Study on the Evolutionary Dynamics of the Symmetric Sequential Game without Dominant and Weakly Dominant Strategies

Yunhao Liu, Wei Zhang

Abstract—**Assuming that the players with bounded rationality are randomly selected as the first movers or the second movers, and neither the first movers nor the second movers have dominant or weakly dominant strategies. This paper studied the group evolution dynamics based on symmetric sequential game with no dominant and weakly dominant strategies. First, it was found that there are no internal rest points in this type of evolutionary game, that is, one or some pure strategies must disappear with evolution, and every pure strategy has the possibility to eventually disappear. Then, the stability of these rest points was analyzed. The results show that when the sequential game has two PNE, there exist two symmetric PNE of the symmetric sequential game in the set of ESS, and there is at most one mixed strategy Nash equilibrium of symmetric sequential game. And when the sequential game has no PNE， there are no symmetric PNE of the symmetric sequential game in the set of ESS. Finally, numerical simulations were performed for the system dynamics when the sequential game path most favorable to the second movers is consistent or inconsistent with that most favorable to first movers in the case where the sequential game has two PNE, as well as for the system dynamics when the sequential game path most unfavorable to the second movers is consistent or inconsistent with that most favorable to the first movers in the absence case where the sequential game has no PNE.**

Index Terms— **Symmetric sequential game, Bounded rationality, Evolutionary game, Dominant and weakly dominant**

I. Introduction

THE traditional evolutionary game model is often based on simultaneous games.However, whether in nature or human society, many games have multiple stages, and the identity or game order of the players are often uncertain. For example, whether an Individual or population of a certain species will take the initiative to attack other discovered Individual or population, and how another Individual or population will react after being attacked or not being attacked. For instance, it or they may counter - attack to defend their territory or migrate for the survival. However, migration also involves risks. It seems that neither side has dominant or weakly dominant strategy. There are also similar situations in sociology and economics.

Regarding the research on the sequential game with many players, Le Roux [1] studied various equilibria in infinite sequential games with many players and many outcomes.Regarding the incomplete rational multi-stage game, Nishimura [2] studied the rational choice behavior of the players in the complete information sequence game, without making rational assumptions about other players participating in the same game. Karwowski et al. [3] studied sequential Stackelberg games with finite rationality. Brihaye et al. [4] consider an N-person non-zero-sum game played on a finite game tree, in which players have the right to repeatedly update their repeated strategies. Tan et al. [5] found that in finite repetition games, intra-group punishment and inter-group conflict weaken intra-group cooperation. Kurokawa [6] studied repeated N-player games with malicious and non-malicious strategies with opt-out options. Battigalli et al. [7] studied belief change, rationality and strategic reasoning in sequential games.

Evolutionary game model is often used to study imperfect rational games. Most of the current researches are based on simultaneous games. Such as the Prisoner's dilemma [8-11], three-strategy simultaneous games [12-14] and the four-strategy cyclic dominance models [15-16]. At present, there are few researches on evolutionary game models based on sequential game. The evolutionary game model in this paper is based on a kind of symmetric sequential game. In this game, the order of moves of the players is random, and there are no dominant and weak dominant strategies.

II. Basic model

This paper considers a symmetric sequential game in which a pair of players are randomly selected from a large enough group to play the sequential game, and each player is randomized with the same probability of 1/2 as the first mover or the second mover. The first mover’ s set of pure strategies is $S_1=\{A,B\}$ and the second mover’ s is $S_2=\{C,D\}$. And the utility of a player with action order is

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Table I: Symmetric sequential game matrix

Up / Down	$A-C$	$A-D$	$B-C$	$B-D$
$A-C$	$\begin{pmatrix} 0.5u_{AC}^1+0.5u_{AC}^2 \\ 0.5u_{AC}^2+0.5u_{AC}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{AC}^1+0.5u_{AD}^2 \\ 0.5u_{AC}^2+0.5u_{AD}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{BC}^1+0.5u_{AC}^2 \\ 0.5u_{BC}^2+0.5u_{AC}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{BC}^1+0.5u_{AD}^2 \\ 0.5u_{BC}^2+0.5u_{AD}^1 \end{pmatrix}$
$A-D$	$\begin{pmatrix} 0.5u_{AD}^1+0.5u_{AC}^2 \\ 0.5u_{AD}^2+0.5u_{AC}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{AD}^1+0.5u_{AD}^2 \\ 0.5u_{AD}^2+0.5u_{AD}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{BD}^1+0.5u_{AC}^2 \\ 0.5u_{BD}^2+0.5u_{AC}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{BD}^1+0.5u_{AD}^2 \\ 0.5u_{BD}^2+0.5u_{AD}^1 \end{pmatrix}$
$B-C$	$\begin{pmatrix} 0.5u_{AC}^1+0.5u_{BC}^2 \\ 0.5u_{AC}^2+0.5u_{BC}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{AC}^1+0.5u_{BD}^2 \\ 0.5u_{AC}^2+0.5u_{BD}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{BC}^1+0.5u_{BC}^2 \\ 0.5u_{BC}^2+0.5u_{BC}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{BC}^1+0.5u_{BD}^2 \\ 0.5u_{BC}^2+0.5u_{BD}^1 \end{pmatrix}$
$B-D$	$\begin{pmatrix} 0.5u_{AD}^1+0.5u_{BC}^2 \\ 0.5u_{AD}^2+0.5u_{BC}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{AD}^1+0.5u_{BD}^2 \\ 0.5u_{AD}^2+0.5u_{BD}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{BD}^1+0.5u_{BC}^2 \\ 0.5u_{BD}^2+0.5u_{BC}^1 \end{pmatrix}$	$\begin{pmatrix} 0.5u_{BD}^1+0.5u_{BD}^2 \\ 0.5u_{BD}^2+0.5u_{BD}^1 \end{pmatrix}$

denoted as u_{jk}^i , $i=1,2$; $j\in S_1$, $k\in S_2$, when the first mover and the second mover adopts strategy j and the secondmover adopts strategy k respectively.

Assume that neither side has dominant or weakly dominant strategy. Let $u_{AC}^1>u_{BC}^1$, $u_{AD}^1<u_{BD}^1$, and $u_{AC}^1>u_{BD}^1$. When $u_{AC}^2<u_{AD}^2$ and $u_{BC}^2>u_{BD}^2$, in the case of given game order, it is not difficult to conclude that the sequential game has no PNE (Pure Strategy Nash Equilibrium).When $u_{AC}^2>u_{AD}^2$ and $u_{BC}^2<u_{BD}^2$,the sequential game has two PNE (Pure Strategy Nash Equilibrium) (A,C) and (B,D) ,one SPNE (Subgame Perfect Nash Equilibrium) (A,C). When the order of the game is uncertain, every player needs to choose pure strategies for playing different roles, which makes the sequential game symmetric for players. Let the players' set of pure strategies in the symmetric sequential game be $S=\times_{i=1,2}S_i=\{j-k\,|\,j\in S_1,k\in S_2\}$.It is not difficult to conclude that there are no dominant and weakly dominant strategies in the symmetric sequential game when the sequential game has no PNE, and there are four PNE, namely $(A-C,A-C)$, $(A-D,B-C)$, $(B-C,A-D)$ $(B-D,B-D)$. The game matrix is shown in Table I, where the upper and lower elements of the utility vectors correspond to the expected utilities of the two players respectively.

III. Evolutionary dynamics

Let the proportion of players who choose strategy $j-k$ in the population be x_{j-k}. At time point t, the population state can be seen as a mixed strategy, expressed as a vector $x(t)=(x_{A-C}(t),x_{A-D}(t),x_{B-C}(t),x_{B-D}(t))\in\Delta$, where the simplex $\Delta=\left\{x\in R_+^4:\sum_{m\in\times_{i=1,2}S_i}x_m=1\right\}$.

The expected utility of various pure strategies are expressed respectively:

$$\begin{aligned}u(A-C,x)&=0.5(x_{A-C}u_{AC}^1+x_{A-D}u_{AD}^1+x_{B-C}u_{AC}^1+x_{B-D}u_{AD}^1)\\&+0.5(x_{A-C}u_{AC}^2+x_{B-C}u_{BC}^2+x_{A-D}u_{AC}^2+x_{B-D}u_{BC}^2)\end{aligned}\tag{1}$$

$$\begin{aligned}u(A-D,x)&=0.5(x_{A-C}u_{AC}^1+x_{A-D}u_{AD}^1+x_{B-C}u_{AC}^1+x_{B-D}u_{AD}^1)\\&+0.5(x_{A-C}u_{AD}^2+x_{A-D}u_{AD}^2+x_{B-C}u_{BD}^2+x_{B-D}u_{BD}^2)\end{aligned}\tag{2}$$

$$\begin{aligned}u(B-C,x)&=0.5(x_{A-C}u_{BC}^1+x_{A-D}u_{BD}^1+x_{B-C}u_{BC}^1+x_{B-D}u_{BD}^1)\\&+0.5(x_{A-C}u_{AC}^2+x_{A-D}u_{AC}^2+x_{B-C}u_{BC}^2+x_{B-D}u_{BC}^2)\end{aligned}\tag{3}$$

$$\begin{aligned}u(B-D,x)&=0.5(x_{A-C}u_{BC}^1+x_{A-D}u_{BD}^1+x_{B-C}u_{BC}^1+x_{B-D}u_{BD}^1)\\&+0.5(x_{A-C}u_{AD}^2+x_{A-D}u_{AD}^2+x_{B-C}u_{BD}^2+x_{B-D}u_{BD}^2)\end{aligned}\tag{4}$$

The expected utility of the player randomly drawn from the group is:

$$u(x,x)=\sum_{m\in\times_{i=F,L}S_i}x_m u(m,x)\tag{5}$$

The Replicator Dynamics [17] of the group taking the pure strategy $m\in\times_{i=1,2}S_i$ is expressed as:

$$dx_m/dt=[u(m,x)-u(x,x)]x_m\tag{6}$$

Possible rest points can be obtained by simultaneous equations (6): $x^1=(1,0,0,0)$, $x^2=(0,1,0,0)$, $x^3=(0,0,1,0)$, $x^4=(0,0,0,1)$, $x^5=\left(0,\frac{L_B}{L_A+L_B},\frac{F_D}{F_C+F_D},\frac{L_AF_C-L_BF_D}{(L_A+L_B)(F_C+F_D)}\right)$,

$x^6=\left(\frac{L_BF_D-L_AF_C}{(L_A+L_B)(F_C+F_D)},\frac{F_C}{F_C+F_D},\frac{L_A}{L_A+L_B},0\right)$,

$x^7=\left(\frac{F_D}{F_C+F_D},\frac{L_BF_C-L_AF_D}{(F_C+F_D)(L_A+L_B)},0,\frac{L_A}{L_A+L_B}\right)$,

$x^8=\left(\frac{L_B}{L_A+L_B},0,\frac{L_AF_D-L_BF_C}{(F_C+F_D)(L_A+L_B)},\frac{F_C}{F_C+F_D}\right)$,

$x^9=\left(0,\frac{F_C+L_B}{F_C+F_D+L_A+L_B},\frac{F_D+L_A}{F_C+F_D+L_A+L_B},0\right)$,

$x^{10}=\left(\frac{F_D+L_B}{F_C+F_D+L_A+L_B},0,0,\frac{F_C+L_A}{F_C+F_D+L_A+L_B}\right)$, where $F_C=u_{AC}^1-u_{BC}^1$, $F_D=u_{BD}^1-u_{AD}^1$, $L_A=u_{AC}^2-u_{AD}^2$, $L_B=u_{BD}^2-u_{BC}^2$. F_C and F_D are positive while L_A and L_B are either both positive or both negative. Four pure strategies can also be represented by x^1, x^2, x^3, x^4.Conclusion 1 can be obtained from the above solutions.

Conclusion 1: There is no inner rest points in evolutionary games based on two-player and two-strategy symmetric sequential game without dominant and weakly dominant strategies. And $\forall m \in \times_{i=1,2} S_i$, $\exists x^0 \in \Delta$ makes $\lim_{t\to\infty} x_m(t)=0$, Where $x^0 \in \Delta$ indicates the initial state.

ESS set is defined as $\Delta^{ESS} = \{x \in \Delta^{SPNE} : u(y,y) < u(x,y), \forall y \in \beta^*(x), y \neq x\}$ [18], where $\Delta^{SPNE} = \{x^1, x^4\} \in \Delta$ is the symmetric PNE strategies set, $\beta^*(x)$ is the optimal response strategies set to the strategy x.

Conclusion 2: In the evolutionary game based on two-player and two-strategy symmetric sequential game without dominant and weakly dominant strategies, when the sequential game has two PNE, $\Delta^{SPNE} \subseteq \Delta^{ESS}$; when the sequential game has no PNE, $S \not\subset \Delta^{ESS}$.

Proof: Let $x_{B-D} = 1 - x_{A-C} - x_{A-D} - x_{B-C}$, establish a three-dimensional Jacobian matrix to linearize the system, and substitute some rest points into the Jacobian matrix to obtain the eigenvalues as shown in Table II.

Table II: JACOBIAN MATRIX EIGENVALUES AND ASYMPTOTIC STABILITY ANALYSIS

Rest point	Eigenvalue			State(Two PNE)	State(No PNE)
	λ_1	λ_2	λ_3		
x^1	$-\frac{L_A}{2}$	$-\frac{F_C}{2}$	$-\frac{L_A+F_C}{2}$	ESS	Unstable
x^2	$\frac{L_A}{2}$	$\frac{F_D}{2}$	$\frac{L_A+F_D}{2}$	Unstable	Unstable
x^3	$\frac{F_C}{2}$	$\frac{L_B}{2}$	$\frac{L_B+F_C}{2}$	Unstable	Unstable
x^4	$\frac{F_D}{2}$	$-\frac{L_B}{2}$	$-\frac{L_B+F_D}{2}$	ESS	Unstable

Table II shows that when the sequential game has two PNE, x^1 and x^4 are ESS, while x^2 and x^3 are unstable points. And $\Delta^{SPNE} = \{x^1, x^4\}$, therefore $\Delta^{SPNE} \subseteq \Delta^{ESS}$. when the sequential game has no PNE, since there are positive eigenvalues, none of these four points are ESS.

Conclusion 3: In the evolutionary game based on two-player and two-strategy symmetric sequential game without dominant and weakly dominant strategies, when the sequential game has two PNE, the ESS set contains at most one mixed strategy, whose support set does not include pure strategies which are ESS. $L_B < F_C$ and $L_A < F_D$ are the necessary conditions for the existence of the mixed strategy.

Proof: Consider a strategy $x \in \Delta^{ESS}$ and another strategy $y \neq x$, if $C(y) \subset C(x)$, $y \notin \Delta^{SNE}$ [19], where $C(z)$ is the support set of strategy z and Δ^{SNE} is the set of symmetric Nash equilibrium strategies. Conversely, if $y \in \Delta^{SNE}$, and if $x \in \Delta^{ESS}$ and $y \neq x$, then $C(y) \notin C(x)$.When there are two PNE in the sequential game, $A-C$ and $B-D$ are ESS, and $\Delta^{ESS} \subseteq \Delta^{SNE}$, so $A-C, B-D \notin C(x^9) = \{A-D, B-C\}$. Therefore, x^9 is the only possible mixed strategy which is ESS. Substituting x^9 into the Jacobian matrix concludes three eigenvalues $\lambda_1^9 = \frac{F_C L_A - F_D L_B}{2(F_C+F_D+L_A+L_B)}$, $\lambda_2^9 = \frac{-F_C L_A + F_D L_B}{2(F_C+F_D+L_A+L_B)}$, $\lambda_3^9 = \frac{(F_D+L_A)(L_B-F_C)}{2(F_C+F_D+L_A+L_B)}$. Because $\lambda_1^9 = -\lambda_2^9$, it is impossible for all three eigenvalues to be negative, If there is no eigenvalue equal to zero, x^9 must be the saddle point. If one or some of the eigenvalues are zero, according to Lyapunov's first method, it is not certain whether is stable or not. At this time, there are three situations: $\lambda_1^9 = \lambda_2^9 = 0$ and $\lambda_3^9 = 0$; $\lambda_1^9 = \lambda_2^9 = 0$ and $\lambda_3^9 \neq 0$; $\lambda_1^9 = -\lambda_2^9 \neq 0$ and $\lambda_3^9 \neq 0$.

When there are two PNE in the sequential game, F_C, F_D, L_A and L_B are all positive. $F_D L_B = F_C L_A$ can be deduced from $\lambda_1^9 = \lambda_2^9 = 0$. Substituting $F_D L_B = F_C L_A$ into $\lambda_3^9 = 0$, $L_A L_B = F_C F_D$ can be obtained. Combining $F_D L_B = F_C L_A$ and $L_A L_B = F_C F_D$, $F_D = -L_A$ can be obtained. The situation of $\lambda_1^9 = \lambda_2^9 = 0$ and $\lambda_3^9 = 0$ is excluded. When $\lambda_1^9 = -\lambda_2^9 \neq 0$ and $\lambda_3^9 = 0$, one of λ_1^9 and λ_2^9 must be positive. In this case, x^9 is definitely not ESS. When $\lambda_1^9 = \lambda_2^9 = 0$ and $\lambda_3^9 \neq 0$, if x^9 is ESS, $\lambda_3^9 < 0$,then $L_B < F_C$ can be obtained. Due to $F_D L_B = F_C L_A$, $L_A < F_D$. ■

IV. NUMERICAL SIMULATION

Firstly, consider the situation where the initial sequential game has two PNE. At this time, the two relatively optimal sequential game paths for the first mover and the second mover are the same. However, since it is uncertain whether the optimal sequential game path of the second mover is consistent with that of the first mover, this situation will be simulated in inconsistent and consistent cases.

Without loss of generality, let $u_{A-C}^1 = 5$, $u_{B-C}^1 = 3$, $u_{A-D}^1 = 2$, $u_{B-D}^1 = 4$, $u_{A-C}^2 = 4$, $u_{A-D}^2 = 2$, $u_{B-C}^2 = 2$. When $u_{B-D}^L = 5$, $u_{B-D}^2 > u_{A-C}^2$, $L_B > 2$ can be obtained. And $x^5 = \left(0, \frac{3}{5}, \frac{1}{2}, \frac{-1}{10}\right)$, $x^6 = \left(\frac{1}{10}, \frac{1}{2}, \frac{2}{5}, 0\right)$, $x^7 = \left(\frac{1}{2}, \frac{1}{10}, 0, \frac{2}{5}\right)$,

$x^{8}=\left(\frac{3}{5},0,\frac{-1}{10},\frac{1}{2}\right)$, $x^{9}=\left(0,\frac{5}{9},\frac{4}{9},0\right)$, $x^{10}=\left(\frac{5}{9},0,0,\frac{4}{9}\right)$, Where x^5 and x^8 are not rest points because they have negative values, and $L_A F_C < L_B F_D$; When $u_{B-D}^{L}=3$, $u_{B-D}^{2}<u_{A-C}^{2}$, $L_B<2$, $x^{5}=\left(0,\frac{1}{3},\frac{1}{2},\frac{1}{6}\right)$, $x^{6}=\left(\frac{-1}{6},\frac{1}{2},\frac{2}{3},0\right)$, $x^{7}=\left(\frac{1}{2},\frac{-1}{6},0,\frac{2}{3}\right)$, $x^{8}=\left(\frac{1}{3},0,\frac{1}{6},\frac{1}{2}\right)$, $x^{9}=\left(0,\frac{3}{7},\frac{4}{7},0\right)$, $x^{10}=\left(\frac{3}{7},0,0,\frac{4}{7}\right)$,Where x^6 and x^7 are not not rest points because they have negative values , and $L_A F_C > L_B F_D$. In addition, x^9 is the saddle point in both cases.

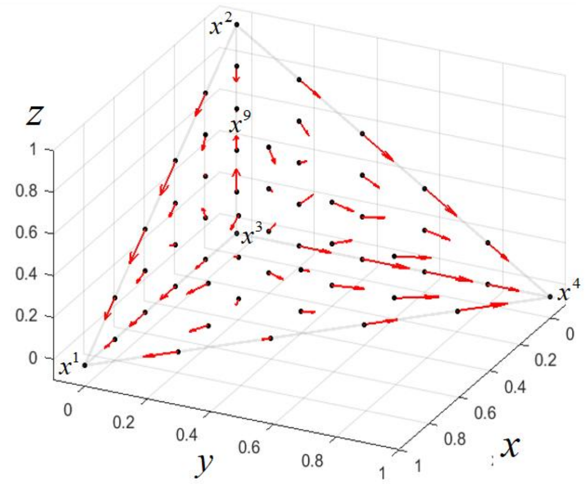

Fig. 1. vector fields evolution simulation when the sequential game has two PNE and $u_{B-D}^{2}>u_{A-C}^{2}$

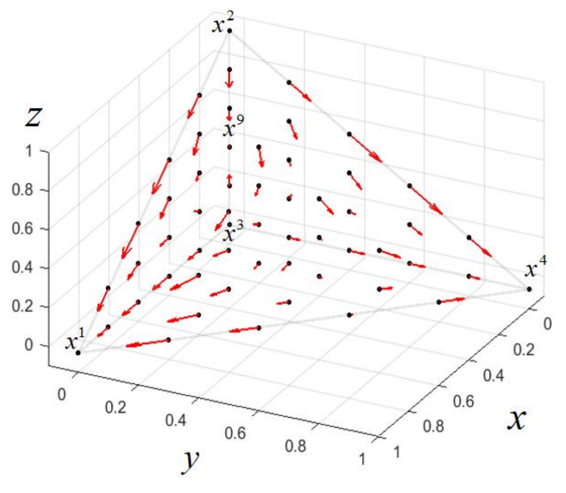

Fig. 2. vector fields evolution simulation when the sequential game has two PNE and $u_{B-D}^{2}<u_{A-C}^{2}$

In Fig.1 and Fig. 2, x represents the frequency of strategy A-C, y represents the frequency of strategy B-D, and z represents the frequency of strategy A-D. And every circular point represents an initial point, the arrows indicate the moving directions starting from that point, and the circular points without arrows are rest points.It is not difficult to find that the vector fields at different initial positions mainly converge towards points x^1 and x^4. When x^9 is a saddle point, it only attracts the vector fields in some directions, while the vector fields in the remaining directions move away from x^9.

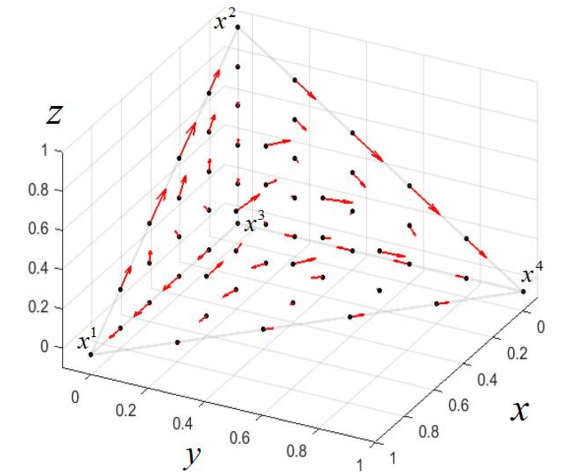

Fig. 3. vector fields evolution simulation when the sequential game has no PNE and $u_{B-D}^{2}>u_{A-C}^{2}$

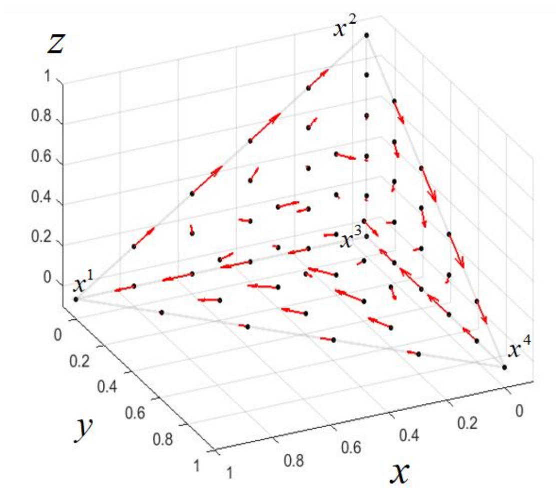

Fig. 4. vector fields evolution simulation when the sequential game has no PNE and $u_{B-D}^{2}<u_{A-C}^{2}$

In addition, consider a situation where there is no PNE in the initial sequential game. In this situation, the optimal sequential game paths of the first mover and the second mover are inconsistent. This situation will be simulated as two cases where the second mover's worst game path is consistent with and inconsistent with the predator's optimal game path.

Without loss of generality, let $u_{A-C}^{1}=5$, $u_{B-C}^{1}=3$, $u_{A-D}^{1}=2$, $u_{B-D}^{1}=4$, $u_{A-C}^{2}=2$, $u_{A-D}^{2}=4$, $u_{B-C}^{2}=4$. Let $a=u_{B-D}^{2}$ take the values of 3 and 1 respectively, so that u_{B-D}^{2} is greater than and less than u_{A-C}^{2} respectively.

In Fig.3 and Fig. 4,There are vector fields in some directions converging towards any one of the points x_1 , x_2 , x_3 and x_4 , while there are also vector fields in some other directions moving away from any one of x_1 , x_2 , x_3 and x_4 . So all of these four rest points are unstable at this time . This is consistent with conclusion 2.

V. CONCLUSION

This paper studied the population evolution dynamics of symmetric sequential games with no dominant and weakly dominant strategies. The players with bounded rationality are randomly selected as the first movers or the second movers with the same probability of 1/2, and neither the first movers nor the second movers have dominant or weakly dominant strategies. Meanwhile, the sequential game path that is most favorable to the first movers and the strategy that the first mover should adopt when the second mover's strategy is given are provided.

The uncertainty of the role makes the sequential game transformed into a game that determines the strategy of two stages at the same time. At this time, there is still no dominant or weakly dominant strategy. It is found that in the evolutionary game based on the two-stage and two-strategy sequential game with no dominant and weakly dominant strategies, there is no internal rest point, that is, at least one pure strategy will disappear with the evolutionary process, and every strategy may disappear. In addition, when the sequential game has two PNE, there exist two symmetric PNE of the symmetric sequential game in the set of ESS, and there is at most one mixed strategy Nash equilibrium of symmetric sequential game. When the sequential game has no PNE， there are no symmetric PNE of the symmetric sequential game in the set of ESS.

Finally, numerical simulations were performed for the system dynamics when the sequential game path most favorable to the second movers is consistent or inconsistent with that most favorable to first movers in the case where the sequential game has two PNE, as well as for the system dynamics when the sequential game path most unfavorable to the second movers is consistent or inconsistent with that most favorable to the first movers in the absence case where the sequential game has no PNE.

Considering that the identities of players may be uncertain in game, in a broad sense, the research framework of this paper is not only applicable to the sequential game, but can also be extended to asymmetric evolutionary game or multi-party evolutionary game with different identities, which awaits further research.

REFERENCES

[1] S. Le Roux, A. Pauly. "Equilibria in multi-player multi-outcome infinite sequential games". Information and Computation, vol.276, pp. 104557, 2021.

[2] H. Nishimura. "Revealed preferences of individual players in sequential games". Journal of Mathematical Economics, vol.96, pp. 102522, 2021.

[3] J. Karwowski, J. Mańdziuk, A. Żychowski. "Sequential Stackelberg Games with bounded rationality". Applied Soft Computing, vol.132, pp. 109846, 2023.

[4] T. Brihaye, G. Geeraerts, M. Hallet, et al. "On the termination of dynamics in sequential games". Information and Computation, vol.272, pp. 104505, 2020.

[5] J. H. W. Tan, F. Bolle. "Intragroup punishment and intergroup conflict aversion weaklyen intragroup cooperation in finitely repeated games". Journal of Behavioral and Experimental Economics, vol.105, pp. 102034, 2023.

[6] S. Kurokawa. "Evolution of spite in an n-player game with an opting-out option". Journal of Theoretical Biology, vol.572, pp. 111585, 2023.

[7] P. Battigalli, E. Catonini, J. Manili. "Belief change, rationality, and strategic reasoning in sequential games". Games and Economic Behavior, vol.142, pp. 527-551, 2023.

[8] J. Wu, B. Zhang, Z. Zhou, Q. He, X. Zheng, R. Cressman, Y. Tao. "Costly punishment does not always increase cooperation". Proc Natl Acad Sci USA, vol.106(41), pp. 17448-17451, 2009.

[9] C. Zhang, Y. Zhu, Z. Chen, J. Zhang. "Punishment in the form of shared cost promotes altruism in the cooperative dilemma games". J Theor Biol, vol.420, pp. 12-17, 2017.

[10] X.Zheng, C. Li, J. Yu, S. Wang, S. Fan, B. Zhang, Y. Tao. "A simple rule of direct reciprocity leads to the stable coexistence of cooperation and defection in the Prisoner's Dilemma game". J Theor Biol, vol.420, pp. 12-17, 2017.

[11] H. Yang, X. Chen. "Promoting cooperation by punishing minority". Appl Math Comput, vol.316, pp. 460-466, 2018.

[12] C. Zhang, Q. Li, Z. Xu, J. Zhang. "Stochastic dynamics of division of labor games in finite populations". Knowledge-Based Systems, vol.155, pp. 11-21, 2018.

[13] H. Xu, C. Tian, S. Fan, J. Li. "Information flows in the market: An evolutionary game approach". Chaos, vol.29, pp. 023126, 2019.

[14] M. Chica, R. Chiong, J. J. Ramasco, H. Abbass. "Effects of update rules on networked N-player trust game dynamics". Commun Nonlinear SCI, vol.19, pp. 104870, 2019.

[15] A. F. Lutz, S. Risau-Gusman, J. J. Arenzon. "Intransitivity and coexistence in four species cyclic games". J Theor Biol, vol.317, pp. 286-292, 2013.

[16] C. Rulquin, J. J. Arenzon. "Globally synchronized oscillations in complex cyclic games". Phys Rev E, vol.89, pp. 032133, 2014.

[17] J. Haigh. "Game Theory and Evolution". Aduances in Applied Probability, vol.7, pp. 8-11, 1975

[18] D. Friedman. "Evolutionary games in economics". Econometrica, vol.59(3), pp. 637 -666, 1991.

[19] P. Taylor, and L. Jonker. "Evolutionary Stable Strategies and Game Dynamics". Mathematical Biosciences, vol.40, pp. 145-156, 1978.