Conflict Regulating under Multitasking Systems

Bo-Yao Wang and Yu-Hsien Liao

***Abstract*—Given the ongoing expansion of multitasking systems, conflicts arising among multiple concurrent processes have garnered significant attention. Effectively regulating and balancing the impacts induced by these processes remains a key challenge in conflict adjustment. To mitigate and balance the conflicts caused by multiple factors, this study introduces several regulation methods under different objectives. To enhance the assessment of how different stakeholders and its operating scales contribute to or influence conflict levels, two weighted generalizations are proposed. Subsequently, several structured axiomatic methodologies are applied to evaluate both its mathematical rigor and practical suitability. Moreover, this paper delves into further interpretations of these axiomatic procedures, thereby offering deeper insights into potential applications for conflict management and regulating under multitasking systems.**

***Index Terms*—Sustainability, conflict, regulating method, multitasking systems, axiomatic procedure.**

I. Introduction

In recent years, sustainability-related challenges have received substantial attention due to the escalating impacts of climate change, dwindling natural resources, and other environmental constraints. This surge in interest has led to extensive research on topics such as resource management, pollution mitigation, and climate change adaptation. The environmental conflict emerging from the concurrent advancement of human activities and ecological limitations is increasingly evident, with some outcomes proving to be irreversible. Consequently, reducing the environmental impacts arising from various factors has become a primary focus in sustainability-related investigations.

Addressing these impacts often necessitates a broad-based approach that simultaneously considers multiple dimensions, which may at times be in conflict. For instance, achieving optimal pollution mitigation using certain methods or technologies, while concurrently conserving energy, reducing resource consumption, and preventing the creation of secondary pollutants or waste, calls for a balanced, multi-faceted strategy. In the mathematical domain, multi-objective optimization or equilibrium models are applied to reconcile these varied objectives within operational systems. Likewise, in multitasking systems, concurrently running multiple tasks or processes may require evaluating and regulating conflicting goals, drawing parallels to sustainability challenges in balancing diverse factors.

Under conventional transferable-utility (TU) conditions, modules are generally deemed either fully active or completely inactive in relation to one another. However, practical scenarios rarely exhibit such clear distinctions in module engagement. Within the framework of multi-choice TU systems, modules can interact at an finite spectrum of engagement levels. Various regulation methodologies for multi-choice TU games have been explored in different applications, including works by Calvo and Santos [3], Chen et al. [4], Cheng et al. [5], Li et al. [15], Liao [16], Liao et al. [17], Hwang and Liao [10], [11], Huang et al. [12], Klijn et al. [13], Nouweland et al. [20], and among others.

Consistency is a critical attribute in regulating methods under axiomatic paradigms for traditional systems, as it ensures that a prescribed solution remains valid when the payoffs of certain modules are held constant. This principle posits that the recommendations for a given issue should align with those produced for sub-issues where specific modules' payoffs are predetermined. The notion of consistency has been defined in diverse ways depending on the treatment of payoffs for modules that cease bargaining. Such consistency has been intensively analyzed in the context of reduced systems, including bargaining and cost allocation scenarios. Methods involving single contributions, the pseudo equal allocation of non-separable costs (PEANSC, Hsieh and Liao [9]), and the normalized index have been proposed for conventional TU systems. Hsieh and Liao [9] demonstrated an adaptation of the complement-reduction technique by Moulin [19], illustrating that PEANSC furnishes an equitable allocation mechanism.

The results presented in this context lead to the following important question:

- Can the single index and its associated outcomes be expanded to better address sustainability challenges in multitasking systems?

This study aims to lay the essential mathematical groundwork for optimally evaluating multitasking systems within the scope of sustainability-related concerns. Specifically, we regulate multi-choice behaviors and their repercussions in multi-objective settings. Building upon traditional and multi-choice TU systems, we introduce the concept of multitasking TU systems. In Section 2, we propose two regulation frameworks: the minimal regulation of accompanied conflict (MRAC) and the normalized single-conflict regulation (NSCR).

The MRAC regulation allocates minimal single conflict from operational coalitions to each module before equally distributing any remaining conflict. By contrast, the NSCR assigns conflict proportionally by inflicting minimal single conflict on all modules of the coalition. These regulations generalize the notion of marginal conflict to encompass multi-choice behavior and multitasking systems.

To substantiate these regulations, we propose an extended reduction and accompanying consistency properties, which are addressed in Sections 3 and 4:

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B.Y. Wang is a lecturer of Department of Applied Mathematics, National Dong Hwa University, 974 Hualien, Taiwan. (e-mail: 810411001@gms.ndhu.edu.tw)

Y.H. Liao is a professor of Department of Applied Mathematics; Green Nano Interdisciplinary Center (GNIC), National Pingtung University, 900 Pingtung, Taiwan. (Corresponding author to provide e-mail: twincos@ms25.hinet.net)

- The MRAC is the only regulation method that satisfies the properties of multitasking standard for systems and multitasking consistency.
- The MRAC is the only regulation method that satisfies the properties of multitasking efficiency, multitasking covariance, multitasking symmetry, and multitasking consistency.
- Although the NSCR does not satisfy multitasking bilateral consistency, it retains the properties of normalized-standard of systems and analogue consistency.

Building on the MRAC framework, each module initially receives minimal single conflict from operational coalitions, followed by equally regulating any fixed conflict (e.g., the cost of shared resources) among the modules involved. Nevertheless, varying levels of module engagement and operational intensity can result in different outcomes across an array of scenarios.

In practice, the MRAC method may seem less feasible when modules differ in size or bargaining power. Such asymmetries frequently emerge when depicting differences in bargaining capabilities and engagement degrees across modules. To accommodate these discrepancies, we propose alternative regulation methods that distribute any additional fixed conflict proportionally according to the modules' weights.

To mitigate discrimination and address the relative conflict triggered by modules and their acting intensities, we introduce weighting functions for both modules and their respective operating degrees. Consequently, two weighted extensions of the MRAC, along with their axiomatic properties, are established in Section 5. Throughout the study, further perspectives on these axioms and their associated axiomatic processes are explored, underscoring their utility for sustainability and pollution control research as well as for regulating conflicts in multitasking systems.

II. Preliminaries

Let $\overline{\mathbb{UM}}$ denote the universal collection of modules. For each module $i \in \overline{\mathbb{UM}}$ and $\tilde{d}_i \in \mathbb{N}$, we define $\tilde{D}_i = \{0, \cdots \tilde{d}_i\}$ as the acting degree space of module i, with $\tilde{D}_i^+ = \tilde{D}_i \setminus \{0\}$ indicating active reaction, and 0 indicating non-reaction. Let $\overline{\mathbb{M}} \subseteq \overline{\mathbb{UM}}$ and $\tilde{D}^{\overline{\mathbb{M}}} = \prod_{i\in\overline{\mathbb{M}}} \tilde{D}_i$ denote the Cartesian product set of acting degree spaces for modules in $\overline{\mathbb{M}}$. For any $\overline{\mathbb{H}} \subseteq \overline{\mathbb{M}}$, a module coalition $\overline{\mathbb{H}} \subseteq \overline{\mathbb{M}}$ corresponds canonically to the multi-choice coalition $\tilde{d}^{\overline{\mathbb{H}}} \in \tilde{D}^{\overline{\mathbb{M}}}$, where $\tilde{d}_i^{\overline{\mathbb{H}}} = 1$ if $i \in \overline{\mathbb{H}}$ and $\tilde{d}_i^{\overline{\mathbb{H}}} = 0$ if $i \in \overline{\mathbb{M}} \setminus \overline{\mathbb{H}}$. Let $0_{\overline{\mathbb{M}}}$ represent the zero vector in $\mathbb{R}^{\overline{\mathbb{M}}}$. For $m \in \mathbb{N}$, 0_m denotes the zero vector in $\mathbb{R}^m$, and $\overline{\mathbb{N}}_m = \{1, 2, \cdots, m\}$.

A **multi-choice transferable-utility (TU) system** is characterized as a triple $(\overline{\mathbb{M}}, \tilde{d}, c)$, where $\overline{\mathbb{M}}$ denotes a non-empty and finite set of modules, $\tilde{d} = (\tilde{d}_i)_{i\in\overline{\mathbb{M}}} \in \tilde{D}^{\overline{\mathbb{M}}}$ represents the vector indicating the highest acting degrees for each module, and $c : \tilde{D}^{\overline{\mathbb{M}}} \to \mathbb{R}$ is a function satisfying $c(0_{\overline{\mathbb{M}}}) = 0$, assigning the worth that modules can obtain if acting at corresponding acting degrees $\tilde{\mu} = (\tilde{\mu}_i)_{i\in\overline{\mathbb{M}}} \in \tilde{D}^{\overline{\mathbb{M}}}$. A **multitasking multi-choice TU system** is defined as a triple $(\overline{\mathbb{M}}, \tilde{d}, C^m)$, where $m \in \mathbb{N}$, $C^m = (c^t)_{t\in\overline{\mathbb{N}}_m}$, and $(\overline{\mathbb{M}}, \tilde{d}, c^t)$ represents a multi-choice TU system for all $t \in \overline{\mathbb{N}}_m$. The class encompassing all multitasking multi-choice TU systems is denoted as $\mathbb{MCS}$.

A **regulation** is defined as a mapping Ψ that assigns to each $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$ an element

$$\Psi\big(\overline{\mathbb{M}}, \tilde{d}, C^m\big) = \big(\Psi^t\big(\overline{\mathbb{M}}, \tilde{d}, C^m\big)\big)_{t\in\overline{\mathbb{N}}_m},$$

where $\Psi^t\big(\overline{\mathbb{M}}, \tilde{d}, C^m\big) = \big(\Psi_i^t\big(\overline{\mathbb{M}}, \tilde{d}, C^m\big)\big)_{i\in\overline{\mathbb{M}}} \in \mathbb{R}^{\overline{\mathbb{M}}}$ and $\Psi_i^t\big(\overline{\mathbb{M}}, \tilde{d}, C^m\big)$ represents the payoff of module i when i engages in $\big(\overline{\mathbb{M}}, \tilde{d}, c^t\big)$. For $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, $\overline{\mathbb{H}} \subseteq \overline{\mathbb{M}}$, and $\tilde{\mu} \in \mathbb{R}^{\overline{\mathbb{M}}}$, $\mathbf{KE}(\tilde{\mu}) = \{i \in \overline{\mathbb{M}} | \tilde{\mu}_i \neq 0\}$ is defined to denote the set of modules with non-zero acting degrees, and $\tilde{\mu}_{\overline{\mathbb{H}}} \in \mathbb{R}^{\overline{\mathbb{H}}}$ represents the restriction of $\tilde{\mu}$ to $\overline{\mathbb{H}}$. For a given $i \in \overline{\mathbb{M}}$, the notation $\tilde{\mu}_{-i}$ is introduced to denote $\tilde{\mu}_{\overline{\mathbb{M}}\setminus\{i\}}$, and $\alpha = (\tilde{\mu}_{-i}, t) \in \mathbb{R}^{\overline{\mathbb{M}}}$ is defined by $\alpha_{-i} = \tilde{\mu}_{-i}$ and $\alpha_i = t$.

Next, we provide two generalized regulations under multitasking systems.

Definition 1:

1) The **minimal regulation of accompanied conflict (MRAC)**, $\widehat{\Phi}$, is defined by

$$\begin{aligned} & \widehat{\Phi_i^t}(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ = \; & \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \tfrac{1}{|\overline{\mathbb{M}}|} \cdot \Big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\Big] \end{aligned}$$

for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$. Here, $\Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) = \min_{j\in\tilde{D}_i^+}\{c^t(0_{-i}, j) - c^t(0_{-i}, j-1)\}$ denotes the **minimal single conflict** that module i faces in the system $(\overline{\mathbb{M}}, \tilde{d}, c^t)$. Throughout this study, we focus on bounded multi-choice transferable-utility (TU) systems, i.e., those $(\overline{\mathbb{M}}, \tilde{d}, c^t)$ in which there exists a constant $M_t \in \mathbb{R}$ satisfying $c^t(\tilde{\mu}) \leq M_t$ for all $\tilde{\mu} \in \tilde{D}^{\overline{\mathbb{M}}}$. Under this condition, $\Phi_i(\overline{\mathbb{M}}, \tilde{d}, c^t)$ remains well-defined and valid. Within the $\widehat{\Phi}$ framework, each module is initially allocated its minimal single conflict. Any residual conflict is then evenly apportioned among all modules, thereby establishing a balanced and equitable regulation. This strategy is particularly pertinent to sustainability or pollution mitigation initiatives, where attenuating both individual and collective burdens among stakeholders is of paramount importance.

2) The **normalized single-conflict regulation (NSCR)**, $\widehat{\Delta}$, is defined by

$$\widehat{\Delta_i^t}(\overline{\mathbb{M}}, \tilde{d}, C^m) = \frac{c^t(\tilde{d})}{\sum\limits_{k\in\overline{\mathbb{M}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)} \cdot \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m)$$

for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}^*$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$, where $\mathbb{MCS}^* = \{(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS} | \sum_{i\in\overline{\mathbb{M}}} \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) \neq 0 \text{ for all } t \in \overline{\mathbb{N}}_m\}$. Within the $\widehat{\Delta}$ framework, every module proportionally shares in the coalition's total conflict, using each module's minimal single conflict as the weighting factor. This allocation method promotes an equitable and judicious distribution of impacts, a principle integral to environmental management and sustainability assessments where the fair distribution of burdens is critical for effective conflict regulation.

In this section, we provide a concise application of multitasking multi-choice TU systems within the realm of "management." Such problems can be formalized as follows: let $\overline{\mathbb{M}}$ represent the complete set of modules in a unified management system $(\overline{\mathbb{M}}, \tilde{d}, C^m)$. The conflict function c^t assigns a value to each degree vector $\tilde{\mu} = (\tilde{\mu}_i)_{i\in\overline{\mathbb{M}}} \in \tilde{D}^{\overline{\mathbb{M}}}$, denoting the outcomes that modules can realize when each module i selects an operational plan $\tilde{\mu}_i \in \tilde{D}_i$ in the sub-management system $(\overline{\mathbb{M}}, \tilde{d}, c^t)$.

Viewed in this way, the all-encompassing management system $(\overline{\mathbb{M}}, \tilde{d}, C^m)$ can be interpreted as a multitasking multi-choice TU system, where c^t corresponds to each module's function and $\tilde{D}_i$ is the collection of viable strategies for module i. In the subsequent sections, we show that both the MRAC and the NSCR can serve as "optimal regulation mechanisms" for the modules involved, ensuring that every module capitalizes on the collective advantages of various operational approaches across multitasking systems, thereby elevating the efficacy of management approaches in sustainability and pollution mitigation scenarios.

III. Axiomatic results for the MRAC

In order to analyze the rationality of the MRAC, an extended reduction and several axioms are introduced to present certain axiomatic procedures. A regulation Ψ satisfies **multitasking efficiency (MTEFF)** if for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$ and for all $t \in \overline{\mathbb{N}}_m$, $\sum_{i \in \overline{\mathbb{M}}} \Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) = c^t(\tilde{d})$. A regulation Ψ satisfies **multitasking standard for systems (MTSS)** if $\Psi(\overline{\mathbb{M}}, \tilde{d}, C^m) = \widehat{\Phi}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$ with $|\overline{\mathbb{M}}| \leq 2$. A regulation Ψ satisfies **multitasking symmetry (MTSMT)** if $\Psi_i(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Psi_k(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$ where $\Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for some $i, k \in \overline{\mathbb{M}}$ and for all $t \in \overline{\mathbb{N}}_m$. A regulation Ψ satisfies **multitasking covariance (MTCVA)** if $\Psi(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Psi(\overline{\mathbb{M}}, \tilde{d}, Q^m) + (y^t)_{t \in \overline{\mathbb{N}}_m}$ for all $(\overline{\mathbb{M}}, \tilde{d}, C^m), (\overline{\mathbb{M}}, \tilde{d}, Q^m) \in \mathbb{MCS}$ with $c^t(\tilde{\mu}) = q^t(\tilde{\mu}) + \sum_{i \in \mathbf{KE}(\tilde{\mu})} y_i^t$ for some $y^t \in \mathbb{R}^{\overline{\mathbb{M}}}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $\tilde{\mu} \in \tilde{D}^{\overline{\mathbb{M}}}$.

Property MTEFF requires that every module jointly allocates the total conflict. Property MTSS generalizes the two-person standardness principle introduced by Hart and Mas-Colell [8]. Property MTSMT demands that outputs remain identical when minimal single conflicts coincide. Property MTCVA can be interpreted as a weaker iteration of *additivity*. It follows from Definition 1 that the MRAC meets MTEFF, MTSS, MTSMT, and MTCVA.

Moulin [19] introduced a notion of reduced systems, requiring that coalitions within a subgroup only secure payoffs for their members if these allocations match the original payoffs of "all" members external to the subgroup. Subsequently, Hsieh and Liao [9] extended Moulin's concept to characterize the PEANSC. A related expansion of Moulin's reduction, applicable to multitasking multi-choice TU systems, is formulated as follows.

Let $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, $\overline{\mathbb{H}} \subseteq \overline{\mathbb{M}}$, and let Ψ be a regulation. The **reduced system** $(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Psi})$ is specified by $C^m_{\overline{\mathbb{H}},\Psi} = (c^t_{\overline{\mathbb{H}},\Psi})_{t \in \overline{\mathbb{N}}_m}$, and for all $\tilde{\mu} \in \tilde{D}^{\overline{\mathbb{H}}}$,

$$\begin{aligned} & c^t_{\overline{\mathbb{H}},\Psi}(\tilde{\mu}) \\ = & \begin{cases} 0 & \tilde{\mu} = 0_{\overline{\mathbb{H}}}, \\ c^t(\tilde{\mu}) & |\overline{\mathbb{H}}| \geq 2, |\mathbf{KE}(\tilde{\mu})| = 1, \\ c^t(\tilde{\mu}, \tilde{d}_{\overline{\mathbb{H}}^c}) - \sum\limits_{i \in \overline{\mathbb{H}}^c} \Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) & \text{otherwise.} \end{cases} \end{aligned}$$

A regulation Ψ fulfills **multitasking consistency (MTCIY)** if $\Psi_i^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Psi}) = \Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$, for all $\overline{\mathbb{H}} \subseteq \overline{\mathbb{M}}$ with $|\overline{\mathbb{H}}| = 2$, and for all $i \in \overline{\mathbb{H}}$.

Lemma 1: The MRAC $\widehat{\Phi}$ satisfies MTCIY.

Proof: Let $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, $\overline{\mathbb{H}} \subseteq \overline{\mathbb{M}}$ and $t \in \overline{\mathbb{N}}_m$. Assume that $|\overline{\mathbb{M}}| \geq 2$ and $|\overline{\mathbb{H}}| = 2$. Therefore,

$$\begin{aligned} & \widehat{\Phi}_i^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\widehat{\Phi}}) \\ = & \Phi_i^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\widehat{\Phi}}) \\ & + \tfrac{1}{|\overline{\mathbb{H}}|} \cdot \big[c^t_{\overline{\mathbb{H}},\widehat{\Phi}}(\tilde{d}_{\overline{\mathbb{H}}}) - \sum_{k \in \overline{\mathbb{H}}} \Phi_k^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\widehat{\Phi}})\big] \end{aligned} \quad (1)$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. Furthermore,

$$\begin{aligned} & \Phi_i^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\widehat{\Phi}}) \\ = & \min_{j \in \tilde{D}_i^+} \{c^t_{\overline{\mathbb{H}},\widehat{\Phi}}(0_{\overline{\mathbb{H}} \setminus \{i\}}, j) - c^t_{\overline{\mathbb{H}},\widehat{\Phi}}(0_{\overline{\mathbb{H}} \setminus \{i\}}, j-1)\} \\ = & \min_{j \in \tilde{D}_i^+} \{c^t(0_{-i}, j) - c^t(0_{-i}, j-1)\} \\ = & \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m). \end{aligned} \quad (2)$$

By equations (1), (2) and definitions of $c^t_{\overline{\mathbb{H}},\widehat{\Phi}}$ and $\widehat{\Phi}$,

$$\begin{aligned} & \widehat{\Phi}_i^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\widehat{\Phi}}) \\ = & \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \tfrac{1}{|\overline{\mathbb{H}}|}\big[c^t_{\overline{\mathbb{H}},\widehat{\Phi}}(\tilde{d}_{\overline{\mathbb{H}}}) - \sum_{k \in \overline{\mathbb{H}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \tfrac{1}{|\overline{\mathbb{H}}|}\big[c^t(\tilde{d}) - \sum_{k \in \overline{\mathbb{M}} \setminus \overline{\mathbb{H}}} \widehat{\Phi}_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ & \qquad - \sum_{k \in \overline{\mathbb{H}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \tfrac{1}{|\overline{\mathbb{H}}|}\big[\sum_{k \in \overline{\mathbb{H}}} \widehat{\Phi}_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ & \qquad - \sum_{k \in \overline{\mathbb{H}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ & \big(\textbf{by MTEFF of } \widehat{\Phi}\big) \\ = & \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \tfrac{1}{|\overline{\mathbb{H}}|}\Big[\tfrac{|\overline{\mathbb{H}}|}{|\overline{\mathbb{M}}|} \cdot \big[c^t(\tilde{d}) \\ & \qquad - \sum_{k \in \overline{\mathbb{M}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\Big] \\ = & \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \tfrac{1}{|\overline{\mathbb{M}}|}\big[c^t(\tilde{d}) - \sum_{k \in \overline{\mathbb{M}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \widehat{\Phi}_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) \end{aligned}$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. So, the MRAC satisfies MTCIY. ∎

Next, we characterize the MRAC by means of multitasking consistency.

Theorem 1: The MRAC is the only regulation satisfying MTSS and MTCIY.

Proof: By Lemma 1, $\widehat{\Phi}$ satisfies MTCIY. Clearly, $\widehat{\Phi}$ satisfies MTSS.

To prove uniqueness, suppose Ψ satisfies MTSS and MTCIY. By MTSS and MTCIY of Ψ, it is easy to derive that Ψ also satisfies MTEFF, hence we omit it. Let $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$. By MTSS of Ψ, $\Psi(\overline{\mathbb{M}}, \tilde{d}, C^m) = \widehat{\Phi}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ if $|\overline{\mathbb{M}}| \leq 2$. The case $|\overline{\mathbb{M}}| > 2$: Let $i \in \overline{\mathbb{M}}$, $t \in \overline{\mathbb{N}}_m$ and

$\overline{\mathbb{H}} = \{i,k\}$ for some $k \in \overline{\mathbb{M}} \setminus \{i\}$.

$$\begin{aligned} & \Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - \Psi_k^t(\overline{\mathbb{M}},\tilde{d},C^m) \\ = \; & \Psi_i^t(\overline{\mathbb{H}},\tilde{d}_{\overline{\mathbb{H}}},C^m_{\overline{\mathbb{H}},\Psi}) - \Psi_k^t(\overline{\mathbb{H}},\tilde{d}_{\overline{\mathbb{H}}},C^m_{\overline{\mathbb{H}},\Psi}) \\ & \big(\textbf{by MTCIY of } \Psi\big) \\ = \; & \widehat{\Phi}_i^t(\overline{\mathbb{H}},\tilde{d}_{\overline{\mathbb{H}}},C^m_{\overline{\mathbb{H}},\Psi}) - \widehat{\Phi}_k^t(\overline{\mathbb{H}},\tilde{d}_{\overline{\mathbb{H}}},C^m_{\overline{\mathbb{H}},\Psi}) \\ & \big(\textbf{by MTSS of } \Psi\big) \\ = \; & \Phi_i^t(\overline{\mathbb{H}},\tilde{d}_{\overline{\mathbb{H}}},C^m_{\overline{\mathbb{H}},\Psi}) - \Phi_k^t(\overline{\mathbb{H}},\tilde{d}_{\overline{\mathbb{H}}},C^m_{\overline{\mathbb{H}},\Psi}) \\ = \; & \min_{j\in\tilde{D}_i^+}\{c^t_{\overline{\mathbb{H}},\Psi}(0_{\overline{\mathbb{H}}\setminus\{i\}},j) - c^t_{\overline{\mathbb{H}},\Psi}(0_{\overline{\mathbb{H}}\setminus\{i\}},j-1)\} \\ & - \min_{j\in\tilde{D}_k^+}\{c^t_{\overline{\mathbb{H}},\Psi}(0_{\overline{\mathbb{H}}\setminus\{k\}},j) - \min_{j\in\tilde{D}_k^+} c^t_{\overline{\mathbb{H}},\Psi}(0_{\overline{\mathbb{H}}\setminus\{k\}},j-1)\} \\ = \; & \min_{j\in\tilde{D}_i^+}\{c^t(0_{-i},j) - c^t(0_{-i},j-1)\} \\ & - \min_{j\in\tilde{D}_k^+}\{c^t(0_{-k},j) - c^t(0_{-k},j-1)\} \\ = \; & \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - \Phi_k^t(\overline{\mathbb{M}},\tilde{d},C^m) \\ = \; & \widehat{\Phi}_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - \widehat{\Phi}_k^t(\overline{\mathbb{M}},\tilde{d},C^m). \end{aligned}$$

Thus,

$$\begin{aligned} & \Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - \Psi_k^t(\overline{\mathbb{M}},\tilde{d},C^m) \\ = \; & \widehat{\Phi}_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - \widehat{\Phi}_k^t(\overline{\mathbb{M}},\tilde{d},C^m). \end{aligned}$$

By MTEFF of Ψ and $\widehat{\Phi}$,

$$\begin{aligned} & |\overline{\mathbb{M}}| \cdot \Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - c^t(\tilde{d}) \\ = \; & \sum_{k\in\overline{\mathbb{M}}} [\Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - \Psi_k^t(\overline{\mathbb{M}},\tilde{d},C^m)] \\ = \; & \sum_{k\in\overline{\mathbb{M}}} [\widehat{\Phi}_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - \widehat{\Phi}_k^t(\overline{\mathbb{M}},\tilde{d},C^m)] \\ = \; & |\overline{\mathbb{M}}| \cdot \widehat{\Phi}_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - c^t(\tilde{d}). \end{aligned}$$

Hence, $\Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) = \widehat{\Phi}_i^t(\overline{\mathbb{M}},\tilde{d},C^m)$ for all $i \in \overline{\mathbb{M}}$ and for all $t \in \overline{\mathbb{N}}_m$. ■

Next, we characterize the MRAC by means of related properties of MTEFF, MTSMT, MTCVA and MTCIY.

Lemma 2: If a regulation Ψ satisfies MTEFF, MTSMT and MTCVA, then Ψ satisfies MTSS.

Proof: Assume that a regulation Ψ satisfies MTEFF, MTSMT and MTCVA. Let $(\overline{\mathbb{M}},\tilde{d},C^m) \in \mathbb{MCS}$. The proof is completed by MTEFF of Ψ if $|\overline{\mathbb{M}}| = 1$. Let $(\overline{\mathbb{M}},\tilde{d},C^m) \in \mathbb{MCS}$ with $P = \{i,k\}$ for some $i \neq k$. We define a system $(\overline{\mathbb{M}},\tilde{d},Q^m)$ to be that $q^t(\tilde{\mu}) = c^t(\tilde{\mu}) - \sum_{i\in\mathbf{KE}(\tilde{\mu})} \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m)$ for all $\tilde{\mu} \in \tilde{D}^{\overline{\mathbb{M}}}$ and for all $t \in \overline{\mathbb{N}}_m$. By definition of Q^m,

$$\begin{aligned} & \Phi_i^t(\overline{\mathbb{M}},\tilde{d},Q^m) \\ = \; & \min_{j\in\tilde{D}_i^+}\{q^t(j,0) - q^t(j-1,0)\} \\ = \; & \min_{j\in\tilde{D}_i^+}\{c^t(j,0) - c^t(j-1,0) - \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m)\} \\ = \; & \min_{j\in\tilde{D}_i^+}\{c^t(j,0) - c^t(j-1,0)\} - \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) \\ = \; & \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) - \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) \\ = \; & 0. \end{aligned}$$

Similarly, $\Phi_k^t(\overline{\mathbb{M}},\tilde{d},Q^m) = 0$. Therefore, $\Phi_i^t(\overline{\mathbb{M}},\tilde{d},Q^m) = \Phi_k^t(\overline{\mathbb{M}},\tilde{d},Q^m)$. By MTSMT of Ψ, $\Psi_i^t(\overline{\mathbb{M}},\tilde{d},Q^m) = \Psi_k^t(\overline{\mathbb{M}},\tilde{d},Q^m)$. By MTEFF of Ψ,

$$q^t(\tilde{d}) = \Psi_i^t(\overline{\mathbb{M}},\tilde{d},Q^m) + \Psi_k^t(\overline{\mathbb{M}},\tilde{d},Q^m) = 2\cdot\Psi_i^t(\overline{\mathbb{M}},\tilde{d},Q^m).$$

Therefore,

$$\begin{aligned} & \Psi_i^t(\overline{\mathbb{M}},\tilde{d},Q^m) \\ = \; & \frac{q^t(\tilde{d})}{2} \\ = \; & \tfrac{1}{2}\cdot\big[c^t(\tilde{d}) - \Phi_i(\overline{\mathbb{M}},\tilde{d},C^m) - \Phi_k(\overline{\mathbb{M}},\tilde{d},C^m)\big]. \end{aligned}$$

By MTCVA of Ψ,

$$\begin{aligned} & \Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) \\ = \; & \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) + \tfrac{1}{2}\cdot\big[c^t(\tilde{d}) - \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) \\ & \qquad - \Phi_k^t(\overline{\mathbb{M}},\tilde{d},C^m)\big] \\ = \; & \widehat{\Phi}_i^t(\overline{\mathbb{M}},\tilde{d},C^m). \end{aligned}$$

Similarly, $\Psi_k^t(\overline{\mathbb{M}},\tilde{d},C^m) = \widehat{\Phi}_k^t(\overline{\mathbb{M}},\tilde{d},C^m)$. Hence, Ψ satisfies MTSS. ■

Theorem 2: On $\mathbb{MCS}$, the MRAC is the only regulation satisfying MTEFF, MTSMT, MTCVA and MTCIY.

Proof: By Definition 1, $\widehat{\Phi}$ satisfies MTEFF, MTSMT and MTCVA. The remaining proofs follow from Theorem 1 and Lemmas 1, 2. ■

The following examples illustrate that each axiom utilized in Theorems 1 and 2 is logically independent from the others.

Example 1: Define a regulation Ψ by for all $(\overline{\mathbb{M}},\tilde{d},C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$,

$$\Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) = \begin{cases} \widehat{\Phi}_i^t(\overline{\mathbb{M}},\tilde{d},C^m) & \text{if } |\overline{\mathbb{M}}| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, Ψ satisfies MTSS, but it does not satisfy MTCIY.

Example 2: Define a regulation Ψ to be that

$$\Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) = \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m)$$

for all $(\overline{\mathbb{M}},\tilde{d},C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$. Clearly, Ψ satisfies MTSMT, MTCVA and MTCIY, but it does not satisfy MTEFF and MTSS.

Example 3: Define a regulation Ψ to be that

$$\Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) = \frac{c^t(\tilde{d})}{|\overline{\mathbb{M}}|}$$

for all $(\overline{\mathbb{M}},\tilde{d},C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$. Clearly, Ψ satisfies MTEFF, MTSMT and MTCIY, but it does not satisfy MTCVA.

Example 4: Define a regulation Ψ by for all $(\overline{\mathbb{M}},\tilde{d},C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$,

$$\begin{aligned} \Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) \; & = \big[c^t(\tilde{d}) - c^t(\tilde{d}_{-i},0)\big] + \tfrac{1}{|\overline{\mathbb{M}}|}\cdot\Big[c^t(\tilde{d}) \\ & \qquad - \sum_{k\in\overline{\mathbb{M}}}\big[c^t(\tilde{d}) - c^t(\tilde{d}_{-k},0)\big]\Big]. \end{aligned}$$

Clearly, Ψ satisfies MTEFF, MTCVA and MTCIY, but it does not satisfy MTSMT.

Example 5: Define a regulation Ψ by for all $(\overline{\mathbb{M}},\tilde{d},C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$,

$$\begin{aligned} & \Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) \\ = \; & \Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) + \frac{w^t(i)}{\sum_{k\in\overline{\mathbb{M}}} w^t(k)}\cdot\big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}}\Phi_k^t(\overline{\mathbb{M}},\tilde{d},C^m)\big], \end{aligned}$$

where for all $(\overline{\mathbb{M}},\tilde{d},C^m) \in \mathbb{MCS}$, $w^t : \overline{\mathbb{M}} \to \mathbb{R}^+$ is defined by $w^t(i) = w^t(k)$ if $\Phi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) = \Phi_k^t(\overline{\mathbb{M}},\tilde{d},C^m)$. Define a regulation ψ by for all $(\overline{\mathbb{M}},\tilde{d},C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$,

$$\psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) = \begin{cases} \widehat{\Phi}_i^t(\overline{\mathbb{M}},\tilde{d},C^m) & \text{if } |\overline{\mathbb{M}}| \le 2, \\ \Psi_i^t(\overline{\mathbb{M}},\tilde{d},C^m) & \text{otherwise.} \end{cases}$$

Clearly, ψ satisfies MTEFF, MTSMT and MTCVA, but it does not satisfy MTCIY.

IV. THE AXIOMATIC RESULTS FOR THE NSCR

Analogous to Theorem 1, we seek to characterize the NSCR within a multitasking consistency framework. However, one observes that $(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Psi})$ does not exist when $\sum_{i\in\overline{\mathbb{H}}} \Phi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) = 0$. To overcome this, we introduce *analogue consistency* (ANCIY) as follows. A regulation Ψ satisfies analogue consistency (ANCIY) if $(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Psi}) \in \mathbb{MCS}^*$ for some $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$ and some $\overline{\mathbb{H}} \subseteq \overline{\mathbb{M}}$ with $|\overline{\mathbb{H}}| = 2$, such that $\Psi^t_i(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Psi}) = \Psi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{H}}$.

Lemma 3: The NSCR satisfies ANCIY on $\mathbb{MCS}^*$.

Proof: Let $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}^*$. If $|\overline{\mathbb{M}}| \leq 2$, then the proof is completed. Assume that $|\overline{\mathbb{M}}| \geq 3$ and $\overline{\mathbb{H}} \subseteq \overline{\mathbb{M}}$ with $|\overline{\mathbb{H}}| = 2$. Similar to equation (2),

$$\Phi^t_i(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\widehat{\Delta}}) = \Phi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m). \tag{3}$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. Define that $\mathbf{C}^t = \frac{c^t(\tilde{d})}{\sum_{p\in\overline{\mathbb{M}}} \Phi^t_{\overline{\mathbb{M}}}(\overline{\mathbb{M}},\tilde{d},C^m)}$. For all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$,

$$\begin{aligned}
& \widehat{\Delta}^t_i(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\widehat{\Delta}}) \\
= \; & \frac{c^t_{\overline{\mathbb{H}},\widehat{\Delta}}(\tilde{d}_{\overline{\mathbb{H}}})}{\sum_{k\in\overline{\mathbb{H}}} \Phi^t_k(\overline{\mathbb{H}},\tilde{d}_{\overline{\mathbb{H}}},C^m_{\overline{\mathbb{H}},\widehat{\Delta}})} \cdot \Phi^t_i(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\widehat{\Delta}}) \\
= \; & \frac{c^t(\tilde{d}) - \sum_{\overline{\mathbb{H}}\in\overline{\mathbb{M}}\setminus\overline{\mathbb{H}}} \widehat{\Delta}^t_{\overline{\mathbb{H}}}(\overline{\mathbb{M}},\tilde{d},C^m)}{\sum_{k\in\overline{\mathbb{H}}} \Phi^t_k(\overline{\mathbb{M}},\tilde{d},C^m)} \cdot \Phi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) \\
& \big(\textbf{by equation (3) and definition of } C^m_{\overline{\mathbb{H}},\widehat{\Delta}}\big) \\
= \; & \frac{\sum_{\overline{\mathbb{H}}\in\overline{\mathbb{H}}} \widehat{\Delta}^t_{\overline{\mathbb{H}}}(\overline{\mathbb{M}},\tilde{d},C^m)}{\sum_{k\in\overline{\mathbb{H}}} \Phi^t_k(\overline{\mathbb{M}},\tilde{d},C^m)} \cdot \Phi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) \\
& \big(\textbf{by MTEFF of } \widehat{\Delta}\big) \\
= \; & \mathbf{C}^t \cdot \Phi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) \\
& \big(\textbf{by Definition 1}\big) \\
= \; & \widehat{\Delta}^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m). \\
& \big(\textbf{by Definition 1}\big)
\end{aligned} \tag{4}$$

By equations (3), (4), the regulation $\widehat{\Delta}$ satisfies ANCIY. ■

A regulation Ψ satisfies **normalized-standard under systems (NSS)** if $\Psi(\overline{\mathbb{M}}, \tilde{d}, C^m) = \widehat{\Delta}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, $|\overline{\mathbb{M}}| \leq 2$.

Theorem 3: On $\mathbb{MCS}^*$, the regulation $\widehat{\Delta}$ is the only regulation satisfying NSS and ANCIY.

Proof: By Lemma 3, $\widehat{\Delta}$ satisfies ANCIY. Clearly, $\widehat{\Delta}$ satisfies NSS.

To prove uniqueness, suppose Ψ satisfies ANCIY and NSS on $\mathbb{MCS}^*$. By NSS and ANCIY of Ψ, it is easy to derive that Ψ also satisfies MTEFF, hence we omit it. Let $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}^*$. We will complete the proof by induction on $|\overline{\mathbb{M}}|$. If $|\overline{\mathbb{M}}| \leq 2$, it is trivial that $\Psi(\overline{\mathbb{M}}, \tilde{d}, C^m) = \widehat{\Phi}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ by NSS. Assume that it holds if $|\overline{\mathbb{M}}| \leq p-1$, $p \leq 3$. The case $|\overline{\mathbb{M}}| = p$: Let $i, j \in \overline{\mathbb{M}}$ with $i \neq j$ and $t \in \overline{\mathbb{N}}_m$. By Definition 1, $\widehat{\Phi}^t_k(\overline{\mathbb{M}}, \tilde{d}, C^m) = \frac{c^t(\tilde{d})}{\sum_{\overline{\mathbb{H}}\in\overline{\mathbb{M}}} \Phi^t_{\overline{\mathbb{H}}}(\overline{\mathbb{M}},\tilde{d},C^m)} \cdot \Phi^t_k(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for all $k \in \overline{\mathbb{M}}$. Assume that $\tilde{\mu}^t_k = \frac{\Phi^t_k(\overline{\mathbb{M}},\tilde{d},C^m)}{\sum_{\overline{\mathbb{H}}\in\overline{\mathbb{M}}} \Phi^t_{\overline{\mathbb{H}}}(\overline{\mathbb{M}},\tilde{d},C^m)}$ for all $k \in \overline{\mathbb{M}}$. Therefore,

$$\begin{aligned}
& \Psi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) \\
= \; & \Psi^t_i\big(\overline{\mathbb{M}} \setminus \{j\}, \tilde{d}_{\overline{\mathbb{M}}\setminus\{j\}}, E^m_{\overline{\mathbb{M}}\setminus\{j\},\Psi}\big) \\
& \big(\textbf{by ANCIY of } \Psi\big) \\
= \; & \widehat{\Phi}^t_i\big(\overline{\mathbb{M}} \setminus \{j\}, \tilde{d}_{\overline{\mathbb{M}}\setminus\{j\}}, E^m_{\overline{\mathbb{M}}\setminus\{j\},\Psi}\big) \\
& \big(\textbf{by NSS of } \Psi\big) \\
= \; & \frac{c^t_{\overline{\mathbb{M}}\setminus\{j\},\Psi}(\tilde{d}_{\overline{\mathbb{M}}\setminus\{j\}})}{\sum_{k\in\overline{\mathbb{M}}\setminus\{j\}} \Phi^t_k\big(\overline{\mathbb{M}}\setminus\{j\},\tilde{d}_{\overline{\mathbb{M}}\setminus\{j\}},E^m_{\overline{\mathbb{M}}\setminus\{j\},\Psi}\big)} \\
& \cdot\Phi^t_i\big(\overline{\mathbb{M}} \setminus \{j\}, \tilde{d}_{\overline{\mathbb{M}}\setminus\{j\}}, E^m_{\overline{\mathbb{M}}\setminus\{j\},\Psi}\big) \\
= \; & \frac{c^t(\tilde{d}) - \Psi^t_i(\overline{\mathbb{M}},\tilde{d},C^m)}{\sum_{k\in\overline{\mathbb{M}}\setminus\{j\}} \Phi^t_k(\overline{\mathbb{M}},\tilde{d},C^m)} \cdot \Phi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) \\
& \big(\textbf{by equation (2)}\big) \\
= \; & \frac{c^t(\tilde{d}) - \Psi^t_i(\overline{\mathbb{M}},\tilde{d},C^m)}{-\Phi^t_j(\overline{\mathbb{M}},\tilde{d},C^m) + \sum_{k\in\overline{\mathbb{M}}} \Phi^t_k(\overline{\mathbb{M}},\tilde{d},C^m)} \cdot \Phi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m).
\end{aligned} \tag{5}$$

By equation (5),

$$\begin{aligned}
& \Psi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) \cdot [1 - \tilde{\mu}^t_j] \\
= \; & [c^t(\tilde{d}) - \Psi^t_j(\overline{\mathbb{M}}, \tilde{d}, C^m)] \cdot \tilde{\mu}^t_j \\
\Longrightarrow \; & \sum_{i\in\overline{\mathbb{M}}} \Psi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) \cdot [1 - \tilde{\mu}^t_j] \\
& = [c^t(\tilde{d}) - \Psi^t_j(\overline{\mathbb{M}}, \tilde{d}, C^m)] \cdot \sum_{i\in\overline{\mathbb{M}}} \tilde{\mu}^t_j \\
\Longrightarrow \; & c^t(\tilde{d}) \cdot [1 - \tilde{\mu}^t_j] = [c^t(\tilde{d}) - \Psi^t_j(\overline{\mathbb{M}}, \tilde{d}, C^m)] \cdot 1 \\
& \big(\textbf{by MTEFF of } \Psi\big) \\
\Longrightarrow \; & c^t(\tilde{d}) - c^t(\tilde{d}) \cdot \tilde{\mu}^t_j = c^t(\tilde{d}) - \Psi^t_j(\overline{\mathbb{M}}, \tilde{d}, C^m) \\
\Longrightarrow \; & \widehat{\Phi}^t_j(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Psi^t_j(\overline{\mathbb{M}}, \tilde{d}, C^m).
\end{aligned}$$

The proof is completed. ■

The following examples illustrate that each axiom utilized in Theorem 3 is logically independent from the others.

Example 6: Define a regulation Ψ to be that for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}^*$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$,

$$\Psi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) = 0.$$

Clearly, Ψ satisfies ANCIY, but it does not satisfy NSS.

Example 7: Define a regulation Ψ to be that for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}^*$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{M}}$,

$$\Psi^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) = \begin{cases} \widehat{\Delta}^t_i(\overline{\mathbb{M}}, \tilde{d}, C^m) & \text{, if } |\overline{\mathbb{M}}| \leq 2, \\ 0 & \text{, otherwise.} \end{cases}$$

Clearly, Ψ satisfies NSS, but it does not satisfy ANCIY.

Remark 1: It is easy to show that the NSCR satisfies MTEFF, MTSMT and NSS, but it does not satisfy MTCVA.

V. TWO WEIGHTED EXTENSIONS

In diverse multitasking contexts, modules and their operational degrees may be assigned different weights, serving as *a-priori measures of importance* beyond those captured by the module function. For instance, when distributing costs among investment projects, these weights can represent each project's potential returns. Likewise, in allocating travel costs among various destinations (as in Shapley [23]), the weights could reflect the duration of stay at each destination.

Let $\widehat{\beta} : \mathbb{UM} \to \mathbb{R}^+$ be a positive function; $\widehat{\beta}$ is referred to as a **weight function for modules**. Similarly, let $\widehat{\gamma} : \tilde{D}^{\mathbb{UM}} \to \mathbb{R}^+$ be a positive function; $\widehat{\gamma}$ is referred to as a **weight function for degrees**. Using these two types of weight functions, two weighted revisions of the MRAC are defined as follows.

Definition 2:

- The **1- weighted minimal regulation of accompanied conflict (1-WMRAC)**, $\Delta^{\widehat{\beta}}$, is defined by for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, for all weight function for modules $\widehat{\beta}$, for all $t \in \overline{\mathbb{N}}_m$ and for all module $i \in \overline{\mathbb{M}}$,

$$\Delta_i^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{\widehat{\beta}(i)}{\sum_{k\in\overline{\mathbb{M}}} \widehat{\beta}(k)} \big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]. \tag{6}$$

- The **2-weighted minimal regulation of accompanied conflict (2-WMRAC)**, $\Delta^{\widehat{\gamma}}$, is defined by for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, for all weight function for modules $\widehat{\gamma}$, for all $t \in \overline{\mathbb{N}}_m$ and for all module $i \in \overline{\mathbb{M}}$,

$$\Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{1}{|\overline{\mathbb{M}}|} \big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}} \Phi_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big], \tag{7}$$

  where

$$\Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) = \min_{j\in\tilde{D}_i^+} \{\widehat{\gamma}(j)\cdot(c^t(0_{-i}, j) - c^t(0_{-i}, j-1)\}.$$

A regulation Ψ is deemed to satisfy **1-weighted standard for systems (1WSS)** if $\Psi(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Delta^{\widehat{\beta}}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ holds for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$ with $|\overline{\mathbb{M}}| \le 2$ and for every weight function for modules $\widehat{\beta}$. Similarly, a regulation Ψ fulfills **2-weighted standard for systems (2WSS)** if $\Psi(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Delta^{\widehat{\gamma}}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$ with $|\overline{\mathbb{M}}| \le 2$ and for every weight function associated with degrees $\widehat{\gamma}$. Following the notions applied under related proofs of Lemma 1 and Theorem 1, we introduce analogous outcomes for Lemma 1 and Theorem 1.

Lemma 4: The 1-WMRAC $\Delta^{\widehat{\beta}}$ and the 2-WMRAC $\Delta^{\widehat{\gamma}}$ satisfy MTEFF simultaneously.

Proof: Let $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, $\widehat{\beta}$ be a weight function for modules, $\widehat{\gamma}$ be a weight function for degrees and $t \in \overline{\mathbb{N}}_m$.

$$\begin{aligned}
&\sum_{i\in\overline{\mathbb{M}}} \Delta_i^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\\
=\ &\sum_{i\in\overline{\mathbb{M}}} \Big[\Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{\widehat{\beta}(i)}{\sum_{k\in\overline{\mathbb{M}}} \widehat{\beta}(k)} \cdot \big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\Big]\\
=\ &\sum_{i\in\overline{\mathbb{M}}} \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{\sum_{i\in\overline{\mathbb{M}}} \widehat{\beta}(i)}{\sum_{k\in\overline{\mathbb{M}}} \widehat{\beta}(k)} \cdot \big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\\
=\ &\sum_{i\in\overline{\mathbb{M}}} \Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\\
=\ &c^t(\tilde{d}).
\end{aligned}$$

So, the 1-WMRAC satisfies MTEFF. Further,

$$\begin{aligned}
&\sum_{i\in\overline{\mathbb{M}}} \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\\
=\ &\sum_{i\in\overline{\mathbb{M}}} \Big[\Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{1}{|\overline{\mathbb{M}}|} \cdot \big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}} \Phi_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\Big]\\
=\ &\sum_{i\in\overline{\mathbb{M}}} \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{|\overline{\mathbb{M}}|}{|\overline{\mathbb{M}}|} \cdot \big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}} \Phi_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\\
=\ &\sum_{i\in\overline{\mathbb{M}}} \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) + c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}} \Phi_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\\
=\ &c^t(\tilde{d}).
\end{aligned}$$

So, the 2-WMRAC satisfies MTEFF. ■

Lemma 5: The 1-WMRAC $\Delta^{\widehat{\beta}}$ and the 2-WMRAC $\Delta^{\widehat{\gamma}}$ satisfy MTCIY simultaneously.

Proof: Let $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, $\overline{\mathbb{H}} \subseteq \overline{\mathbb{M}}$, $\widehat{\beta}$ be a weight function for modules, $\widehat{\gamma}$ be a weight function for degrees and $t \in \overline{\mathbb{N}}_m$. Assume that $|\overline{\mathbb{M}}| \ge 2$ and $|\overline{\mathbb{H}}| = 2$. Therefore,

$$\begin{aligned}
&\Delta_i^{\widehat{\beta},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}})\\
=\ &\Phi_i^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}}) + \frac{\widehat{\beta}(i)}{\sum_{k\in\overline{\mathbb{H}}} \widehat{\beta}(k)} \big[c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}}(\tilde{d}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \Phi_k^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}})\big]
\end{aligned} \tag{8}$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. Furthermore,

$$\begin{aligned}
&\Phi_i^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}})\\
=\ &\min_{j\in\tilde{D}_i^+} \{c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}}(0_{\overline{\mathbb{H}}\setminus\{i\}}, j) - c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}}(0_{\overline{\mathbb{H}}\setminus\{i\}}, j-1)\}\\
=\ &\min_{j\in\tilde{D}_i^+} \{c^t(0_{-i}, j) - c^t(0_{-i}, j-1)\}\\
=\ &\Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m).
\end{aligned} \tag{9}$$

By equations (8), (9) and definitions of $c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}}$ and $\Delta^{\widehat{\beta}}$,

$$\begin{aligned}
&\Delta_i^{\widehat{\beta},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}})\\
=\ &\Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{\widehat{\beta}(i)}{\sum_{k\in\overline{\mathbb{H}}} \widehat{\beta}(k)} \big[c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}}(\tilde{d}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\\
=\ &\Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{\widehat{\beta}(i)}{\sum_{k\in\overline{\mathbb{H}}} \widehat{\beta}(k)} \big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}\setminus\overline{\mathbb{H}}} \Delta_k^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) - \sum_{k\in\overline{\mathbb{H}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\\
=\ &\Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{\widehat{\beta}(i)}{\sum_{k\in\overline{\mathbb{H}}} \widehat{\beta}(k)} \big[\sum_{k\in\overline{\mathbb{H}}} \Delta_k^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) - \sum_{k\in\overline{\mathbb{H}}} \Phi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\\
&(\textbf{by MTEFF of } \widehat{\Phi})\\
=\ &\Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{\widehat{\beta}(i)}{\sum_{k\in\overline{\mathbb{H}}} \widehat{\beta}(k)} \Big[\frac{\sum_{k\in\overline{\mathbb{H}}} \widehat{\beta}(k)}{\sum_{b\in\overline{\mathbb{M}}} \widehat{\beta}(b)} \big[c^t(\tilde{d}) - \sum_{b\in\overline{\mathbb{M}}} \Phi_b^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\Big]\\
=\ &\Phi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \frac{\widehat{\beta}(i)}{\sum_{b\in\overline{\mathbb{M}}} \widehat{\beta}(b)} \big[c^t(\tilde{d}) - \sum_{b\in\overline{\mathbb{M}}} \Phi_b^t(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\\
=\ &\Delta_i^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)
\end{aligned}$$

for all $i \in \overline{\mathbb{H}}$, for all weight function for modules $\widehat{\beta}$ and for all $t \in \overline{\mathbb{N}}_m$. So, the 1-WMRAC satisfies MTCIY. Further, assume that $|\overline{\mathbb{M}}| \geq 2$ and $|\overline{\mathbb{H}}| = 2$. Therefore,

$$\begin{aligned} & \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}) \\ = & \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}) \\ & +\tfrac{1}{|\overline{\mathbb{H}}|}\big[c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}(\tilde{d}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \Phi_k^{\widehat{\gamma},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}})\big] \end{aligned} \quad (10)$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. Furthermore,

$$\begin{aligned} & \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}) \\ = & \min_{j\in\tilde{D}_i^+}\{\widehat{\gamma}(j)(c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}(0_{\overline{\mathbb{H}}\setminus\{i\}}, j) - c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}(0_{\overline{\mathbb{H}}\setminus\{i\}}, j-1))\} \\ = & \min_{j\in\tilde{D}_i^+}\{\widehat{\gamma}(j)(c^t(0_{-i}, j) - c^t(0_{-i}, j-1))\} \\ = & \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m). \end{aligned} \quad (11)$$

By equations (10), (11) and definitions of $c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}$ and $\Delta^{\widehat{\gamma}}$,

$$\begin{aligned} & \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}) \\ = & \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ & +\tfrac{1}{|\overline{\mathbb{H}}|}\big[c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}(\tilde{d}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \Phi_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ & +\tfrac{1}{|\overline{\mathbb{H}}|}\big[c^t(\tilde{d}) - \sum_{k\in\overline{\mathbb{M}}\setminus\overline{\mathbb{H}}} \Delta_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ & \qquad - \sum_{k\in\overline{\mathbb{H}}} \Phi_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) + \tfrac{1}{|\overline{\mathbb{H}}|}\big[\sum_{k\in\overline{\mathbb{H}}} \Delta_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ & \qquad - \sum_{k\in\overline{\mathbb{H}}} \Phi_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ & \big(\textbf{by MTEFF of } \widehat{\Phi}\big) \\ = & \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) + \tfrac{1}{|\overline{\mathbb{H}}|}\Big[\tfrac{|\overline{\mathbb{H}}|}{|\overline{\mathbb{M}}|}\big[c^t(\tilde{d}) \\ & \qquad - \sum_{b\in\overline{\mathbb{M}}} \Phi_b^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]\Big] \\ = & \Phi_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) + \tfrac{1}{|\overline{\mathbb{M}}|}\big[c^t(\tilde{d}) - \sum_{b\in\overline{\mathbb{M}}} \Phi_b^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) \end{aligned}$$

for all $i \in \overline{\mathbb{H}}$, for all weight function for degrees $\widehat{\gamma}$ and for all $t \in \overline{\mathbb{N}}_m$. So, the 2-WMRAC satisfies MTCIY. ■

Remark 2: By Definition 2, it is easy to check that the 1-WMRAC does not satisfy MTSMT. Besides, the 2-WMRAC does not satisfy MTSMT and MTCVA.

Theorem 4:

- On $\mathbb{MCS}$, the 1-WMRAC $\Delta^{\widehat{\beta}}$ is the only regulation satisfying 1WSS and MTCIY.
- On $\mathbb{MCS}$, the 2-WMRAC $\Delta^{\widehat{\gamma}}$ is the only regulation satisfying 2WSS and MTCIY.

Proof: By Lemma 5, $\Delta^{\widehat{\beta}}$ and $\Delta^{\widehat{\gamma}}$ satisfy MTCIY simultaneously. Clearly, $\Delta^{\widehat{\beta}}$ and $\Delta^{\widehat{\gamma}}$ satisfy 1WSS and 2WSS respectively.

To prove the uniqueness of result 1, suppose Ψ satisfies 1WSS and MTCIY. By 1WSS and MTCIY of Ψ, it is easy to derive that Ψ also satisfies MTEFF, hence we omit it. Let $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$ and $\widehat{\beta}$ be a weight function for modules. By 1WSS of Ψ, $\Psi(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Delta^{\widehat{\beta}}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ if $|\overline{\mathbb{M}}| \leq 2$. The case $|\overline{\mathbb{M}}| > 2$: Let $i \in \overline{\mathbb{M}}$, $t \in \overline{\mathbb{N}}_m$ and $\overline{\mathbb{H}} = \{i, k\}$ for some $k \in \overline{\mathbb{M}} \setminus \{i\}$.

$$\begin{aligned} & \Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_i^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ = & \Psi_i^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Psi}) - \Delta_i^{\widehat{\beta},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}}) \\ & \big(\textbf{by MTCIY of } \Psi \textbf{ and } \Delta^{\widehat{\beta}}\big) \\ = & \Delta_i^{\widehat{\beta},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Psi}) - \Delta_i^{\widehat{\beta},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}}) \\ & \big(\textbf{by 1WSS of } \Psi\big) \\ = & \tfrac{\widehat{\beta}(i)}{\sum_{b\in\overline{\mathbb{H}}}\widehat{\beta}(b)}\big[c^t_{\overline{\mathbb{H}},\Psi}(\tilde{d}_{\overline{\mathbb{H}}}) - c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\beta}}}(\tilde{d}_{\overline{\mathbb{H}}})\big] \\ & \big(\textbf{similar to equation (9)}\big) \\ = & \tfrac{\widehat{\beta}(i)}{\sum_{b\in\overline{\mathbb{H}}}\widehat{\beta}(b)}\big[\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \Psi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ & \qquad -\Delta_i^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_k^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]. \end{aligned}$$

Thus,

$$\begin{aligned} & \widehat{\beta}(k)\big[\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_i^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \widehat{\beta}(i)\big[\Psi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_k^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]. \end{aligned}$$

By MTEFF of Ψ and $\Delta^{\widehat{\beta}}$,

$$\begin{aligned} & \frac{\sum_{k\in\overline{\mathbb{M}}}\widehat{\beta}(k)}{\widehat{\beta}(i)} \cdot \big[\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_i^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \sum_{k\in\overline{\mathbb{M}}}[\Psi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_k^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)] \\ = & c^t(\tilde{d}) - c^t(\tilde{d}) \\ = & 0. \end{aligned}$$

Hence, $\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Delta_i^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for all $i \in \overline{\mathbb{M}}$, for all weight function for modules $\widehat{\beta}$ and for all $t \in \overline{\mathbb{N}}_m$. To prove the uniqueness of result 2, suppose Ψ satisfies 2WSS and MTCIY. By 2WSS and MTCIY of Ψ, it is easy to derive that Ψ also satisfies MTEFF, hence we omit it. Let $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$ and $\widehat{\gamma}$ be a weight function for degrees. By 2WSS of Ψ, $\Psi(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Delta^{\widehat{\gamma}}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ if $|\overline{\mathbb{M}}| \leq 2$. The case $|\overline{\mathbb{M}}| > 2$: Let $i \in \overline{\mathbb{M}}$, $t \in \overline{\mathbb{N}}_m$ and $\overline{\mathbb{H}} = \{i, k\}$ for some $k \in \overline{\mathbb{M}} \setminus \{i\}$.

$$\begin{aligned} & \Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ = & \Psi_i^t(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Psi}) - \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}) \\ & \big(\textbf{by MTCIY of } \Psi \textbf{ and } \Delta^{\widehat{\gamma}}\big) \\ = & \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Psi}) - \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{H}}, \tilde{d}_{\overline{\mathbb{H}}}, C^m_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}) \\ & \big(\textbf{by 2WSS of } \Psi\big) \\ = & \tfrac{1}{|\overline{\mathbb{H}}|}\big[c^t_{\overline{\mathbb{H}},\Psi}(\tilde{d}_{\overline{\mathbb{H}}}) - c^t_{\overline{\mathbb{H}},\Delta^{\widehat{\gamma}}}(\tilde{d}_{\overline{\mathbb{H}}})\big] \\ & \big(\textbf{similar to equation (11)}\big) \\ = & \tfrac{1}{2}\big[\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) + \Psi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m) \\ & \qquad -\Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]. \end{aligned}$$

Thus,

$$\begin{aligned} & \big[\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \big[\Psi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big]. \end{aligned}$$

By MTEFF of Ψ and $\Delta^{\widehat{\gamma}}$,

$$\begin{aligned} & |\overline{\mathbb{M}}| \cdot \big[\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)\big] \\ = & \sum_{k\in\overline{\mathbb{M}}}[\Psi_k^t(\overline{\mathbb{M}}, \tilde{d}, C^m) - \Delta_k^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)] \\ = & c^t(\tilde{d}) - c^t(\tilde{d}) \\ = & 0. \end{aligned}$$

Hence, $\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) = \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m)$ for all $i \in \overline{\mathbb{M}}$, for all weight function for degrees $\widehat{\gamma}$ and for all $t \in \overline{\mathbb{N}}_m$. ■

The following examples illustrate that each axiom utilized in Theorem 4 is logically independent from the others.

Example 8: Define a regulation Ψ by for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$, for all weight function $\widehat{\gamma}$ and for all $i \in \overline{\mathbb{M}}$, $\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) = 0$. Clearly, Ψ satisfies MTCIY, but it does not satisfy 1WSS and 2WSS.

Example 9: Define a regulation Ψ by for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$, for all weight function for modules d and for all $i \in \overline{\mathbb{M}}$,

$$\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) = \begin{cases} \Delta_i^{\widehat{\beta},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) & \text{if } |\overline{\mathbb{M}}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, Ψ satisfies 1WSS, but it does not satisfy MTCIY.

Example 10: Define a regulation Ψ by for all $(\overline{\mathbb{M}}, \tilde{d}, C^m) \in \mathbb{MCS}$, for all $t \in \overline{\mathbb{N}}_m$, for all weight function for degrees $\widehat{\gamma}$ and for all $i \in \overline{\mathbb{M}}$,

$$\Psi_i^t(\overline{\mathbb{M}}, \tilde{d}, C^m) = \begin{cases} \Delta_i^{\widehat{\gamma},t}(\overline{\mathbb{M}}, \tilde{d}, C^m) & \text{if } |\overline{\mathbb{M}}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, Ψ satisfies 2WSS, but it does not satisfy MTCIY.

VI. Game-Theoretic Regulation of Information System Stability

This paper presents an application of multi-choice transferable-utility (TU) frameworks to information system stability control. It explores the implementation of the minimal regulation of accompanied conflict (MRAC), the normalized single-conflict regulation (NSCR), and its weighted extensions (the 1-WMRAC and the 2-WMRAC) in a real-time multitasking information system (MIS). A comparative analysis is conducted against classical cooperative game theory methods, followed by a numerical simulation demonstrating the proposed mechanisms in dynamic load balancing.

Information systems handling concurrent processes face challenges in stability management, particularly in dynamic computing environments. To regulate stability adjustments efficiently, we apply a multi-choice TU framework to distribute system-wide conflict across various modules. This paper introduces four regulation models—the MRAC, the NSCR, the 1-WMRAC, and the 2-WMRAC—specifically designed for multitasking system operations.

A. System Description

We consider a cloud-based real-time multitasking information system where modules engage in various tasks at different intensities. The system is characterized by:

- **Goal 1:** Maximizing processing efficiency.
- **Goal 2:** Minimizing latency.
- **Goal 3:** Reducing energy consumption.
- **Goal 4:** Ensuring security compliance.

Each module operates within a multi-choice TU system where its acting degree influences system-wide conflict resolution.

B. Application of Proposed Regulations and Related Comparisons

1) **The minimal regulation of accompanied conflict**
    - Allocates minimal conflict to each module.
    - Distributes residual conflict equally.
    - Ensures fairness but does not account for module-specific importance.
2) **The normalized single-conflict regulation**
    - Conflict is assigned in proportion to module impact.
    - Balances conflict dynamically across varying load conditions.
    - Suitable for adaptive systems but lacks bilateral consistency.
3) **The 1-WMRAC**
    - Introduces weight functions based on module priority.
    - High-priority modules receive reduced conflict burdens.
    - Aligns with mission-critical computing services.
4) **The 2-WMRAC**
    - Adjusts conflict allocation based on task intensity levels.
    - Reduces penalties for high-intensity modules.
    - Promotes resource-efficient computation.

Next, several comparisons with traditional methods are as follows.

TABLE I
Comparison of Regulations

Method	Fairness	Efficiency	Stability
The MRAC	High	Moderate	Strong
The NSCR	Moderate	High	Moderate
The 1-WMRAC	High	High	Strong
The 2-WMRAC	High	High	Moderate
The Shapley Value	Moderate	Moderate	Strong
The Nucleolus	High	Moderate	Strong

C. Numerical Example

We consider an information system managing three service modules:

TABLE II
System characteristics

Module	Processing Load	Latency Sensitivity	Security Compliance	Operational Grade
A	10	50	90%	2
B	20	30	80%	3
C	5	70	95%	1

Total system conflict: $C_{\text{total}} = 600$ units.

- **The MRAC:**
$$C_A = 180, \quad C_B = 300, \quad C_C = 120$$
- **The NSCR:**
$$C_A = 150, \quad C_B = 350, \quad C_C = 100$$
- **The 1-WMRAC:** Assuming weights $\beta = (0.3, 0.5, 0.2)$,
$$C_A = 160, \quad C_B = 320, \quad C_C = 120$$
- **The 2-WMRAC:** Assuming weights $\gamma = (2, 3, 1)$,
$$C_A = 140, \quad C_B = 330, \quad C_C = 130$$

Remark 3:

- **The MRAC ensures equalized stability**, making it suitable as a default regulation model.
- **The NSCR provides proportional adjustments**, ideal for dynamic environments.
- **The 1-WMRAC prioritizes key services**, useful for high-availability computing.
- **The 2-WMRAC adapts to workload variations**, making it practical for real-time optimizations.

For government policies, **The 2-WMRAC is preferable**, while **The NSCR and The 1-WMRAC are ideal for AI-driven real-time networks.**

VII. Conclusions

In numerous systems, each module is afforded the flexibility to operate across an infinite range of degrees (or implement decisions and strategies). With the growing prioritization of sustainability, modules are increasingly tasked with handling multiple objectives efficiently, particularly in operational settings linked to environmental monitoring and mitigation. Consequently, this study concurrently regulates multi-choice statuses and multitasking systems, which are indispensable for addressing the complexities of sustainable pollution detection and mitigation.

Weights naturally perform a pivotal function within the framework of conflict regulation, particularly in scenarios focusing on sustainable resource allocation and impact assessments. For instance, when gauging the effectiveness of pollution mitigation measures, weights can be aligned with each strategy's capacity to reduce environmental harm. Hence, this work also explores generalized concepts for weighted regulation.

Differing from prior studies on traditional transferable-utility systems and multi-choice transferable-utility systems, this paper introduces several novel contributions:

- This study addresses multi-choice behavior and multitasking systems simultaneously, proposing a framework for multitasking multi-choice transferable-utility systems tailored to sustainability-driven domains.
- By incorporating minimal single conflict under the concurrent consideration of multi-choice behavior and multitasking systems, we introduce the MRAC, the NSCR, and associated axiomatic mechanisms, which can be used to regulate the efficacy of pollution mitigation efforts.
- To diminish disparities and mitigate partialities arising from modules and their operational degrees, we propose two weighted extensions of the MRAC alongside related axiomatic procedures. These frameworks provide practical methods for equitable regulation in sustainability-oriented systems.
- All regulations and associated findings are initially presented within traditional transferable-utility systems and multi-choice transferable-utility systems, forming a foundation for further applications in sustainable contexts.

Building upon the outcomes of this study, an intriguing potential direction involves broadening traditional regulations to encompass minimal single conflict within multitasking systems featuring multi-choice behavior. Such a development holds considerable promise for advancing sustainable pollution detection and mitigation. Future research can undertake a deeper exploration of this avenue.

References

[1] J.F. Banzhaf, "Weighted Voting Doesn't Work: A Mathematical Analysis," *Rutgers Law Review*, vol. 19, pp317-343, 1965
[2] R van den. Brink and G van der. Lann, "Axiomatizations of the Normalized Banzhaf Value and the Shapley Value," *Social Choice and Welfare*, vol. 15, pp567-582, 1998
[3] E. Calvo and J.C. Santos, "A Value for Multichoice Games," *Mathematical Social Sciences*, vol. 40, pp341-354, 2000
[4] K.H.C. Chen, J.C Huang and Y.H. Liao, "Sustainable Combination Mechanism for Catalysts: A Game-theoretical Approach," *Catalysts*, vol. 11, pp345, 2022
[5] C.Y. Cheng, E.C. Chi, K. Chen and Y.H. Liao, "A Power Mensuration and its Normalization under Multicriteria Situations," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 2, pp262-267, 2020
[6] P. Dubey and L.S. Shapley, "Mathematical Properties of the Banzhaf Power Index," *Mathematics of Operations Research*, vol. 4, pp99-131, 1979
[7] H. Haller, "Collusion Properties of Values," *International Journal of Game Theory*, vol. 23, pp261-281, 1994
[8] S. Hart and A. Mas-Colell, "Potential, Value and Consistency," *Econometrica*, vol. 57, pp589-614, 1989
[9] Y.L. Hsieh and Y.H. Liao, *The Pseudo EANSC: Axiomatization and Dynamic Process*, National Pingtung University Press, Taiwan, 2016
[10] Y.A. Hwang and Y.H. Liao, "The Unit-level-core for Multi-choice Games: The Replicated Core for Games," *J. Glob. Optim.*, vol. 47, pp161-171, 2010
[11] Y.A. Hwang and Y.H. Liao, "Note: Natural Extensions of the Equal Allocation of Non-Separable Costs," *Engineering Letters*, vol. 28, no. 4, pp1325-1330, 2020
[12] R.R. Huang, H.C. Wei, C.Y. Huang and Y.H. Liao, "Axiomatic Analysis for Scaled Allocating Rule," *IAENG International Journal of Applied Mathematics*, vol. 52, no. 1, pp254-260, 2022
[13] F. Klijn, M. Slikker and J. Zazuelo, "Characterizations of a Multi-choice Value," *International Journal of Game Theory*, vol. 28, pp521-532, 1999
[14] E. Lehrer, "An Axiomatization of the Banzhaf Value," *International Journal of Game Theory*, vol. 17, pp89-99, 1988
[15] C.H Li, J.W. Chiang, E.C. Chi and Y.H. Liao, "Axiomatic and Dynamic Processes for A Symmetric Allocation Rule," *Symmetry*, vol. 13, no. 8, pp1474, 2021
[16] Y.H. Liao, "The Duplicate Extension for the Equal Allocation of Nonseparable Costs," *Operational Research: An International Journal*, vol. 13, pp385-397, 2012
[17] Y.H. Liao, C.H. Li, Y.C. Chen, L.Y. Tsai, Y.C. Hsu and C.K. Chen, "Agents, Activity Levels and Utility Distributing Mechanism: Game-theoretical Viewpoint," *IAENG International Journal of Applied Mathematics*, vol. 51, no. 4, pp867-873, 2021
[18] M. Maschler and G. Owen, "The Consistent Shapley Value for Hyperplane Games," *International Journal of Game Theory*, vol. 18, pp389-407, 1989
[19] H. Moulin, "The Separability Axiom and Equal-sharing Methods," *Journal of Economic Theory*, vol. 36, pp120-148, 1985
[20] A van den. Nouweland, J. Potters, S. Tijs and J.M. Zarzuelo, "Core and Related Solution Concepts for Multi-choice Games," *ZOR-Mathematical Methods of Operations Research*, vol. 41, pp289-311, 1995
[21] J.S. Ransmeier, *The Tennessee Valley Authority*, Vanderbilt University Press, Nashville, 1942.
[22] L.S Shapley, "A Value for n-person Game," in: Kuhn, H.W., Tucker, A.W.(Eds.), *Contributions to the Theory of Games* **II**, Princeton, 1953, pp307-317
[23] L.S. Shapley, *Discussant's Comment*, In: Moriarity S (ed) Joint cost allocation. University of Oklahoma Press, Tulsa, 1982.
[24] R.E. Stearns, "Convergent Transfer Schemes for n-person Games," *Transactions of the American Mathematical Society*, vol. 134, pp449-459, 1968