-1} \leq 2 + 2 + \frac{3}{n-1}$ as $k - 1 \leq n - 1$
So, $\frac{\sum_{v \in V(G^*)} \deg v - 1}{\Delta - 1} \leq 4 + \frac{4}{n-1}$ and hence $\psi_D(G^*) \leq 4$ for $n \geq 5$.
To find the maximum similar degree partition of $G^*$, we initiate by considering a single vertex partition class $\{v\}$ as a class in our similar degree partition.
Then $V"$ may form another partition class in the above considered similar degree partition.
Now, the possibility of getting 4 partitions will be achieved when $U$ can be partitioned into two classes of degree sum either $n$ or $n-1$. ($n+1$ is not possible as $k \leq n$.
So, for $k = n-1$ or $n$, we consider two cases.
Whenever $k$ is even
$\pi_4 = \left\{V', V", \left\{u_1, u_2, \dots, u_{\frac{k}{2}}\right\}, \left\{u_{\frac{k}{2}+1}, u_{\frac{k}{2}+2}, \dots, u_k\right\}\right\}$ forms our required maximum similar degree partition of degree sum $n, n, k, k$ respectively.
And whenever $k$ is odd,
$\pi_4 = \left\{V', V" - \{v_1, v_2\} \cup \{u_k\},\right.$
$\left.\left\{u_1, u_2, \dots, u_{\frac{k-1}{2}}, v_1\right\}, \left\{u_{\frac{k-1}{2}+1}, u_{\frac{k-1}{2}+2}, \dots, u_{k-1}, v_2\right\}\right\}$forms our required maximum similar degree partition of degree sum $n, n, k, k$ respectively.
Hence, $\psi_D(G^*) = 4$ if $n - 1 \leq k \leq n$.
For $k < n - 1$, we can have the maximum of 3 partition classes in any similar degree partition.
Now, $\pi_3 = \{V', V", U\}$ will form a maximum similar degree partition if $D(U) = n$ or $n - 1$ ($n + 1$ is not possible).
$i.e., 2k = n$ or $n - 1 \Rightarrow k = \frac{n}{2}$ or $\frac{n-1}{2}$ provided $k$ is an integer.
Next, we consider the similar degree partition with more than one vertex in all the partition classes.
In this case, first we observe that $D(V_i) > n$ for $i = 1,2,3$.
As $\sum_{v \in V(G^*)} \deg v = 2n + 2k$, $2k > n \Rightarrow k > \left\lceil\frac{n}{2}\right\rceil$ as $k$ is an integer.
Let $W_1 = \left\{u_1, u_2, \dots, u_{\left\lfloor\frac{n}{2}\right\rfloor}\right\}$ and $W_2 = \left\{u_{\left\lfloor\frac{n}{2}\right\rfloor+1}, u_{\left\lfloor\frac{n}{2}\right\rfloor+2}, \dots, u_k\right\}$.
It is evident that the sum of the degrees of the vertices in $W_1$ is $n$ or $n - 1$ according as n is even or odd.
Now, if $|W_2| = 3t$ where $t = k - \left\lfloor\frac{n}{2}\right\rfloor$, then $W_2$ can be divided into 3 partitions say $X_1, X_2, X_3$ of same cardinality $t$.
Then $\pi_3 = \{V' \cup X_1, V" \cup X_2, W_1 \cup X_3\}$ will be a maximum similar degree partition with the degree sum as $n + 2t$ each when $n$ is even and the degree sum as $n + 2t, n + 2t, n + 2t - 1$ respectively when n is odd.
Next, if $|W_2| = 3t + 1$ where $t = k - \left\lfloor\frac{n}{2}\right\rfloor$, then $W_2$ can be divided into 3 partitions say $X_1, X_2, X_3$ such that $|X_1| = t, |X_2| = t + 1$ and $|X_3| = t$.
Then $\pi_3 = \{V' \cup X_1, V" - \{v_1\} \cup X_2, W_1 \cup \{v_1\} \cup X_3\}$ is a maximum similar degree partition with the degree sum as $n + 2t, n + 2t + 1, n + 2t + 1$ respectively when $n$ is even and the degree sum as $n + 2t, n + 2t + 1, n + 2t$ respectively when n is odd.
Suppose $|W_2| = 3t + 2$ where $t = k - \left\lfloor\frac{n}{2}\right\rfloor$, then $W_2$ can be divided into 3 partitions say $X_1, X_2, X_3$ such that $|X_1| = t, |X_2| = t + 1$ and $|X_3| = t + 1$.
In this case,$\pi_3 = \{V' \cup \{v_1\} \cup X_1, V" - \{v_1\} \cup X_2, W_1 \cup X_3\}$ is a maximum similar degree partition with the degree sum as $n + 2t + 1, n + 2t + 1, n + 2t + 2$ respectively when $n$ is even and the degree sum as $n + 2t + 1$ each when n is odd.
Hence, $\psi_D(G^*) = 3$ if $\left\lceil\frac{n-1}{2}\right\rceil \leq k \leq n - 2$.
In the remaining cases, $i.e.$, for $k < \left\lceil\frac{n-1}{2}\right\rceil$,
let $W_1 = \left\{u_1, u_2, \dots, u_{\left\lfloor\frac{k}{2}\right\rfloor}\right\}$ and $W_2 = \left\{u_{\left\lfloor\frac{k}{2}\right\rfloor+1}, u_{\left\lfloor\frac{k}{2}\right\rfloor+2}, \dots, u_k\right\}$.
If $k$ is even, $\pi_2 = \{V' \cup W_1, V" \cup W_2\}$ forms a perfect maximum similar degree partition for $G^*$ with the degree sum $n + k$.
If $k$ is odd, $\pi_2 = \{V' \cup \{v_1\} \cup W_1, V" - \{v_1\} \cup W_2\}$ forms a perfect maximum similar degree partition for $G^*$ with the degree sum $n + k$.
Hence, $\psi_D(G^*) = 2$ if $1 \leq k < \left\lceil\frac{n-1}{2}\right\rceil$. □

**Theorem 13** Let G be a regular graph of odd degree $r > 4$ and order n. Let $G^*$ be a graph obtained by subdividing k edges of $G$, then
$$\psi_D\ (G^*) = \begin{cases} n + \dfrac{2k}{r-1} & if\ 2k \equiv 0\ (mod\ (r-1)) \\ n + \dfrac{2k}{r+1} & else\ if\ 2k \equiv 0\ (mod\ (r+1)) \\ \max\left\{n, \dfrac{n}{2} + \dfrac{2k}{r}\right\} & else\ if\ k \equiv 0\ (mod\ n) \\ & and\ k \equiv 0 (mod\ r) \\ n & else\ if\ k \equiv 0\ (mod\ n) \\ \dfrac{n}{2} + \dfrac{2k}{r} & else\ if\ k \equiv 0 (mod\ r) \\ 1 & otherwise \end{cases}$$

**Proof** Observe that n is even as r is odd.
$\psi_D(G^*)$ is the maximized partition number with the difference in their degree sums never exceeding 1.
$G^*$ is a graph with degree sequence $(r^n, 2^k)$ where $1 \leq k \leq \frac{nr}{2}$.
Let $V(G^*) = \{v_i, u_j\ /1 \leq i \leq n, 1 \leq j \leq k\}$ such that $\deg v_i = r$ and $\deg u_j = 2$.

To achieve the optimality in partition number, we start of considering the possibilities of having a partition class as $\{v_i\}$, for any $1 \leq i \leq n$.
As $r$ is mentioned to be odd, there is no possibility of perfect similar degree partition with vertices $u_j$'s.
Therefore, they must be classified into partition classes with degree sum to be either $r-1$ or $r+1$ but not both.
In this case, we have

$$\psi_D(G^*) = \begin{cases} n + \dfrac{2k}{r-1} & if\ 2k \equiv 0\ (mod\ (r-1)) \\ n + \dfrac{2k}{r+1} & if\ 2k \not\equiv 0\ (mod\ (r-1)) \\ & and\ 2k \equiv 0\ (mod\ (r+1)) \end{cases}$$

Next, we think of the possibility of having a partition class with both $v_i's\ and\ u_j's$.
At this time, we may get a chance of having both $k \equiv 0 (mod\ r)\ and\ k \equiv 0(mod\ n)$, the degree sum may be $r + \frac{2k}{n}$ or $2r$.
Hence, $\psi_D(G^*) = \max\left\{n, \frac{n}{2} + \frac{2k}{r}\right\}$ if $k \equiv 0(mod\ r)\ and\ k \equiv 0(mod\ n)$.
But when $k \equiv 0\ (mod\ n)$ but $k \not\equiv 0(mod\ r)$, considering optimality, the degree sum of the partition class must be $r + \frac{2k}{n}$.
Hence, $\psi_D(G^*) = n$ if $2k \not\equiv 0\ (mod\ (r \pm 1))$ and if $k \equiv 0\ (mod\ n)$ provided $k \not\equiv 0(mod\ r)$
Finally, we discuss on the probability of having partition class with at least two $v_i's$.
It must be noted that the partition, if possible, with $v_i's\ and\ u_j's$ is maximum only when there are at most two $v_j's$ in a partition class.
Hence, the partition class degree sum is optimum when it is $2r$.
This gives us that, $\psi_D\ (G^*) = \frac{n}{2} + \frac{2k}{r}$ if
$k \equiv 0(mod\ r)\ but\ k \not\equiv 0(mod\ n)$ and $2k \not\equiv 0\ (mod\ (r \pm 1))$.
All the other possible partitions will not be maximum as they are covered by any one of the above said cases. Hence, we conclude that

$$\psi_D\ (G^*) = \begin{cases} n + \dfrac{2k}{r-1} & if\ 2k \equiv 0\ (mod\ (r-1)) \\ n + \dfrac{2k}{r+1} & else\ if\ 2k \equiv 0\ (mod\ (r+1)) \\ \max\left\{n, \dfrac{n}{2} + \dfrac{2k}{r}\right\} & else\ if\ k \equiv 0\ (mod\ n) \\ & and\ k \equiv 0(mod\ r) \\ n & else\ if\ k \equiv 0\ (mod\ n) \\ \dfrac{n}{2} + \dfrac{2k}{r} & else\ if\ k \equiv 0(mod\ r) \\ 1 & otherwise \end{cases}$$

□

**Theorem 14** Let G be regular graph of even degree $r \geq 4$ and order n. Let $G^*$ be a graph obtained by subdividing k edges of G, then

$$\psi_D(G^*) = \begin{cases} n + \dfrac{2k}{r} & if\ 2k \equiv 0\ (mod\ r) \\ n & else\ if\ k \equiv 0\ (mod\ n) \\ 1 & otherwise \end{cases}$$

**Proof** Since all vertices are of even degree, only perfect similar degree partition exists if any.
We know $\psi_D(G^*) \leq \left\lfloor \frac{\sum_{v \in V(G^*)} \deg v}{r} \right\rfloor$ (Theorem 3)
$\Rightarrow \psi_D(G^*) \leq n + \frac{2k}{r}$.
Also, no two vertices of $G$ can be in a same partition class of any maximum similar degree partition (if such partition exists), otherwise the maximum cannot be achieved.
Hence the degree sum of every partition class in a similar degree partition must be either $r\ or\ r + 2l$ if such partition exists.
**Case i** Let the degree sum of the partition class be $r$.
In this case, every single vertex of $G$ should form a partition class and the newly added vertices should form $2k$ partition classes.
This is possible only when $2k \equiv 0(mod\ r)$.
So, $\psi_D(G^*) = n + \frac{2k}{r}$ if $2k \equiv 0(mod\ r)$.
**Case ii** Let the degree sum of the partition class be $r + 2l$.
As discussed in the above theorem, this is possible only when $l = \frac{k}{n}.\ i.e., k \equiv 0(mod\ n)$.
In all the remaining cases $\psi_D(G^*) = 1$. □

**Corollary 15** Let $G$ be a graph obtained from $K_n$ by subdividing $k$ edges. Then

$$\psi_D(G) = \begin{cases} n + \dfrac{2k}{n-2} & if\ 2k \equiv 0\ (mod\ (n-2)) \\ n + \dfrac{2k}{n-1} & else\ if\ 2k \equiv 0\ (mod\ (n-1)) \\ n + \dfrac{2k}{n} & else\ if\ 2k \equiv 0\ (mod\ n) \\ \max\left\{n, \dfrac{n}{2} + \dfrac{2k}{n-1}\right\} & else\ if\ k \equiv 0\ (mod\ n), \\ & n \equiv 0(mod\ 2) and \\ & k \equiv 0(mod\ (n-1)) \\ n & else\ if\ k \equiv 0\ (mod\ n) \\ \dfrac{n}{2} + \dfrac{2k}{n-1} & else\ if\ k \equiv 0(mod\ (n-1)) \\ & and\ n \equiv 0\ (mod\ 2) \\ 1 & otherwise \end{cases}$$

□

**Corollary 16** Let $G$ be a graph obtained from Kneser graph $K(n,k)$ by subdividing $t$ edges. Then for $k \leq \frac{p}{2}$, whenever $\binom{n-k}{k}$ is odd,

$$\psi_D\ (G) = \begin{cases} \dbinom{n}{k} + \dfrac{2t}{k} & if\ 2t \equiv 0\ (mod\ k) \\ \dbinom{n}{k} + \dfrac{2t}{k+2} & else\ if\ 2t \equiv 0\ (mod\ (k+2)) \\ \max\left\{\dbinom{n}{k}, \dfrac{\binom{n}{k}}{2} + \dfrac{2t}{k+1}\right\} & else\ if\ t \equiv 0\ \left(mod\ \dbinom{n}{k}\right) \\ & and\ t \equiv 0(mod\ (k+1)) \\ \dbinom{n}{k} & else\ if\ t \equiv 0\ \left(mod\ \dbinom{n}{k}\right) \\ \dfrac{\binom{n}{k}}{2} + \dfrac{2t}{k+1} & else\ if\ t \equiv 0(mod\ (k+1)) \\ 1 & otherwise \end{cases}$$

And whenever $\binom{n-k}{k}$ is even,

$$\psi_D(G)=\begin{cases}\binom{n}{k}+\dfrac{2t}{\binom{n-k}{k}} & if\ 2t\equiv 0\left(mod\ \binom{n-k}{k}\right)\\ \binom{n}{k} & else\ if\ t\equiv 0\left(mod\ \binom{n}{k}\right)\\ 1 & otherwise\end{cases}$$

**Proof** It may be noted that $K(n,k)$ is a null graph whenever $k>\frac{n}{2}$.

Let $\binom{n-k}{k}$ be odd.

We know that $\binom{n}{r}$ is odd iff $r=n-1$ and $n$ is odd.

$\binom{n-k}{k}$ is odd iff $(n-k)-k=1\Rightarrow n=2k+1$ and $n-k$ is odd. So, $n-k=k+1$ and $\binom{n-k}{k}=\binom{k+1}{k}=k+1$.

$\therefore \psi_D(G)$

$$=\begin{cases}\binom{n}{k}+\dfrac{2t}{k} & if\ 2t\equiv 0\ (mod\ k)\\ \binom{n}{k}+\dfrac{2t}{k+2} & else\ if\ 2t\equiv 0\left(mod\ (k+2)\right)\\ \max\left\{\binom{n}{k},\dfrac{\binom{n}{k}}{2}+\dfrac{2t}{k+1}\right\} & else\ if\ t\equiv 0\left(mod\ \binom{n}{k}\right)\\ & and\ t\equiv 0\left(mod\ (k+1)\right)\\ \binom{n}{k} & else\ if\ t\equiv 0\left(mod\ \binom{n}{k}\right)\\ \dfrac{\binom{n}{k}}{2}+\dfrac{2t}{k+1} & else\ if\ t\equiv 0\left(mod\ (k+1)\right)\\ 1 & otherwise\end{cases}$$

The result is obvious when $\binom{n-k}{k}$ is even. □

**Corollary 17** Let $G$ be a graph obtained from $Q_n$ by subdividing $k$ edges. Then

$\psi_D(G)$

$$=\begin{cases}2^n+\dfrac{2k}{n-1} & if\ 2k\equiv 0\left(mod\ (n-1)\right)\\ 2^n+\dfrac{2k}{n} & else\ if\ 2k\equiv 0\ (mod\ n)\\ 2^n+\dfrac{2k}{n+1} & else\ if\ 2k\equiv 0\left(mod\ (n+1)\right)\\ \max\left\{2^n,2^{n-1}+\dfrac{2k}{n}\right\} & else\ if\ k\equiv 0(mod\ n),\\ & k\equiv 0\ (mod\ 2^n)\\ & and\ n\equiv 0\ (mod\ 2)\\ 2^n & else\ if\ k\equiv 0(mod\ 2^n)\\ 2^{n-1}+\dfrac{2k}{n} & else\ if\ k\equiv 0(mod\ n)\\ & and\ n\equiv 0\ (mod\ 2)\\ 1 & otherwise\end{cases}$$

□

**Theorem 18** Let $G$ be a regular graph of degree $r\geq 2$ and order n. If $G^*$ is a graph obtained from $G$ by removing 2 edges incident with the same vertex, then

$$\psi_D(G^*)=\begin{cases}n-1 & if\ r=2,3\\ \dfrac{n-1}{2} & if\ n\equiv 1\ (mod\ 2)\ and\ r=4\\ 3 & if\ n=6\\ 1 & otherwise\end{cases}$$

**Proof** The degree sequence of $G^*$ is $(r-2,(r-1)^2,r^{n-3})$. Let $V(G^*)=\{v_1,v_2,\dots,v_n\}$ with $\deg_{G^*} v_1=r-2$, $\deg_{G^*} v_2=\deg_G^* v_3=r-1$ and $\deg_{G^*} v_i=r$ for $4\leq i\leq n$. Also, $\sum_{v\in V(G^*)}\deg v=nr-4$.

To have a maximum similar degree partition, we start with the possibility of having a partition class to be a singleton set $\{v_i\}$ for any $i,4,\leq i\leq n$.

The only possibility of partitioning vertices $\{v_1,v_2\}$ occurs when $2r-3=r-1\Rightarrow r=2$ or $2r-3=r\Rightarrow r=3$. $i.e.$, G is cycle or $3-regular$.

Hence, when $r=2$ or 3,

$\pi_3(G^*)=\{\{v_1,v_2\},\{v_3\},\{v_i\}\ /\ 4\leq i\leq n\}$ is s maximum similar degree partition for $G^*$ implying that $\psi_D(G^*)=n-1$.

Nest we consider the partition class of the form $V_i=\{v_i,v_j\},1\leq i,j\leq n$

**Case i** Let the degree sum of $V_1=2r$

Then all other partition class must have degree sum $2r$ or $2r\pm 1$, which could be possible when there is a partition class $V_t=\{v_1,v_2,v_3\}$

i.e., when $3r-4=2r-1\Rightarrow r=3$, which is a contradiction.

$3r-4=2r\Rightarrow r=4$ and

$3r-4=2r+1\Rightarrow r=5$

Hence, when $r=4$ or 5 the remaining vertices must form partition class among themselves contributing two in each class.

i.e., when $n-3$ is even or $n$ is odd.

Since n is odd, r cannot be 5.

$\therefore$ we conclude that $\psi_D(G^*)=\frac{n-3}{2}+1=\frac{n-1}{2}$ if $n\equiv 1\ (mod\ 2)\ and\ r=4$.

**Case ii** Let the degree sum of $V_1=2r-1$

Then all the other partition classes must be of degree sum $2r-2$ or $2r-1$ otherwise $2r-1$ or $2r$.

The case of having $2r$ is discussed in Case i.

It can be noted that the perfect similar degree sum cannot be $2r-1$, as even a partition class $V_s=\{v_1,v_2,v_j\},j\geq 4$ matches up to $2r-1$, $(3r-3=2r-1\Rightarrow r=4)$, the remaining vertices cannot form a partition class of degree sum $2r$.

Therefore, we restrict to the possibility of having a partition class of degree sum $2r-2$.

Hence, there is a partition $V_m=\{v_1,v_j\},j\geq 4$.

By definition of $\psi_D(G^*)$, all other partition classes must of degree sum $2r-2$, which is possible when $n=6$

i.e., when $n=6,\pi_3=\{\{v_1,v_6\},\{v_2,v_5\},\{v_3,v_4\}\}$ forms a partition class in this case.

**Case iii** Let the degree sum of $V_1=2r-2$

Then all the partition classes must be of degree sum $2r-3$ or $2r-2$ otherwise $2r-2$ or $2r-1$.

The latter case is discussed in Case ii.

The former case is possible only when there are two partition classes in the similar degree partition say $V_1$ and $V_2$ with $D(V_1)=2r-2$ and $D(V_2)=2r-3$. So, $V_1=\{v_1,v_4\}$ and $V_2=\{v_2,v_3\}$. Hence, the considered graph is a graph of order 4.

That is, $r < 4$, which has been discussed already.
All the other possible partitions will not be maximum as they are covered any one of the above said cases.
Hence in all the remaining cases, $\psi_D(G^*) = 1$ □

**Proposition 19** Let $G^*$ be a graph obtained from a complete graph $K_n$ $(n \geq 4)$ by removing $n-2$ edges incident with a vertex. Then $\psi_D(G^*) = n-1$.
**Proof** The degree sequence of $G^*$ is $\left((n-1),(n-2)^{n-1},1\right)$.
Let $\deg_{G^*} v_1 = n-1$, $\deg_{G^*} v_i = n-2$ for $2 \leq i \leq n-1$ and $\deg_{G^*} v_n = 1$. It is noticeable that $\psi_D(G^*) < n$.
So, $\pi_{n-1} = \left\{\{v_i\},\{v_2,v_n\}\right\}$ forms a maximum similar degree partition for $G^*$ with degree sum as $n-1$ and $n-2$ where $i = 1,3,4,\dots,n-1$. □

**Theorem 20** Let $V(K_n) = \{v_1,v_2,\dots,v_n\}$. For $k = 1,2,\dots,\left\lfloor\frac{n}{2}\right\rfloor - 1$, define a family $G_n^{(k)}$ of graphs as follows: Set $G_n^{(0)} = K_n, V\left(G_n^{(k)}\right) = V(K_n)$ and $\mathrm{E}\left(\mathrm{G}_n^{(k)}\right) = \mathrm{E}\left(\mathrm{G}_n^{(k-1)}\right) \setminus$
$\{v_{2k-1}v_{2i}, v_{2k-1}v_{2k+j} / 1 \leq i \leq k-1, 1 \leq j \leq n-2k\}$
$\{v_{2k-1}v_{2i}, v_{2k-1}v_{2k+j} / 1 \leq i \leq k-1, 1 \leq j \leq n-2k\}$. Then $\psi_D\left(G_n^{(k)}\right) = n-k$.
**Proof** The degree sequence of $G_n^{(k)}$ is $((n-k)^k,(n-k-1)^{n-2k},1^k)$.
In $G_n^{(k)}$, $\deg_{G_n^{(k)}} v_{2t} = n-k$ for $1 \leq t \leq k$
$\deg_{G_n^{(k)}} v_{2t+1} = 1$ for $1 \leq t \leq k$ and
$\deg_{G_n^{(k)}} v_s = n-k-1$ for $2k+1 \leq s \leq n$.
It can be easily verified that $\frac{\sum_{v \in V\left(G_n^{(k)}\right)} \deg v - 1}{\Delta - 1} = (n-k) + \frac{2k-1}{n-k-1}$.
As $k \leq \left\lfloor\frac{n}{2}\right\rfloor - 1$, $2k-1 \leq \begin{Bmatrix} n-3 & if\ n\ is\ even \\ n-4 & if\ n\ is\ odd \end{Bmatrix}$.
So, $\frac{2k-1}{n-k-1} \leq \frac{n-3}{n-2} < 1$ as $n \geq 4$. $\therefore \psi_D(G^*) \leq n-k$.
$\pi_{n-k} = \left\{\{v_x\},\{v_{2t-1},v_{n-t+1}\}\right\}$ where $x = 2,4,\dots,2k,2k+1,\dots,n-k$ and $1 \leq t \leq k$ forms a maximum similar degree partition for $G^*$.
$\therefore \psi_D\left(G_n^{(k)}\right) \leq n-k$. □

**Theorem 21** [8] $\psi_D(I_n) = \left\lfloor\frac{n}{2}\right\rfloor + 1$.
**Proof** Let $V(I_n) = \{v_1,v_2,\dots,v_n\}$ and $d(v_i) = i$ for $1 \leq i \leq n-1$ and $d(v_n) = \left\lfloor\frac{n}{2}\right\rfloor$.
In the case when n is even, $\pi_{\frac{n}{2}+1} = \left\{V_1,V_2,\dots,V_{\frac{n}{2}+1}\right\}$ where $V_1 = \{v_1,v_{n-3}\}$, $V_2 = \{v_2,v_{n-4}\},\dots$, $V_{\frac{n}{2}-2} = \left\{v_{\frac{n}{2}-2},v_{\frac{n}{2}}\right\}, V_{\frac{n}{2}-1} = \left\{v_{\frac{n}{2}-1},v_n\right\}, V_{\frac{n}{2}} = \{v_{n-1}\}$ and $V_{\frac{n}{2}+1} = \{v_{n-2}\}$ forms a similar degree partition with all classes having degree sum $n-2$ except $V_{\frac{n}{2}}$ and $V_{\frac{n}{2}-1}$ have degree sum as $n-1$.
Hence $\psi_D(I_n) = \frac{n}{2}+1$ when n is even.
As in the above case, when n is odd, $\pi_{\frac{n+1}{2}} = \left\{V_1,V_2,\dots,V_{\frac{n+1}{2}}\right\}$ where $V_1 = \{v_1,v_{n-2}\}$,
$V_2 = \{v_2,v_{n-3}\},\dots, V_{\frac{n-3}{2}} = \left\{v_{\frac{n-3}{2}},v_{\frac{n+1}{2}}\right\}, V_{\frac{n-1}{2}} = \left\{v_{\frac{n-1}{2}},v_n\right\}$ *and* $V_{\frac{n+1}{2}} = \{v_{n-1}\}$ serves as a perfect similar degree partition with degree sum of all classes being n-1 and also forcing $\psi_D(I_n) = \frac{n+1}{2}$.
Therefore, we conclude that $\psi_D(I_n) = \left\lfloor\frac{n}{2}\right\rfloor + 1$. □

**Theorem 22** $\psi_D\left(S_p(P_n)\right) = n + \left\lfloor\frac{n-2}{2}\right\rfloor$ for $n \geq 3$.
**Proof** Assume $n \geq 3$. Let $V(P_n) = \{v_1,v_2,\dots,v_n\}$ and $V\left(S_p(P_n)\right) = \{v_1,v_2,\dots,v_n,w_1,w_2,\dots,w_n\}$.
We know $\deg v_i = 4$ where $2 \leq i \leq n-1$, $\deg v_j = 2$ where $j = 1,n$, $\deg w_i = 2$ where $2 \leq i \leq n-1$ and $\deg w_j = 1$ where $j = 1,n$.
Now any degree partition of $S_p(P_n)$ must have the degree sum atleast 3.
To maximize $\psi_D\left(S_p(P_n)\right)$, the degree sum is either 3 or 4.
Let $V_1 = \{v_1,w_1\}, V_i = \{v_i\}$ for $2 \leq i \leq n-1, V_n = \{v_n,w_n\}$ and $V_{n+j} = \{w_{2j},w_{2j+1}\}$ for $1 \leq j \leq \frac{n-2}{2}$ if n is even.
Then $V_j, 1 \leq j \leq n + \frac{n-2}{2}$ forms a degree partition of $S_p(P_n)$, which gives $\psi_D\left(S_p(P_n)\right) = n + \frac{n-2}{2}$ when n is even.
If n is odd, let the partition class be $V_1 = \{w_{n-1},w_1,w_n\}, V_i = \{v_i\}$ for $2 \leq i \leq n-1$, $V_{n+j-1} = \{w_{2j},w_{2j+1}\}$ for $1 \leq j \leq \frac{n-3}{2}$ and $V_{n+\left\lfloor\frac{n-2}{2}\right\rfloor} = \{v_1,v_n\}$ which gives $\psi_D\left(S_p(P_n)\right) = n + \left\lfloor\frac{n-2}{2}\right\rfloor$ when n is odd. □

**Theorem 23** Let G be a $r-regular$ graph with $r \geq 2$. Then
$\psi_D\left(S_p(G)\right) = \begin{Bmatrix} n + \frac{n}{2}\ if\ n \equiv 0\ (mod\ 2) \\ n \quad if\ n \equiv\ 1\ (mod\ 2) \end{Bmatrix}$.
**Proof** Let $G$ be a regular graph of order $n$.
Let $V(G) = \{v_1,v_2,\dots,v_n\}$ and $V\left(S_p(G)\right) = \{v_1,v_2,\dots,v_n,w_1,w_2,\dots,w_n\}$.
Then $\deg v_i = 2r$ and $\deg w_i = r$ for $1 \leq i \leq n$.
If n is even, the partition classes are given as $V_i = \{v_i\}$ for $1 \leq i \leq n$ $V_{n+j} = \{w_{2j},w_{2j-1}\}$ for $1 \leq j \leq \frac{n}{2}$ which gives to $\psi_D\left(S_p(G)\right) = n + \frac{n}{2}$.
If n is odd, let $V_i = \{v_i,w_i\ /\ 1 \leq i \leq n\}$. Then $V_1,V_2,\dots,V_n$ forms a degree partition of $S_p(G)$.
Therefore, $\psi_D\left(S_p(G)\right) = n$ if n is odd. □

**Theorem 24** For any $n \geq 2, \psi_D\left(S_p(K_{1,n})\right) = 3$.
**Proof** Let $V(K_{1,n}) = \{v,v_1,v_2,\dots,v_n\}$ such that $\deg v = n$ and $\deg v_i = 1$ for $1 \leq i \leq n$ in $K_{1,n}$.
Now $V\left(S_p(K_{1,n})\right) = \{v,v_1,v_2,\dots,v_n,w,w_1,w_2,\dots w_n\}$ where $\deg v = 2n$, $\deg v_i = 2$ for $1 \leq i \leq n$, $\deg w = n$ and $\deg w_i = 1$ for $1 \leq i \leq n$ in $S_p(K_{1,n})$.
Let the partition classes of $S_p(K_{1,n})$ be given as $V_1 = \{w,w_i\}, V_2 = \{v\}$ and $V_3 = \{v_i\}$ where $1 \leq i \leq n$ which gives $\psi_D\left(S_p(K_{1,n})\right) \geq 3$.
Also, by theorem 2,
$$\psi_D\left(S_p(K_{1,n})\right) \leq \left\lfloor\frac{6n-1}{2n-1}\right\rfloor = \left\lfloor\frac{6n-3}{2n-1} + \frac{2}{2n-1}\right\rfloor.$$
So, for $n \geq 2, \psi_D\left(S_p(K_{1,n})\right) \leq 3$.
Hence the theorem. □

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