A Smooth Algorithm for Solving Convex Inequalities and Its Applications

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***Abstract*—Solving large-scale systems of nonlinear equations and inequalities poses significant challenges in computational optimization. This paper proposes a novel smoothing approximation framework to tackle this issue. Initially, convex inequalities are reformulated as a non-differentiable minimax problem. Subsequently, we demonstrate that approximate solutions can be obtained through a smoothing approximation technique. To solve the resulting approximation problem, a Newton-type algorithm is employed. Furthermore, we analyze key properties of the approximate function and establish global convergence of the proposed algorithm under mild assumptions. Numerical experiments validate the method's efficacy and efficiency.**

***Index Terms*—Operational research, Optimization, Smooth function, Inequality problem, Algorithm.**

I. Introduction

SYSTEMS of nonlinear equalities and inequalities play a critical role in diverse real-world applications, including applied mathematics, computer science, data analysis, image reconstruction, and set separation problems. These systems are fundamental for modeling, design, and analytical processes, particular in numerical solutions of partial differential equations (PDEs), power systems, nonlinear complementarity problems, and unconstrained optimization [1], [2], [3], [4]. A prominent example is the Graph Realization Problem (GRP) [5], which has attracted considerable attention in theoretical computer science. Additionally, parameters ensuring the GUUB property in phase plane control are expressed via nonlinear inequalities [6]. The simultaneous stabilization of multiple linear time-invariant (LTI) systems can also be interpreted as solving specific nonlinear inequalities [7]. Similarly, determining the operational area of robotic manipulators involves solving specific nonlinear inequalities [8], [3]. For a comprehensive review of nonlinear inequalities and their applications, readers are encouraged to consult related literature, including [5], [9] and the references therein.

As noted in [10], [11], a growing body of research explores on related areas, including constrained optimization problems, convex variational inequalities, complementary problems, and equilibrium problems. For detailed insights, readers are referred to [12], [13], [14]. Convex inequality problems present significant difficulties due to their non-convex,non-smoothness, and lack of Lipschitz continuity [15], [16].

Over the past decade, smoothing function methods have become prominent numerical techniques for solving nonlinear inequalities, attracting significant research attention (see, e.g., [17], [18], [19], [20], [21], [22]). These approaches transform the original problem into a nonsmooth function $F(x)$, then approximate it through by a smooth function $f(x,\varepsilon)$ with a smoothing parameter ε. The generalized derivative $\partial F(x)$ is then approximated by $f'(x,\varepsilon)$. Notable rooted in the least-squares framework, these methods maintain robust convergence properties while encountered nonsmoothness challenges. However, the performance and efficiency are heavily depended on the specific function developed, and the computational complexity may increase exponential in the worst case, presenting implementation barriers. These inherent limitations underscore the critical need for developing more effective approximation functions to advance practical applications in nonlinear inequality.

Manuscript received August 29, 2024; revised May 17, 2025.

This work was supported in part by the Beijing Municipal Social Science Foundation under Grant 22GLC068.

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In this paper, we focus on smoothing algorithms and introduce a novel continuous approach for solving convex inequalities. The main contributions of this work are twofold:

(I) We present a novel smoothing function and develop a numerical algorithm for solving nonlinear inequalities. The algorithm is implemented via the MATLAB function *fmincon*, effectively identifies solutions to these inequalities while maintaining computational efficiency.

(II) The proposed numerical algorithm can be easily employed to calculate solutions for nonlinear inequalities by utilizing an optimization search criterion.

The remainder of this paper is structured as follows. Section 2 introduces a smooth approximation and presents its properties of. Section 3 analyzes the convergence properties of the algorithm. Section 4 provides a demonstration of numerical experiments, and Section 5 concludes the findings and potential directions for future research.

II. Smoothing Function and its Properties

Consider the equalities and inequalities

$$\begin{cases} f_I(x)\le 0, & I=\{1,2,\cdots,l\},\\ f_E(x)=0, & E=\{l+1,l+2,\cdots,m\}.\end{cases} \tag{1}$$

where $f_i:\mathbf{R^n}\longrightarrow\mathbf{R}, i=1,2,\cdots,m(m\ge 2)$, are continuously differentiable and convex, and $x\in\mathbf{R^n}$. When I is empty set, the system (1) reduces to a system of equations; whereas E is empty it is a system of inequalities.

To depict and solve the system (1), we can transform problem (1) into the following equivalent optimization problems

$$\min_{x\in\mathbf{R^n}}\sum_{i=1}^{m}(f_i(x))^+, \tag{2}$$

where $f_i(x)^+=\max\{0,f_i(x)\}, i=1,2,\cdots,m$.

The function in (2) presents a non-smooth optimization problem. This non-smoothness limits the applicability of differentiable optimization methods. To address this challenge, various smoothing techniques have been proposed and widely discussed in related literature to solve problem (2)(see, for example [20], [21], [22], [23]).

This paper focus on smoothing algorithms, proposing a novel continuous approach for solving convex inequalities. The smoothing method involves reformulating these inequalities into an equivalent, non-differentiable optimization problem. And the proposed methodology demonstrates superior numerical stability and computational efficiency compared to conventional techniques, supported by both theoretical analysis and numerical simulation. As noted in [22], [19], the main idea of smoothing-type algorithm for solving inequalities is to reformulate system (1) as a system of smoothing equations via projection function.

Let

$$f_i(x)^+ = \max\{0, f_i(x)\}, i = 1, 2, \cdots, m. \quad (3)$$

We notice that the inequalities (7) is equivalent to the following optimization problem

$$\min_{x\in\mathbf{R^n}} \sum_{i=1}^{m} f_i(x)^+. \quad (4)$$

Thus, the inequalities are formulated as an equivalent unconstraint minimizing problem. However, formulation (4) is non-differentiable and difficult to solve, rending classical optimization methods inapplicable.We construct a differentiable approximation by smoothing technique, simplifying its application. To establish the properties of the smooth function, we present the following assumptions.

Assumption 1. $f_i(x)(i = 1, 2, ..., m)$ satisfies the following:

$$\lim_{\|x\|\to+\infty} f_i(x) = +\infty(i = 1, 2, \cdots, m).$$

Assumption 2. $f_i(x)(i = 1, 2, ..., m)$ is a convex and differentiable function.

Under Assumption 1, there exists at least a global minima of problem (4). And Assumption 2 makes the analysis simpler, and it is also a common assumption in the convex inequalities literature (see, for example, [1], [9]).

Next, we connect the maximum function $\max\{0, t\}$ to a parametric smoothing procedure with an adjustable parameter r, yielding the smooth function $\theta_r(\cdot)$. Under the mild conditions, the proposed smoothing function exhibits important mathematical properties such as strong convexity and, in many case, infinite differentiability, enabling the inequality problem to be equivalently transformed into a smooth nonlinear programming problem. As $r \to 0$, the smoothing function converges to the original non-smooth maximum function, as will be rigorously proved later.

We introduce a real value function $\theta_r(\cdot)$ map $\mathbf{R}$ into $\mathbf{R}^+$ to approximate the maximum function.

Define a function $\theta_r(t)$ by

$$\theta_r(t) = \frac{t + \sqrt{t^2 + 4r^2}}{2}, \quad (5)$$

where $r > 0$ is an adjustable parameter.

Remark 1 The function $\frac{t+\sqrt{t^2+4r^2}}{2}$ has a wide range of applications and is commonly used in the field of analysis and control theory (see [24] and reference therein). In this paper, we apply this function to approximate the convex inequalities and analyze its properties for the first time. Furthermore, the upper bound between the approximation function and the original function is proposed. Next, we introduce some important properties of the real value function $\theta_r(\cdot)$.

Proposition 1 Let $\theta_r(\cdot) : \mathbf{R} \longrightarrow \mathbf{R}^+$ be the function given as above. Then the function $\theta_r(t)$ is strictly convex and differentiable for all $r \in (0, 1]$, and $\theta_r^{'}(0) > 0$.

Proof Based on $\theta_r(t)$ defined in (9), we have

$$\theta_r^{'}(t) = \frac{1}{2}(1 + \frac{t}{\sqrt{t^2 + 4r^2}}), \theta_r^{''}(t) = \frac{2r^2}{(t^2 + 4r^2)^{\frac{3}{2}}} > 0.$$

This shows that $\theta_r(t)$ is strictly convex and differentiable for any $t \in \mathbf{R}$ and $r \in (0, 1]$. In addition, we have $\theta_r^{'}(0) = \frac{1}{2} > 0$.

Proposition 2 Let $\theta_r(\cdot) : \mathbf{R} \longrightarrow \mathbf{R}^+$ be the function given as $\theta_r(t) = \frac{t+\sqrt{t^2+4r^2}}{2}$, then we get that $\lim_{r\to0^+} \theta_r(t) = \max\{0, t\}$. That is, the function $\theta_r(t)$ uniformly converges to the maximum function $\max\{0, t\}$ when the adjustable parameter r approaches to zero.

Proof We can compute that

$$\begin{aligned}\lim_{r\to0^+} \theta_r(t) &= \lim_{r\to0^+} \frac{t + \sqrt{t^2 + 4r^2}}{2} \\ &= \lim_{r\to0^+} \frac{t + |t|}{2}\end{aligned}$$

If $t < 0$, then we have $|t| = -t$, which indicates $\lim_{r\to0^+} \frac{t+|t|}{2} = \lim_{r\to0^+} \frac{t-t}{2} = 0$; otherwise, if $t \geq 0$, we have $|t| = t$, which gives $\lim_{r\to0^+} \frac{t+|t|}{2} = \lim_{r\to0^+} \frac{t+t}{2} = t$. Hence, $\lim_{r\to0^+} \theta_r(t) = max\{0, t\}$.

Proposition 3 For any $t \in (-\infty, +\infty)$, the approximation function $\theta_r(\cdot)$ increase in r. Moreover, for $0 < r_2 < r_1 < 1$, we have $0 < \theta_{r_1}(\cdot) - \theta_{r_2}(\cdot) < r_1 - r_2$.

Proof Since $\frac{\partial\theta_r(t)}{\partial r} = \frac{2r}{\sqrt{t^2+4r^2}} > 0$, which follows $\theta_r(\cdot)$ is increasing in r.

Further, we get that

$$\begin{aligned}\theta_{r_1}(t) - \theta_{r_2}(t) &= \frac{4r_1^2 - 4r_2^2}{2(\sqrt{t^2 + 4r_1^2} + \sqrt{t^2 + 4r_2^2})} \\ &\leq \frac{4r_1^2 - 4r_2^2}{2(2r_1 + 2r_2)} = r_1 - r_2.\end{aligned}$$

Figure 1 shows the smoothing function $\theta_r(t)$ approaches function $\max\{0, t\}$ under different parameters r. It can be

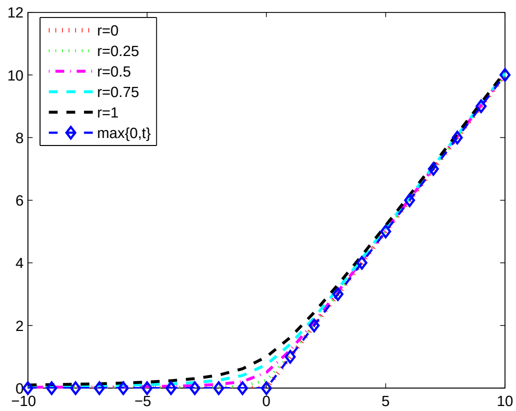

Fig. 1. The pattern of $\theta_r(t)$ and $\max\{0, t\}$ for different r.

observed that a smaller value of the parameter r leads to a higher degree of approximation. Specially, $\theta_r(t)$ exactly approaches $\max\{0,t\}$ when $r=0$. However, if parameter r is zero, the function $\theta_r(t)$ is non-smooth.

Based on the discussion aforementioned, the function $\theta_r(\cdot)$ with a adjustable parameter r is applied here to replace the plus function of (8) and obtain a differentiable optimization problem

$$\max_x \phi_r(x) = \sum_{i=1}^{m} \frac{f_i(x) + \sqrt{f_i^2(x) + 4r^2}}{2} \tag{6}$$

This problem is a strongly convex, unconstrained minimization problem, ensuring the existence of a unique optimal solution. We demonstrate that the solution of problem (1) can be obtained by solving problem (9) as r approaches zero.

Remark 2 Similar algorithmic framework has been discussed in [20] and [21] for solving the system of inequalities. One main feature of this work is that the method proposed is conceptually simple and numerically stable.

Remark 3 Huang and Zhang [22] converted inequalities into a system of smooth equations, focusing on the case $n=m$ and introducing slack variable for $n<m$. In contrast, our approach reformulates the inequalities into an optimization problem and presents a novel smooth function, which is applicable regardless of whether $n=m$ or $n<m$ or $n>m$.

We begin with two simple lemmas that is the basis of our theoretical analysis. Since their proof can be found in [25] (Theorem 1.3.10) , we here omit the proof due to space.

Lemma 1 Suppose that $g_i(x): V \longrightarrow R^+ (i=1,2,\cdots,l)$ are convex functions with convex set $V \subseteq \mathbf{R}^n$ and nonnegative real number $\lambda_i (i=1,2,\cdots,l)$, then $\sum_{i=1}^{l} \lambda_i g_i(x)$ is also convex function in convex set V.

Lemma 2 Suppose that $g_i(x): V \longrightarrow R^+ (i=1,2,\cdots,l)$ are differentiable function with $V \subseteq \mathbf{R}^n$ being a set and $\lambda_i (i=1,2,\cdots,l)$ are real number, then $\sum_{i=1}^{l} \lambda_i g_i(x)$ is also differentiable function in set V.

Threorem 1 Let $\phi_r(x)$ be the function defined as (3), then $\phi_r(x)$ is strictly convex and differentiable.

Proof By the expression of $\phi_r(x)$, together with Lemma 1 and Lemma 2, we immediately obtain the desired conclusion.

Threorem 2 For any x and $r>0$ it holds

$$\phi_r(x) - \sum_{i=1}^{m} \max\{0, f_i(x)\} \le mr.$$

If $r_1>0$, $r_2>0$ and $r_1 \ge r_2$, we get that

$$0 \le \phi_{r_1}(x) - \phi_{r_2}(x) \le m(r_1 - r_2).$$

Proof According to (10), we have

$$\begin{aligned} &\phi_r(x) - \sum_{i=1}^{m} \max\{0, f_i(x)\} \\ &= \frac{1}{2}\sum_{i=1}^{m} \frac{4r^2}{\sqrt{f_i^2(x)+4r^2} + \sqrt{f_i^2(x)}} \\ &\le \frac{1}{2}\sum_{i=1}^{m} \frac{4r^2}{2r} = mr. \end{aligned}$$

Note that $\phi_r(x)$ is increasing in r, we get that $\phi_{r_1}(x) - \phi_{r_2}(x) \ge 0$ for $r_1 \ge r_2$. Furthermore, we have

$$\begin{aligned} \phi_{r_1}(x) - \phi_{r_2}(x) &= \sum_{i=1}^{m} \frac{4r_1^2 - 4r_2^2}{2(\sqrt{f_i^2(x)+4r_1^2} + \sqrt{f_i^2(x)+4r_2^2})} \\ &\le \sum_{i=1}^{m} \frac{4r_1^2 - 4r_2^2}{2(2r_1+2r_2)} = m(r_1 - r_2). \end{aligned}$$

Theorem 1 indicates that the approximate function is differentiable convex function. Thus unconstraint optimization method can be applied to solve this problem.

III. Algorithm and Convergence for the Inequalities

Based on the above discussions, the inequality problem can be converted into an unconstraint smoothing optimization problem:

$$\min \phi_r(x) = \sum_{i=1}^{m} \frac{f_i(x) + \sqrt{f_i^2(x) + 4r^2}}{2}$$

By making use of the result of the previous section and taking advantage of the differentiable of the objective function of problem (10), we prescribe a Newton algorithm with Armijo stepsize that makes the algorithm globally convergent.

Let

$$F(x) = \sum_{i=1}^{m} \max_{1\le i\le m} \{f_i(x), 0\}$$

Algorithms is as follows:

Algorithm 1 The algorithm for smoothing approximation

Ensure: $x^{(0)}$, $r_0>0, l\in(0,1)$ and $\epsilon>0$.

Step 1(Initialization) Given $r_0>0, l\in(0,1)$, and $\epsilon>0$. Select a initial point $x^{(0)}$ and let $k:=0$.

Step 2(Termination)

if $|F(x^{(k)}) - \phi_{r_k}(x^{(k)})| > \epsilon$ **then**

Step 3 Solve problem (10) and denote by $\overline{x}^{(k)}$ its optimal solution:

3.1 Newton direction d^k: Solve the linear equation

$$\nabla_{xx}^2 \phi_r(x^{(k)}) d^k = -\nabla_x \phi_r(x^{(k)})$$

3.2 Armijo stepsize: Select $\lambda_k = \{1, \frac{1}{2}, \frac{1}{4}, \cdots\}$such that

$$\begin{aligned} &\phi_r(x^{(k)})d^k - \phi_r(x^{(k)}) + \lambda_k d^k) \\ &\ge -\delta\lambda_k[\nabla_x \phi_r(x^{(k)})]^T d^k \end{aligned}$$

where $\delta \in (0, \frac{1}{2})$.

Step 4 Update the parameter $r_{k+1} = lr_k$.

Step 5 Set $x^{(k+1)} = \overline{x}^{(k)}, k := k+1$ and go to Step 2.

end if

To guarantee the algorithm is well-defined, we next give a convergent theorem of solution.

Threorem 3 Let the level set $L\{x|\phi_r(x) \le \phi_{r_0}(x^{(0)})\}$ is bounded and the function $\phi_r(x)$ is defined as (3), then the limited point of $\{x^{(k)}\}$ is the optimal solution of the inequality problem.

Proof Noting that $\phi_r(x)$ is convex function, according to the convergence theorem of convex programming we know that the limited point of x^* which deriving from the algorithm is the global solution of the function $\phi_r(x)$. Considering the Theorem 1 and Theorem 2, this limit also is the optimal solution of the inequalities.

TABLE I
SIMULATIONS FOR EXAMPLES

Examples	ST	NI	SOL	F
Example 1	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	5	$\begin{pmatrix} -1.623521 \\ 0.216929 \end{pmatrix}$	$\begin{pmatrix} -0.998610 \\ -0.976563 \\ -11.048299 \\ -3.353867 \\ -4.765113 \\ -1.787725 \end{pmatrix}$
Example 2	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	5	$\begin{pmatrix} 0.800814 \\ 0.551510 \\ 0.224163 \\ 0.000002 \end{pmatrix}$	$\begin{pmatrix} -0.199186 \\ -0.897924 \\ -0.448489 \\ -0.800009 \\ -0.775837 \\ -0.502467 \end{pmatrix}$
Example 3	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$	5	$\begin{pmatrix} 0.030293 \\ 0.009044 \end{pmatrix}$	$\begin{pmatrix} -0.999001 \\ -0.999001 \end{pmatrix}$
Example 4	$\begin{pmatrix} 0.5 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	3	$\begin{pmatrix} 0.706607 \\ 0.706899 \\ 1.000000 \\ 0.992802 \\ 0.992802 \end{pmatrix}$	$\begin{pmatrix} -0.293393 \\ 2.076067 \\ -0.293100 \\ 5.002926 \\ -0.071977 \end{pmatrix}$

IV. NUMERICAL EXPERIMENTS

To demonstrate the effectiveness and efficiency of the smoothing method, we provide examples. The algorithms are implemented in MATLAB Online (https://matlab.mathworks.com/) using the following notations: ST denotes the value of starting point $x^{(0)}$, SOL represents the feasible solution obtained by the algorithm, and NI indicates the number of iterations. The other parameters are set as follows:

$$r_0 = 1, l = \frac{1}{4}, \varepsilon = 10^{-6}$$

Example 1[26] $f_1(x) = \sin(x_1) \le 0, f_2(x) = -\cos(x_2) \le 0, f_3(x) = x_1 - 1 \le 0, f_4(x) = x_2 - \frac{\pi}{2} - 2 \le 0, f_5(x) = x_1 - \pi \le 0, f_6(x) = -x_2 - \frac{\pi}{2} \le 0.$
Example 2[27] $f_1(x) = x_1 - 1 \le 0, f_2(x) = 10(x_2 - x_1^2) \le 0, f_3(x) = x_2 - 1 \le 0, f_4(x) = 10(x_3 - x_2^2) \le 0, f_5(x) = x_3 - 1 \le 0, f_6(x) = 10(x_4 - x_3^2) \le 0.$
Example 3[26] $f_1(x) = x_1^2 + x_2^2 - 1 \le 0, f_2(x) = -x_1^2 - x_2^2 + 0.999^2 \le 0.$
Example 4[26] $f_1(x) = x_1 + x_3 - 1.6 \le 0, f_2(x) = 1.333x_2 + x_4 - 3 \le 0, f_3(x) = -x_3 - x_4 + x_5 \le 0, f_4(x) = x_1^2 - x_3^2 - 1.25 = 0, f_5(x) = x_2^{1.5} + 1.5x_4 - 3 \le 0.$
Examples 1 and 2 correspond to the case where $n < m$, while examples 3 and 4 addresses the scenario where $n = m$. The results are shown in Table 1.

Compared to [26], which obtains a feasible solution for Example 1 in six iterations, our method achieves the same in five iterations. For Examples 3 and 4, [26] requires eight and four iterations, respectively, while our method achieves comparable efficiency. Table 2 provides a comparison of iterations with other methods. From Table 2, it is evident that all tested problems were solved with fewer iterations, demonstrating the effectiveness and applicability of our method.

TABLE II
COMPARISON OF ITERATIONS

Example	Iterations		Feasible solution	
	Ours	Reference [26]	Ours	Reference [26]
Example 1	5	6	$\begin{pmatrix} -1.623521 \\ 0.216929 \end{pmatrix}$	$\begin{pmatrix} -0.0294 \\ 1.5416 \end{pmatrix}$
Example 3	5	8	$\begin{pmatrix} 0.030293 \\ 0.009045 \end{pmatrix}$	$\begin{pmatrix} -0.6188 \\ 0.7853 \end{pmatrix}$
Example 4	3	4	$\begin{pmatrix} 0.706607 \\ 0.706899 \\ 1.000000 \\ 0.992802 \\ 0.992802 \end{pmatrix}$	$\begin{pmatrix} 0.5557 \\ 1.3242 \\ 0.9703 \\ 0.9840 \\ 1.1546 \end{pmatrix}$

To further demonstrate the robustness and practicality of the proposed algorithm for large-scale problems, we present Example 5. This example is designed to test the algorithm's performance on high-dimensional systems, highlighting its efficiency and scalability. The results show that the algorithm maintains its effectiveness even as the problem size increases, confirming its suitability for real-world applications where large-scale inequalities are common. Additionally, the algorithm's insensitivity to initial points and its ability to handle cases where $n \neq m$ further underscore its versatility and robustness.

Example 5[27] (Rosenbrocks function) $f_{2k-1}(x) = x_k - 1 \le 0, f_{2k}(x) = 10(x_{k+1} - x_k^2) \le 0, k = 1, 2, \cdots, 19.$
The problem has 20 variables and 38 equations. We choose different initial points $x_0 = (0, 0, \cdots, 0)^T$, $x_0 = (1, 1, \cdots, 1)^T$, $x_0 = (10, 10, \cdots, 10)^T$, $x_0 = (-1, -1, \cdots, -1)^T$ and $x_0 = (-10, -10, \cdots, -10)^T$, which denote by $[0]^n$, $[1]^n$, $[10]^n$, $[-1]^n$ and $[-10]^n$, respectively. The numerical results are listed in Table 3.

V. CONCLUSION

In this paper, we propose a novel smooth formulation that convert traditional inequality into a smooth unconstrained optimization problem, which is efficiently solved using a Newton-Armijo algorithm. This approach overcomes key limitations such as ill-conditioning and computational complexity while delivering high computational efficiency and superior performance. Moreover, the implementation of this algorithm is relatively simple. Future work will focus on exploring alternative smooth functions and further enhancing convergence speed.

REFERENCES

[1] D. J. W., "Newton's method for nonlinear inequalities," *Numerische Mathematik*, vol. 21, no. 7, pp. 381–387, 1973.

[2] R. M. F. N. C. Nguyen, P. Fernandez and J. Peraire, "Accelerated residual descent methods for the iterative solutions of systems of equations," *SIAM Journal on Scientific Computing*, vol. 40, no. 5, pp. 3157–3179, 2018.

[3] P. M. S. Y. D. Lera, Daniela, "Space-filling curves for numerical approximation and visualization of solutions to systems of nonlinear inequalities with applications in robotics," *Applied Mathematics and Computation*, vol. 390, February 2021.

[4] K. Z. Y. X. Zhihua Chen, Yong Guo, "A numerical algorithm for solving power-exponent type nonlinear inequalities with applications in calculating stabilizing parameters for lti systems," *IFAC-PapersOnLine*, vol. 55, no. 3, pp. 263–268, 2022.

TABLE III
SIMULATIONS FOR EXAMPLE 5

ST	SOL	ST	SOL	ST	SOL
	0.894726		0.995128		0.994287
	0.738793		0.982720		0.980996
	0.484395		0.957440		0.953633
	0.178177		0.903360		0.896082
	0.000001		0.796501		0.785068
	0.000000		0.617491		0.596175
	0.0000001		0.350470		0.339016
	0.000000		0.104180		0.085009
	0.000000		0.005427		0.004026
$[0]^n$	0.000000	$[1]^n$	0.003173	$[10]^n$	0.003169
	0.499998		0.501102		0.502459
	0.499998		0.501102		0.502459
	0.499998		0.501102		0.502459
	0.499998		0.501102		0.502459
	0.499998		0.501102		0.502459
	0.499998		0.501102		0.502459
	0.499998		0.501102		0.502459
	0.499998		0.501102		0.502459
	0.499998		0.501102		0.502459
	0.499998		0.501102		0.502459
ST	SOL	ST	SOL		
	0.994287		0.992760		
	0.980996		0.968677		
	0.953633		0.920543		
	0.896083		0.830151		
	0.785068		0.670508		
	0.596175		0.420265		
	0.339016		0.151371		
	0.085009		0.014847		
	0.004026		0.003252		
$[-1]^n$	0.003169	$[-10]^n$	0.003188		
	0.502459		0.499434		
	0.502459		0.499434		
	0.502459		0.499434		
	0.502459		0.499434		
	0.502459		0.499434		
	0.502459		0.499434		
	0.502459		0.499434		
	0.502459		0.499434		
	0.502459		0.499434		
	0.502459		0.499434		

[5] Y. Y. M. C. So, "A semidefinite programming approach to tensegrity theory and realizability of graphs," *In Proceedings of the 17th Annual ACMCSIAM Symposium on Discrete Algorithm (SODA)*, pp. 766–775, 2006.

[6] X. Y. Chen, Z., "Stability analysis of the closed-loop system of a phase-plane controlled rigid satellite," *Aerospace Control and Application*, vol. 40, no. 1, pp. 1–14, 2018.

[7] K. F. S. e. a. S., M., "Simultaneous control of linear systems by genetic algorithms in state and output feedback," *Iranian journal of science and technology: IJST, Transaction A. Science*, vol. 37, no. 1, pp. 35–43, 2013.

[8] P. M. R. L. T. A. Evtushenko, Y., "Approximating a solution set of nonlinear inequalities," *Journal of Global Optimization*, vol. 71, no. 1, pp. 129–145, 2018.

[9] P. K. A. Bjorklund and R. R. Williams, "Solving systems of polynomial equations over gf(2) by a parity-counting self-reduction," *In Proceedings of the 46th International Colloquium on Automata, Languages, and Programming (ICALP 2019)*, 2019, patras, Greece.

[10] S. M., "On the solution of nonlinear inequalities in a finite number of iterations," *Numerische Mathematik*, vol. 46, no. 2, pp. 229–236, 1985, patras, Greece.

[11] M. C. Jiang Lihua, Yin Zhixiang, "L-m method fo the system of nonlinear inequalities," *Journal of Hefei University of Technology*, vol. 32, no. 11, pp. 1669–1772, 2009.

[12] O. M. C. Chen, "Smoothing methods for convex inequalities and linear complementarity problems," *Mathematical Programming*, vol. 71, no. 1, pp. 51–69, 1995.

[13] P. T. H. Bintong Chen, "Smooth approximations to nonlinear complementarity problems," *SIAM Journal on Optimization*, vol. 7, no. 2, pp. 403–420, 1997.

[14] I. Konnov, "The method of pairwise variations with tolerances for linearly constrained optimization problems," *Journal of Nonlinear and Variational Analysis*, no. 1, pp. 25–41, 2017.

[15] C. X. Bian, W., "Optimality and complexity for constrained optimization problems with nonconvex regularization," *Mathematics of Operations Research*, vol. 42, no. 4, pp. 1063–1084, 2017.

[16] H. L. Haeser, Gabriel and Y. Ye, "Optimality condition and complexity analysis for linearly-constrained optimization without differentiability on the boundary," *Mathematical Programming*, no. 5, pp. 1–37, 2018.

[17] Y. J. Zeng W. J., "Successive projection for solving systems of nonlinear equations/inequalities," *Doi:10.48550/arXiv.2012.07555*, Dec. 2020.

[18] N. Y. Rodomanov A., "Subgradient ellipsoid method for nonsmooth convex problems," *Mathematical Programming*, vol. 199, no. 1, pp. 305–341, 2023.

[19] Q. N. M. X. H. Huang, H., "A nonmonotone smoothing-type algorithm for a system of inequalities associated with circular cones," *Asia-Pacific Journal of Operational Research*, vol. 40, no. 2, 2023.

[20] Z. P. Li Xingsi, "An entropic smoothing method for solving convex inequality problems," *Numerical Mathematics: A Journal of Chinese Universities*, vol. 26, no. 1, pp. 25–29, 2004.

[21] W. Ruopeng, "Adjustable entropy method for solving convex inequality problem," *Systems Engineering and Electronics*, vol. 20, no. 5, pp. 1111–1114, 2009.

[22] W. W. Zhenghai Huang, Ying Zhang, "A smoothing-type algorithm for solving system of inequalities," *Journal of Computational and Applied Mathematics*, vol. 22, pp. 355–363, 2008.

[23] Y. J. Shaohua Pan, Tao Tan, "A global continuation algorithm for solving binary quadratic program- ming probelms," *Computational Optimization & Applications*, vol. 41, no. 3, pp. 349–362, 2008.

[24] D. P. F. A. Canada, A., *Ordinary Differential Equations*, 2nd ed. Elsevier Science Ltd, 2006.

[25] S. W. Yuan Yaxiang, *Optimization Theorem and Method*, 1st ed. Chinese Press, 2002.

[26] Z. H. Ying Zhang, "A nonmonotone smoothing-type algorithm for solving a system of equalities and inequalities," *Journal of Computational and Applied Mathematics*, vol. 233, no. 9, pp. 2312–2321, 2010.

[27] Y. Xinshe, "Test problems in optimization," *Engineering optimization*, 2010.