Some Important Inequalities about the Additive Schwarz Preconditioners

Qianfeng Qian, Lingzhi Qian, Huiping Cai*

Abstract—**There are some important inequalities regarding the additive Schwarz preconditioners, which can be used to estimate stability, convergence, and error estimates in model problems. In this work, we prove these important inequalities about the additive Schwarz preconditioners in detail, which is widely used in domain decomposition method.**

Index Terms—**Discrete Sobolev inequality, Sobolev space, Schwarz preconditioners, Domain decomposition method.**

I. Introduction

THE additive Schwarz preconditioners are widely used in the domain decomposition methods. The additive Schwarz type preconditioners have received more and more attention recently[1], [2], [6], [11], [13]. For example, the overlapping additive Schwarz preconditioners are usually used in the domain decomposition with overlap[8], [9]. Two-level additive Schwarz preconditioners for the systems of linear equations resulting from conforming and nonconforming finite element approximations of elliptic boundary value problems are developed in[3], [4], [5], [10]. Restricted additive Schwarz preconditioners with harmonic overlap are developed for the symmetric positive definite problems and elliptic equations[7], [10], [12]. Recently, in the field of numerical solution of elliptic interface problems, research combining the Nitsche extended finite element method with a two-level additive Schwarz preconditioner has provided an effective algorithm to handle complex interface problems without the need for precisely matched meshes[14]. There are some important inequalities about the additive Schwarz preconditioners. Detailed and ingenious proofs of these important inequalities are given in this paper. The inequalities are widely used in domain decomposition methods and additive Schwarz preconditioners.

To facilitate the discussion, let's begin with some notations. Let Ω be divided into polygonal subdomains $\Omega_1, \cdots, \Omega_J$ such that

$$\begin{aligned} &\Omega_j \cap \Omega_l = \emptyset, \qquad \text{if} \quad j \neq l, \\ &\overline{\Omega} = \bigcup_{j=1}^{J} \overline{\Omega}_j, \\ &\partial\Omega_j \cap \partial\Omega_l = \emptyset, \text{a vertex or an edge, if} \quad j \neq l, \end{aligned}$$

and $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω which is aligned with the boundaries of the subdomains.

Let $\Gamma_j = \partial\Omega_j \backslash \partial\Omega$. The interface (skeleton) of the domain decomposition $\Omega_1, \cdots, \Omega_J$ is $\Gamma = \bigcup_j^J \Gamma_j$. The set of nodes of $\mathcal{T}_h$ which belong to Γ_j ($resp.$ Γ) is denoted by $\Gamma_{j,h}$ ($resp.$ Γ_h).

We assume that the subdomains satisfy the following shape regularity assumption: there exist a reference polygonal domain $D_1, \cdots, D_K$ of unit diameter and a positive number H such that for each subdomain Ω_j there is a reference polgon D_k and a C^1 diffeomorphism $\Phi_{j,k} : \overline{D}_k \to \overline{\Omega}_j$ which satisfies the estimates

$$|\nabla\Phi_{j,k}(x)| \lesssim H \ \forall x \in \overline{D}_k \quad \text{and} \quad |\nabla(\Phi^{-1})_{j,k}(x)| \lesssim H^{-1} \ \forall x \in \overline{\Omega}_j.$$

The shape regularity condition implies that all the estimates involving the subdomains follow from corresponding estimates on the reference domains , which also implies

$$\text{diam}\Omega_j \approx H.$$

So, we only need to study the inequalities on the reference domains $D_1, \cdots, D_k$, and then the corresponding inequalities on the subdomains $\Omega_1, \cdots, \Omega_J$ can be obtained easily by using scaling arguments. For simplicity, we consider a reference domain which is an unit square, denoted by D. The following discussions can be extended to other reference domains D_k easily. $\mathcal{T}_{\hat{h}}$ is a quasi-uniform triangulation of D corresponding to $\mathcal{T}_h$, and $\hat{V}_{\hat{h}}$ is the $\mathcal{P}_1$ finite element space.

Definition 1.1: The fractional order Sobolev semi-norm $|\cdot|_{H^{1/2}(\partial D)}$, which measures the smoothness of functions on the boundary ∂D , is defined by

$$|v|^2_{H^{1/2}(\partial D)} = \int_{\partial D}\left[\int_{\partial D}\frac{|v(x)-v(y)|^2}{|x-y|^2}ds(x)\right]ds(y),$$

where ds is the differential of the arc-length. The space $H^{1/2}(\partial D)$ consists of functions $v \in L^2(\partial D)$ such that $|v|_{H^{1/2}(\partial D)} < \infty$, and we define

$$\|v\|^2_{H^{1/2}(\partial D)} = \|v\|^2_{L^2(\partial D)} + |v|^2_{H^{1/2}(\partial D)}.$$

Definition 1.2: The finite space on ∂D, denoted by $\mathcal{L}_{\hat{h}}(\partial D)$ is defined by

$$\mathcal{L}_{\hat{h}}(\partial D) = \{v \in C(\partial D) : v \text{ is piecewise linear with respect to the subdivision of } \partial D \text{ induced by } \mathcal{T}_{\hat{h}}\},$$

Manuscript received October 21, 2024, revised April 14, 2025. This work was supported in part by the the NSF of China (No. 11861054), Natural Science Foundation of Guangxi (No. 2020GXNSFAA297223), Innovation Project of Guangxi Guaduate Education(No. JGY2021028) and English curriculum construction project of Guangxi Normal University (No. 2021XJQYW04).

Qianfeng Qian is a graduate student of the Guangxi Normal University, Guilin Guangxi 541000, P.R. China (e-mail:2023011858@stu.gxnu.edu.cn).

Lingzhi Qian is a professor of the College of Mathematics and Statistics in Guangxi Normal University, Guilin 541006, P.R. China (corresponding author, e-mail: qianlz@mailbox.gxnu.edu.cn).

*Huiping Cai is an associate professor of the College of Mathematics and Statistics in Guangxi Normal University, Guilin 541006, P.R. China (corresponding author, e-mail: caihp1103@sina.com).

(a) Point P is on the line AB.

(b) Point P is outside the line AB.

Fig. 1: Position relation between point P and line AB.

which is the restriction of $\hat{V}_{\hat{h}}$ to ∂D.

Now, we give the inequalities below firstly, and then try to prove in the next section.

Theorem 1.1: (**Discrete Sobolev Inequality**)

$$\|v\|_{L^\infty} \le C(1+|\ln h|)^{\frac{1}{2}}\|v\|_{H^1(\Omega)}, \quad \forall v \in V_h$$

where the positive constant C is independent of h, $V_h \subset H^1(\Omega)$.

Theorem 1.2: Let $v \in \mathcal{L}_{\hat{h}}(\partial D)$, E be an open edge of ∂D, and $v_E \in \mathcal{L}_{\hat{h}}(\partial D)$ be defined by

$$v_E(p) = \begin{cases} v(p) & \text{if the node } p \in E, \\ 0 & \text{if the node } p \in \partial D \backslash E. \end{cases}$$

Then the following estimate holds:

$$|v_E|^2_{H^{1/2}(\partial D)} \lesssim |v|^2_{H^{1/2}(\partial D)} + (1+|\ln \hat{h}|)\|v\|^2_{L^\infty(\partial D)}.$$

Theorem 1.3: Let p be a node on ∂D, and define $v_p \in \mathcal{L}_{\hat{h}}(\partial D)$ such that it vanishes at all the nodes except p and $v_p(p) = 1$. Then we have $|v_p|_{H^{1/2}(\partial D)} \approx 1$.

II. Preliminaries

Lemma 2.1: Suppose A and B are two points on line AB.
(1) If P is on the line AB , A and B are on one side of P (cf. Figure 1(a)), then

$$\int_A^B \frac{1}{|PX|^2}dX = \frac{1}{|PA|} - \frac{1}{|PB|}.$$

(2) If P is outside of the line AB (cf. Figure 1(b)), then

$$\int_A^B \frac{1}{|PX|^2}dX = \frac{\theta}{d},$$

where d is the distance between point P and line AB, and θ is the arc of $\angle APB$.

Proof : (1) We establish the coordinate system so that the coordinates of P, A, X, B is $(x_P, 0)$, $(x_A, 0)$, $(x, 0)$, $(x_B, 0)$, and $x_B \ge x \ge x_A \ge x_P \ge 0$.
Then

$$\begin{aligned} \int_A^B \frac{1}{|PX|^2}dX &= \int_{x_A}^{x_B} \frac{1}{(x-x_P)^2}dx \\ &= \left.\frac{-1}{x-x_P}\right|_{x_A}^{x_B} \\ &= \frac{1}{|PA|} - \frac{1}{|PB|}. \end{aligned}$$

(2) We also establish the coordinate system so that the coordinates of P, A, B is (x_P, y_P), $(x_A, 0)$, $(x, 0)$ $(x_B, 0)$, and $x_B \ge x \ge x_A \ge x_p$.
Then

$$\begin{aligned} \int_A^B \frac{1}{|PX|^2}dX &= \int_{x_A}^{x_B} \frac{1}{d^2+(x-x_P)^2}dx \\ &= \frac{1}{d}\int_{x_A}^{x_B} \frac{1}{1+(\frac{x-x_P}{d})^2}d(\frac{x-x_p}{d}) \\ &= \frac{1}{d}\arctan(t)\Big|_{(x_A-x_P)/d}^{(x_B-x_P)/d} \\ &= \frac{\theta}{d}. \end{aligned}$$

Lemma 2.2: Let $0 = a_0 < a_1 < \cdots < a_n = 1$ be a quasi-uniform partition of the unit interval I so that $a_j - a_i \approx \rho = 1/n$ for $1 \le j \le n$, and $\mathcal{L}_\rho$ be the space of continuous functions on $[0,1]$ which are piecewise linear with respect to this partition. Given any $v \in \mathcal{L}_\rho$, we define $v_* \in \mathcal{L}_\rho$ by

$$v_*(a_j) = \begin{cases} v(a_j) & 1 \le j \le n-1, \\ 0 & j = 0, n. \end{cases}$$

then we have

$$|v_*|_{H^{1/2}(I)} \le C\left(|v|_{H^{1/2}(I)} + \|v\|_{L^\infty(I)}\right)$$

for all $v \in \mathcal{L}_\rho$, where C is a positive constant independent of ρ.

Proof : We define $v^* = v - v_*$, which is in $\mathcal{L}_\rho$, equals v at a_0, a_n and vanishes at all the other a_j.

$$\begin{aligned} |v^*|^2_{H^{1/2}(I)} &= \int_{a_0}^{a_n}\left[\int_{a_0}^{a_n}\frac{|v^*(x)-v^*(y)|^2}{|x-y|^2}dx\right]dy \\ &= \left\{\int_{a_0}^{a_1}\int_{a_0}^{a_1} + \int_{a_{n-1}}^{a_n}\int_{a_{n-1}}^{a_n}\right\}\frac{|v^*(x)-v^*(y)|^2}{|x-y|^2}dxdy \\ &+ 2\int_{a_0}^{a_1}\left[(v^*(y))^2\int_{a_1}^{a_{n-1}}\frac{1}{|x-y|^2}dx\right]dy \\ &+ 2\int_{a_{n-1}}^{a_n}\left[(v^*(y))^2\int_{a_1}^{a_{n-1}}\frac{1}{|x-y|^2}dx\right]dy \\ &+ 2\int_{a_0}^{a_1}\int_{a_{n-1}}^{a_n}\frac{|v^*(x)-v^*(y)|^2}{|x-y|^2}dxdy \\ &\triangleq \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4. \end{aligned}$$

From the definition of $v^*(x)$, we observe that

$$v^*(x) = \begin{cases} \frac{v(a_0)}{a_0-a_1}(x-a_1) & \text{if } x \in [a_0, a_1], \\ 0 & \text{if } x \in (a_1, a_{n-1}), \\ \frac{v(a_n)}{a_n-a_{n-1}}(x-a_{n-1}) & \text{if } x \in [a_{n-1}, a_n], \end{cases} \tag{1}$$

so

$$\begin{aligned} \Pi_1 &= \int_{a_0}^{a_1}\int_{a_0}^{a_1}\left|\frac{v(a_0)}{a_1-a_0}\right|^2 dxdy + \int_{a_{n-1}}^{a_n}\int_{a_{n-1}}^{a_n}\left|\frac{v(a_n)}{a_n-a_{n-1}}\right|^2 dxdy \\ &= (v(a_0))^2 + (v(a_n))^2 \\ &\le C\|v\|^2_{L^\infty(I)}, \end{aligned}$$

from lemma 2.1(1), we have

$$\begin{aligned} \Pi_2 &= 2\int_{a_0}^{a_1}\left[\left|\frac{v(a_0)}{a_0-a_1}(y-a_1)\right|^2\left(\frac{1}{|y-a_1|} - \frac{1}{|y-a_{n-1}|}\right)\right]dy \\ &= 2\int_{a_0}^{a_1}\left|\frac{v(a_0)}{a_0-a_1}\right|^2\left[|y-a_1| - \frac{(y-a_1)^2}{|y-a_{n-1}|}\right]dy \\ &\le 2\left|\frac{v(a_0)}{a_0-a_1}\right|^2\left[(-\frac{(a_1-y)^2}{2})\Big|_{a_0}^{a_1} - \frac{1}{3}\frac{1}{a_{n-1}-a_0}(y-a_1)^3\Big|_{a_0}^{a_1}\right] \\ &\le C\|v\|^2_{L^\infty(I)}, \end{aligned}$$

similarly, we have

$$\begin{aligned}\Pi_3 &= 2\int_{a_{n-1}}^{a_n}\left[\left|\frac{v(a_n)}{a_n-a_{n-1}}(y-a_{n-1})\right|^2\left(\frac{1}{|y-a_1|}-\frac{1}{|y-a_{n-1}|}\right)\right]dy\\ &= 2\int_{a_{n-1}}^{a_n}\left|\frac{v(a_n)}{a_n-a_{n-1}}\right|^2\left[-|y-a_{n-1}|+\frac{(y-a_{n-1})^2}{|y-a_1|}\right]dy\\ &\le 2\left|\frac{v(a_n)}{a_n-a_{n-1}}\right|^2\left[\left(-\frac{(y-a_{n-1})^2}{2}\right)\Bigg|_{a_{n-1}}^{a_n}\right]+\\ &\left(\frac{1}{3}\frac{1}{a_n-a_{n-1}}(y-a_{n-1})^3\right)\Bigg|_{a_{n-1}}^{a_n}\\ &\le C\|v\|^2_{L^\infty(I)},\end{aligned}$$

and

$$\begin{aligned}\Pi_4 &\le C\|v\|^2_{L^\infty(I)}\int_{a_0}^{a_1}\int_{a_{n-1}}^{a_n}\frac{1}{|x-y|^2}dxdy\\ &\le C\|v\|^2_{L^\infty(I)}\frac{|a_1-a_0||a_n-a_{n-1}|}{|a_{n-1}-a_1|^2}\\ &\le C\|v\|^2_{L^\infty(I)}\frac{\rho^2}{(n-1)\rho^2}\\ &\le C\|v\|^2_{L^\infty(I)}.\end{aligned}$$

Then

$$\begin{aligned}|v_*|_{H^{1/2}(I)} &\le |v|_{H^{1/2}(I)}+|v^*|_{H^{1/2}(I)}\\ &\le C(|v|_{H^{1/2}(I)}+\|v\|_{L^\infty(I)}).\end{aligned}$$

Lemma 2.3: Following the notation in lemma2.2, we have

$$\int_0^1 v_*^2(x)\left(\frac{1}{x}+\frac{1}{1-x}\right)dx \le C(1+|\ln\rho|)\|v\|^2_{L^\infty(I)},$$

where C is a positive constant independent of ρ.

Proof : Firstly, by using the expression of $v^*(x)$ (cf. 1), we have

$$\begin{aligned}\int_0^1\frac{v_*^2(x)}{x}dx &= \int_{a_0}^{a_1}\frac{v^2(a_1)}{(a_1)^2}xdx+\int_{a_1}^{a_n}\frac{v_*^2(x)}{x}dx\\ &\le \frac{v^2(a_1)}{2}+\|v\|^2_{L^\infty(I)}\int_{a_1}^{a_n}\frac{1}{x}dx\\ &\le C(1+|\ln\rho|)\|v\|^2_{L^\infty(I)},\end{aligned}$$

similarly,

$$\int_0^1\frac{v_*^2(x)}{1-x}dx \le C(1+|\ln\rho|)\|v\|^2_{L^\infty(I)}.$$

Substituting them into the earlier expression completes the proof of the lemma.

Lemma 2.4: [6] Jensen's inequality, let $r \le q < \infty$, we have

$$(\sum_i |a_i|^q)^{\frac{1}{q}} \le (\sum_i |a_i|^r)^{\frac{1}{r}}. \tag{2}$$

Lemma 2.5: Let $\rho \le diamK \le h$, where $0 < h \le 1$, and P be a finite dimensional subspace of $W_p^l(K)\cap W_q^m(K)$, where $1 \le p \le \infty$, $1 \le q \le \infty$ and $0 \le m \le l$. Then there exists C such that for all $v \in P$, we have

$$\|v\|_{W_p^l(K)} \le Ch^{m-l+\frac{n}{p}-\frac{n}{q}}\|v\|_{W_q^m(K)}. \tag{3}$$

Proof : Here, we only provide a general outline of the proof, and for the detailed proof, please refer to reference [6]. Considering the relationship between p, q, and ∞, there are two cases as follows: the first case is $p=\infty$, by scaling, one can directly prove the result. The second case is $p<\infty$ which includes $q \le p < \infty$, $p < q < \infty$ and $p < q = \infty$, by applying Jensen's inequality, Hölder's inequality, and scaling, the lemma can be proven.

III. Proofs of the main inequalities

Let's prove Theorem 1.1 firstly. We observe that Ω has the cone property, i.e., each point $x \in \Omega$ is the vertex of a cone K_x congruent to the cone (or sector) K defined in polar coordinates by

$$K = \{(r,\theta) : 0 < r < d < \infty, 0 < \theta < \omega < 2\pi\}.$$

Without loss of generality we may assume that $h < \frac{d}{2}$ (In fact, the Discrete Sobolev inequality is trivial with applying the inverse estimate in lemma 2.5 when $h \ge \frac{d}{2}$.

Let T be a triangle of $\mathcal{T}_h$ and c be the centroid of T. For simplicity we may take c to be the origin and K_c to be K. The quasi-uniformity of $\mathcal{T}_h$ implies that there exists a number η which is independent of T and h such that $0 < \eta < 1$ and the cone

$$K_\eta = \{(r,\theta) : 0 < r < \eta h, 0 < \theta < \omega\}$$

is a subset of T.

Let $v \in V_h$ be arbitrary and $\alpha = v(c)$. It follows from the fundamental theorem of calculus that

$$\alpha = v(r,\theta) - \int_0^r \frac{\partial v}{\partial r}(\rho,\theta)d\rho \quad for\ \frac{d}{2} < r < d, \tag{4}$$

and hence

$$\alpha^2 \le 2v^2(r,\theta) + 2(\int_0^r \frac{\partial v}{\partial r}(\rho,\theta)d\rho)^2 \quad for\ \frac{d}{2} < r < d. \tag{5}$$

To bound the integral on the right-hand side of (5) , we proceed as follows

$$\begin{aligned}&\int_0^r \frac{\partial v}{\partial r}(\rho,\theta)d\rho = \int_0^{\eta h}\frac{\partial v}{\partial r}(\rho,\theta)d\rho + \int_{\eta h}^r\frac{\partial v}{\partial r}(\rho,\theta)d\rho\\ &\le \eta h|v|_{W^1_\infty}(T) + [\int_{\eta h}^r(\frac{\partial v}{\partial r}(\rho,\theta))^2\rho d\rho]^{\frac{1}{2}}\ln(\frac{d}{\eta h})^{\frac{1}{2}}.\end{aligned} \tag{6}$$

By substituting the estimate from (6) into (5) , we obtain the following inequality

$$\begin{aligned}\alpha^2\int_0^\omega\int_{\frac{d}{2}}^d rdrd\theta &\le 2\int_0^\omega\int_{\frac{d}{2}}^d v^2(r,\theta)rdrd\theta\\ &+ 4(\eta h)^2|v|^2_{W^1_\infty(T)}\int_0^\omega\int_{\frac{d}{2}}^d rdrd\theta\\ &+ 4\ln(d/\eta h)\\ &* \int_0^\omega\int_{\frac{d}{2}}^d[\int_{\eta h}^r(\frac{\partial v}{\partial r}(\rho,\theta))^2\rho d\rho d\theta]rdr\\ &\le 2\int_0^\omega\int_{\frac{d}{2}}^d v^2(r,\theta)rdrd\theta\\ &+ 4(\eta h)^2|v|^2_{W^1_\infty(T)}\frac{3}{8}\omega d^2\\ &+ 4\ln(d/\eta h)|v|^2_{W^1_\infty(T)}\\ &* \int_0^\omega\int_{\frac{d}{2}}^d\frac{1}{2}[r^2-(\eta h)^2]drd\theta,\end{aligned} \tag{7}$$

which implies, by the inverse estimate in lemma 2.5,

$$|v(c)| \le C_1(1+|\ln h|)^{\frac{1}{2}}\|v\|_{H^1(\Omega)}. \tag{8}$$

Let x be an arbitrary point in T. The inverse estimate in lemma 2.5 implies that

$$|v(x)-v(c)| \le h|v|_{W^1_\infty(T)} \le C_2|v|_{H^1(T)}, \tag{9}$$

where the positive constant C_2 is independent of h and x.

The Discrete Sobolev Inequality follows from (8), (9) and the arbitrariness of T and x.

Having established Theorem 1.1 , we now turn to the proof of Theorem 1.2 , which addresses the behavior of functions on the boundary ∂D.

Proof : Let a and b be the two endpoints of E, E_a and E_b be the two edges of ∂D neighboring E, and $F = \partial D \backslash \overline{E \cup E_a \cup E_b}$ (cf. Figure 2(a)).
Then we have

$$\begin{aligned}
|v_E|^2_{H^{1/2}(\partial D)} &= \int_{\partial D}\left[\int_{\partial D}\frac{|v_E(x)-v_E(y)|^2}{|x-y|^2}ds(x)\right]ds(y)\\
&= \int_{E\cup E_a\cup E_b\cup F}\int_{E\cup E_a\cup E_b\cup F}\cdots\quad ds(x)ds(y)\\
&= \Big(\int_E\int_E+\int_E\int_{E_a\cup E_b\cup F}+\int_{E_a\cup E_b\cup F}\int_E\\
&+\int_{E_a\cup E_b\cup F}\int_{E_a\cup E_b\cup F}\Big)\cdots\ ds(x)ds(y)\\
&= \int_E\left[\int_E\frac{|v_E(x)-v_E(y)|^2}{|x-y|^2}ds(x)\right]ds(y)\\
&+2\int_E|v_E(y)|^2\left[\int_{E_a\cup E_b\cup F}\frac{1}{|x-y|^2}ds(x)\right]ds(y),
\end{aligned} \tag{10}$$

because of $v_E(x)=0$ on $E_a\cup E_b\cup F$.
From figure 2(a) and lemma 2.1(2), we can find that

$$\begin{aligned}
\int_{E_a\cup E_b\cup F}\frac{1}{|x-y|^2}ds(x) &= \frac{\theta_1}{|a-y|}+\frac{\theta_2}{|b-y|}+\frac{\theta_3}{1}\\
&\approx \frac{1}{|a-y|}+\frac{1}{|b-y|}+1,
\end{aligned} \tag{11}$$

because of $\frac{\pi}{4}\le\theta_1,\theta_2,\theta_3\le\frac{\pi}{2}$.
Substituting 11 into 10 and applying lemma 2.2 and lemma 2.3, we have

$$\begin{aligned}
|v_E|^2_{H^{1/2}(\partial D)} &\approx \int_E\left[\int_E\frac{|v_E(x)-v_E(y)|^2}{|x-y|^2}ds(x)\right]ds(y)\\
&+\int_E v_E^2(y)\Big(\frac{1}{|a-y|}+\frac{1}{|b-y|}\Big)ds(y)+\|v_E\|^2_{L^2(E)}\\
&\lesssim |v|^2_{H^{1/2}(\partial D)}+(1+|\ln\hat{h}|)\|v\|^2_{L^\infty(\partial D)}.
\end{aligned}$$

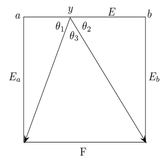
At last, we prove Theorem 1.3.

Proof : Let a and b be the two nodes next to p, $\hat{h}_1=|a-p|$, $\hat{h}_2=|b-p|$, $I=[-\hat{h}_2,\hat{h}_1]$ and ϕ is defined by

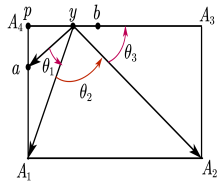
$$\phi(x)=\begin{cases}1-(x/\hat{h}_1) & \text{for } 0\le x\le\hat{h}_1,\\ 1+(x/\hat{h}_2) & \text{for } -\hat{h}_2\le x\le 0.\end{cases}$$

There are two cases (cf. Figure 2 (b) and (c)).
Case 1: the node p is a vertex of the reference domain D ;
Case 2: the node p is not a vertex of the reference domain D.

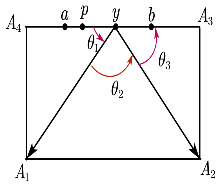
$$\begin{aligned}
|v_p|^2_{H^{1/2}(\partial D)} &= \int_{pa\cup pb}\int_{pa\cup pb}\frac{|v_p(x)-v_p(y)|^2}{|x-y|^2}ds(x)ds(y)\\
&+2\int_{pa\cup pb}|v_p(y)|^2\left[\int_{\partial D\backslash(pa\cup pb)}\frac{1}{|x-y|^2}ds(x)\right]ds(y)\\
&\triangleq \Phi_1+\Phi_2.
\end{aligned}$$



(a) The positions of the various physical quantities.



(b) Point P is a vertex on the reference domain D.



(c) Point P is not a vertex on the reference domain D.

Fig. 2: The positional relationship between point p and the reference domain D.

Obviously, in both cases, then

$$\Phi_1=\int_I\left[\int_I\frac{|\phi(x)-\phi(y)|^2}{|x-y|^2}dx\right]dy.$$

In case 2, using lemma 2.1

$$\begin{aligned}
\Phi_2 &= 2\int_{pa\cup pb}|v_p(y)|^2\Big(\frac{1}{|y-b|}-\frac{1}{|y-A_3|}+\frac{1}{|y-a|}\\
&\quad-\frac{1}{|y-A_4|}+\frac{\theta_3}{|y-A_3|}+\theta_2+\frac{\theta_1}{|y-A_4|}\Big)dy,\\
&= 2\int_{pa\cup pb}|v_p(y)|^2\Big(\frac{1}{|y-b|}+\frac{1}{|y-a|}+\frac{\theta_3-1}{|y-A_3|}+\theta_2\Big)\\
&\quad+\Big(\frac{\theta_1-1}{|y-A_4|}\Big)dy,
\end{aligned}$$

because of

$$\begin{aligned}
&\frac{\pi}{4}\le\theta_1,\theta_2,\theta_3\le\frac{\pi}{2},\\
&|y-b|\le|y-A_3|\le 1,\quad |y-a|\le|y-A_4|\le 1,
\end{aligned}$$

then

$$\begin{aligned}
\Phi_2 &\approx \int_{pa\cup pb}|v_p(y)|^2\Big(\frac{1}{|y-b|}+\frac{1}{|y-a|}\Big)dy+\int_{pa\cup pb}|v_p(y)|^2dy\\
&\approx \|\phi\|^2_{L^2(I)}+\int_I\phi^2(y)\Big(\frac{1}{y+\hat{h}_2}+\frac{1}{\hat{h}_1-y}\Big)dy.
\end{aligned}$$

In case 1, from figure 2(b), for any $y\in pb$, and using lemma 2.1 we have

$$\int_{\partial D\backslash(pa\cup pb)}\frac{1}{|x-y|^2}ds(x)=\frac{1}{|y-b|}+\frac{\theta_3-1}{|y-A_3|}+\frac{\theta_1}{|y-p|}+\theta_2.$$

We define $|p-a|=m, |A_1-a|=n, |y-p|=d$ for simplicity, obviously $m+n=|A_1-p|=1$, then we have

$$\frac{\theta_1}{|y-p|}=\frac{\arctan\frac{m+n}{d}-\arctan\frac{m}{d}}{d}=\frac{\arctan\frac{nd}{d^2+m}}{d},$$

because of

$$\arctan x\le x,$$

then

$$\frac{\theta_1}{|y-p|}\le\frac{n}{d^2+m}\le\frac{|A_1-a|}{|p-a|},$$

then

$$\Phi_2\le C\int_{pa\cup pb}|v_p(y)|^2\left(\frac{1}{|y-b|}+\frac{1}{|y-a|}\right)dy+\int_{pa\cup pb}|v_p(y)|^2dy,$$

obviously,

$$\Phi_2\ge C\int_{pa\cup pb}|v_p(y)|^2\left(\frac{1}{|y-b|}+\frac{1}{|y-a|}\right)dy+\int_{pa\cup pb}|v_p(y)|^2dy,$$

then we have

$$\Phi_2\approx\|\phi\|^2_{L^2(I)}+\int_I\phi^2(y)\left(\frac{1}{y+\hat h_2}+\frac{1}{\hat h_1-y}\right)dy.$$

Above all, we have

$$|v_p|^2_{H^{1/2}(\partial D)}\approx\|\phi\|^2_{L^2(I)}+\int_I\left[\int_I\frac{|\phi(x)-\phi(y)|^2}{|x-y|^2}dx\right]dy+\int_I\phi^2(y)\left(\frac{1}{y+\hat h_2}+\frac{1}{\hat h_1-y}\right)dy,$$

then a direct calculation then yields the theorem.

The next we calculate the above relation, we denote $I_1=[-\hat h_2,0]$, $I_2=[0,\hat h_1]$, then, we have

$$\begin{aligned}|v_p|^2_{H^{1/2}(\partial D)}\approx{}&\|\phi\|^2_{L^2(I)}+\int_I\left[\int_I\frac{|\phi(x)-\phi(y)|^2}{|x-y|^2}dx\right]dy\\&+\int_I\phi^2(y)\left(\frac{1}{y+\hat h_2}+\frac{1}{\hat h_1-y}\right)dy\\={}&\int_{I_1\cup I_2}\phi^2(x)dx+\int_{I_1}\left[\int_{I_1}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^2}dx\right]dy\\&+\int_{I_2}\left[\int_{I_2}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^2}dx\right]dy\\&+2\int_{I_2}\left[\int_{I_1}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^2}dx\right]dy\\&+\int_{I_1\cup I_2}\phi^2(y)(\frac{1}{y+\hat h_2}+\frac{1}{\hat h_1-y})dy\\\triangleq{}&\Psi_1+\Psi_2+\Psi_3+\Psi_4+\Psi_5.\end{aligned}$$

Obviously, Ψ_1,Ψ_2,Ψ_3 are constants, Ψ_4 and Ψ_5 we can directly calculate the result through the definite integral of variable substitution, the results also are constants. Substituting them into the earlier expression completes the proof of the theorem.

IV. Conclusion

This paper provides a detailed proof of the important inequality theorems regarding the additive Schwarz preconditioners, which offer significant theoretical foundations for the design and analysis of the additive Schwarz preconditioners.

References

[1] P. F. Antonietti, P. Houston, G. Pennesi, E. Sli, "An agglomeration-based massively parallel non-overlapping additive Schwarz preconditioner for high-order Discontinuous Galerkin methods on polytopic grids," Math. Comput., vol.89, pp. 2047-2083, 2020.

[2] P. F. Antonietti, M. Sarti, M. Verani, L. T. Zikatanov, "A uniform additive Schwarz preconditioned for high-order discontinuous Galerkin approximations of elliptic problems," J. Sci. Comput., vol.70, pp. 608-630, 2017.

[3] S. C. Brenner, "Two-level additive Schwarz preconditioners for non-conforming finite element methods," Math. Comp., vol.65, pp. 897-921, 1996.

[4] S. C. Brenner, "Lower bounds for two-level additive Schwarz preconditioners with small overlap," SIAM J. Sci. Comput., vol.21, no.5, pp. 1657-1669, 2000.

[5] S. C. Brenner, K. Wang, "Two-level additive Schwarz preconditioners for C^0 interior penalty methods," Numer. Math., vol.102, pp. 231-255,2005.

[6] S. C. Brenner, L. R. Scott, "The mathematical theory of finite element methods," vol. 15 of Texts in Applied Mathematics,3rd edn. Springer, New York, 2008.

[7] X. C. Cai, M. Dryja, M. Sarkis, "Restricted additive Schwarz preconditioners with harmonic overlap for symmetric positive definite linear systems, " SIAM J. Numer. Anal., vol.41, pp. 1209-1231, 2003.

[8] D. Cho, L. Pavarino, S. Scacchi, "Overlapping additive Schwarz preconditioners for isogeometric collocation discretizations of linear elasticity, " Comput. Math. Appl., vol. 93, pp. 66-77 , 2021.

[9] Q. T. Le Gia, I. H.Sloan, T. Tran, "Overlapping additive Schwarz preconditioners for elliptic PDEs on the unit sphere," Math. Comput., vol. 78 , pp.79-101 , 2009.

[10] R. Li, H. Yang, C. Yang, "Parallel multilevel restricted Schwarz preconditioners for implicit simulation of subsurface flows with Peng-Robinson equation of state, " J. Comput. Phys., vol. 422 , 2020.

[11] S. T. Liu, "Additive Schwarz-Type preconditioner for Hermite cubic splines," Appl. Math. Lett., vol.17, no.11, pp. 1267-1275, 2003.

[12] Z. Y. Liu, Y. N. He, "Restricted additive Schwarz preconditioner for elliptic equations with jump coefficients," Adv. Appl. Math. Mech., vol. 8, pp. 1072-1083, 2016.

[13] F. Wang, J. R. Chen, W. Xu, Z. L. Li, "An additive Schwarz preconditioner for the mortar-type rotated Q_1 FEM for elliptic problems with discontinuous coefficients," Appl. Numer. Math., vol.59, pp. 1657-1667, 2009.

[14] H. Y. Chu, Y. Cai, F. Wang, and J. R. Chen, "A two-level additive Schwarz preconditioner for the Nitsche extended finite element approximation of elliptic interface problems," Comput. Math. Appl., vol.147, pp. 111-120, 2023.