Solution of System of Linear Balances Using Minor Rank in the Symmetrized Max-Plus Algebra

Suroto, Ari Wardayani and Najmah Istikaanah

Abstract—**System of linear balances in symmetrized max-plus algebra has a similar role with system of linear equations in conventional algebra. Therefore, this study aimed to discuss the solution to system of linear balances in symmetrized max-plus algebra for arbitrary coefficient matrix. The solution was characterized using minor rank of the coefficient matrix, which was partitioned to position the submatrix corresponding to minor rank in the upper-left corner. Additionally, the guaranteed existence of this balanced inverse submatrix was used to construct solution to system of linear balances. The results showed that solution to the system could be characterized based on minor rank of the coefficient matrix, such as full-row rank, full-column rank, or neither.**

Index Terms—**Balanced inverse, minor rank, system of linear balances, symmetrized max-plus algebra**

I. Introduction

THE max-plus algebra is defined as the set $\mathbb{R} \cup \{-\infty\}$, equipped with two binary operations, namely maximum (denoted as "max") for addition and conventional addition (denoted as "plus") for multiplication. In this context, $\mathbb{R}$ represents the set of all real numbers and the algebraic structure is denoted by $\mathbb{R}_{\max}$. In comparison with conventional algebra, not every element in max-plus algebra has an additive inverse, except for the zero element [7].

The absence of additive inverse in max-plus algebra can be addressed through a symmetrization process. This process introduces a balance relation, denoted by ∇, to define additive inverse-like elements in max-plus algebra. The result of symmetrization process is called symmetrized max-plus algebra, represented by $\mathbb{S}$. Typically, the structure of $\mathbb{S}$ comprises three different components, which include positive, negative, and balance parts. Furthermore, $\mathbb{R}_{\max}$ can be interpreted as the positive parts of $\mathbb{S}$, providing a broader framework for algebraic operations [3].

The system of linear balances in $\mathbb{S}$ serves a similar purpose to system of linear equations in conventional linear algebra. Detailed discussions on system of linear equations in linear algebra are available in [8], highlighting their numerous applications in daily scenarios. Many problems can be formulated and solved using system of linear equations. For instance, the system of linear equations in $\mathbb{R}_{\max}$ was discussed in [3], and the solution was determined using the concept of greatest subsolution. Further studies on max-plus interval systems of linear equations are presented in [9], while the investigation into system of fuzzy numbers in max-plus equations is discussed in [10].

The study of system of linear balances $A \otimes x \nabla b$, where A is an $n \times n$ matrix, has been examined in [3,1], but this discussion is limited to square coefficient matrix. Solving system of linear balances using $A \otimes X \otimes A \nabla A$, where A is an $m \times n$ matrix, as a generalized inverse in conventional algebra, has been addressed in [2]. However, this approach is considered inefficient for determining the solution to system of linear balances. Furthermore, the application of Cholesky decomposition to determine solutions for system of linear balances with $A \otimes x \nabla b$, where A is an $n \times n$ matrix has been explored in [12].

Discussion about expands the solution of system of linear balances $A \otimes x \nabla b$ presented in [3] to accommodate an arbitrary coefficient matrix A. The existence of a balanced inverse for submatrix of A to construct solutions of system of linear balances is determined using minor rank, as described in [11]. The obtained result simplifies the calculation process for determining the solution, compared to the method described in [2]. Additionally, the solution in [3] is observed to be a special case of broader results developed in this study

Based on the organization of this study, Section 1 provides an introduction, explaining the motivation for studying the topics. Section 2 discusses symmetrized max-plus algebra and its connection to conventional algebra. Sections 3, 4, and 5 present the main result, including the existence of balanced inverse of matrix in symmetrized max-plus algebra, characterization of solution for system of linear balances according to minor rank, and application of

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system, respectively. Finally, Section 6 contains the conclusion and the summary of key findings.

II. Preliminaries

Max-plus algebra is a mathematical system defined as $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, equipped with two binary operations, namely addition and multiplication. In this context, $\mathbb{R}$ represents the set of all real numbers, and the operations are defined as follows:

$$a \oplus b = \max(a, b)$$
$$a \otimes b = a + b$$

where $\max(a, -\infty) = a$ and $a + (-\infty) = -\infty$, for every $a, b \in \mathbb{R}_{\max}$. In max-plus algebra, the zero element is denoted as $\mathcal{E} = -\infty$, while the unity element is denoted as $e = 0$. It is important to be aware that every non-zero element in $\mathbb{R}_{\max}$ has no additive inverse

Max-plus algebraic symmetrization can be used to derive a negative form, similar to the process of expanding natural numbers into integers, in order to obtain a balanced element. For a comprehensive discussion on the symmetrisation process in max-plus algebra, please refer to [3]. The addition and multiplication in $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ are defined as follows:

$$(a, b) \oplus (c, d) = (a \oplus c, b \oplus d)$$
$$(a, b) \otimes (c, d) = (a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c)$$

for all $(a, b), (c, d) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max}$. The zero element is $(\mathcal{E}, \mathcal{E})$, the unity element is $(0, \mathcal{E})$, and the zero element is an absorbent for multiplication.

Definition 1 [3]
Let $u = (a, b), v = (c, d) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max}$. The balance relation, denoted by ∇ is defined as follows:

$$u \nabla v \textit{ iff } a \oplus d = b \oplus c$$

The balance relation is both reflexive and symmetric but not transitive. This implies that the relation can not be classified as equivalence. Consequently, it is not feasible to define the quotient set of $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ using ∇.

Definition 2 [3]
Let $u = (a, b), v = (c, d) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max}$. Relation $\mathcal{B}$ in $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ is defined as $u \, \mathcal{B} \, v$ iff

$$\begin{cases} (a, b) \nabla (c, d) & \text{, for } a \neq b \text{ and } c \neq d \\ (a, b) = (c, d) & \text{, for } a = b \text{ or } c = d \end{cases}$$

There are three types of equivalence classes generated by $\mathcal{B}$, including $\overline{(w, -\infty)}$ called max-positive (shortened to w), $\overline{(-\infty, w)}$, known as max-negative (shortened to $\ominus w$), and $\overline{(w, w)}$ referred to as balanced (shortened to $w^{\bullet}$). Max-zero class is denoted by $\overline{(\mathcal{E}, \mathcal{E})}$ and simply written as $\mathcal{E}$. The quotient set of $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ by $\mathcal{B}$ is denoted as

$$(\mathbb{R}_{\max} \times \mathbb{R}_{\max})/\mathcal{B} \stackrel{\text{def}}{=} \mathbb{S}$$

where the zero element is denoted by $\mathcal{E} = \overline{(\mathcal{E}, \mathcal{E})}$ and the unity element is $e = \overline{(0, \mathcal{E})}$. Additionally, $\mathbb{S}$ is called symmetrized max-plus algebra. The set of all max-positive or zero class is denoted by $\mathbb{S}^{\oplus}$, the set of all max-negative or zero class is represented as $\mathbb{S}^{\ominus}$, and the set of all balanced class is denoted by $\mathbb{S}^{\bullet}$. The set of all signed element is represented by $\mathbb{S}^{\vee} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$. Then, $\mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \cup \mathbb{S}^{\bullet} = \mathbb{S}$, $\mathbb{S}^{\oplus} \cap \mathbb{S}^{\ominus} \cap \mathbb{S}^{\bullet} = \left\{\overline{(\mathcal{E}, \mathcal{E})}\right\}$ and $\mathbb{S}^{\vee}_{*} = \mathbb{S}^{\vee} \backslash \mathbb{S}^{\bullet}$ represents the set of all elements that have an inverse multiplication. Several conceptual analogies between symmetrized max-plus algebra and conventional algebra are shown in Table 1.

TABLE I
ANALOGY OF CONCEPTS OF CONVENTIONAL ALGEBRA AND SYMMETRIZED MAX-PLUS ALGEBRA

Conventional Algebra	Symmetrized Max-Plus Algebra
$+$	$\oplus$
$\times$	$\otimes$
$-$	$\ominus$
$=$	∇
0	$a^{\bullet}$
$\mathbb{R}^{+}$	$\mathbb{S}^{\oplus}$
$\mathbb{R}^{-}$	$\mathbb{S}^{\ominus}$

Theorem 3 [3]
If $x, y \in \mathbb{R}_{\max}$, then

$$x \oplus (\ominus\, y) = \begin{cases} x & \text{, for } x > y \\ \ominus\, y & \text{, for } x < y \\ x^{\bullet} & \text{, for } x = y \end{cases}$$

Theorem 4 [3]
For all $a, b, c \in \mathbb{S}$, $a \ominus c \nabla b$ if and only if $a \nabla b \oplus c$.

Theorem 5 [3] **Weak Substitution**
For all $a, b, c \in \mathbb{S}$ and $x \in \mathbb{S}^{\vee}$, if $x \nabla a$ and $c \otimes x \nabla b$, then $c \otimes a \nabla b$.

Theorem 6 [3] **Reduction of Balance**
If $a \nabla b$ then $a = b$, for $a, b \in \mathbb{S}^{\vee}$.

The properties in Theorems 5 and 6 are called weak substitution and reduction of balance in symmetrized max-plus algebra, respectively. A balancing matrix over symmetrized max-plus algebra is similar to an equality matrix in conventional algebra. Specifically, for all $A, B \in \mathbb{S}^{m \times n}$, $A \nabla B$ iff $a_{ij} \nabla b_{ij}$ for $i = 1,2, \ldots, m$ and $j = 1,2, \ldots, n$.

The relationship between symmetrized max-plus algebra and conventional algebra is discussed in [5]. This connection is used to solve problems in symmetrized max-plus algebra through conventional algebraic method. The mapping that defines this link is explained in the following definition.

Definition 7 [5]
A mapping $\mathcal{F}$ with domain of $\mathbb{S} \times \mathbb{R}_0 \times \mathbb{R}_0^{+}$ is defined as

$$\mathcal{F}(a, \mu, s) = \begin{cases} |\mu| e^{as} & \text{, if } a \in \mathbb{S}^{\oplus} \\ -|\mu| e^{|a|_{\oplus} s} & \text{, if } a \in \mathbb{S}^{\ominus} \\ \mu e^{|a|_{\oplus} s} & \text{, if } a \in \mathbb{S}^{\bullet} \end{cases}$$

where $a \in \mathbb{S}, \mu \in \mathbb{R}_0, s \in \mathbb{R}_0^{+}$.

If f and g are functions, then f is asymtotically equivalent to g in the neighborhood of ∞, denoted as $f \sim g$ for $x \to \infty$.

Definition 8 [5]
Let $f(s) \sim v e^{|a|_{\oplus} s}$ in the neighborhood of ∞. The reverse function $\mathcal{R}$ is defined as follows:

$$\mathcal{R}(f) = \begin{cases} |a|_{\oplus} & \text{, if } v \text{ positive} \\ \ominus\, |a|_{\oplus} & \text{, if } v \text{ negative} \end{cases}.$$

III. The Balanced Inverse

This section discusses the inverse of matrix over symmetrized max-plus algebra in a "balanced" sense. The existence of balanced inverse for matrix over symmetrized max-plus algebra is demonstrated using the mapping in Definition 7 and 8. The following definition explains the balanced inverse of matrix in symmetrized max-plus algebra.

Definition 9

Let $A \in \mathbb{S}^{n\times n}$. If there exists a matrix $B \in \mathbb{S}^{n\times n}$ such that $A \otimes B \nabla I_n$ and $\otimes A \nabla I_n$, then A is said to be balanced invertible and B is balanced inverse of A. Furthermore, the balanced inverse of A is denoted by A_∇^{-1}.

First, the determinants of matrix in symmetrized max-plus algebra are discussed, along with their relationship to the determinants of a conventional matrix.

Lemma 10

Let $\bigoplus_{j=1}^{n}\left(\prod_{i=1}^{m} x_{ij}\right) = \mathcal{E}$ where $x_{ij} \in \mathbb{S}$ for $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$. Then, each of $\prod_{i=1}^{n} x_{ij}$ contains $x_{ij} = \mathcal{E}$.

Proof. Since $\bigoplus_{j=1}^{n}\left(\prod_{i=1}^{m} x_{ij}\right) = \mathcal{E}$ then each of $\left(\prod_{i=1}^{m} x_{ij}\right)$ is $\mathcal{E}$. Consequently, $x_{1j} = \mathcal{E}$ or $x_{2j} = \mathcal{E}$ or ... or $x_{mj} = \mathcal{E}$, and each $\prod_{i=1}^{n} x_{ij}$ contains $x_{ij} = \mathcal{E}$. ∎

Theorem 11

Let $A \in \mathbb{S}^{n\times n}$ and $\tilde{A} = \mathcal{F}(A, M_A, \cdot)$ where $M_A \in \mathbb{R}_0^{n\times n}$ is the matrix which corresponds to A by mapping in Definition 7. If $\det(\tilde{A}) = 0$ then $\det(A)\,\nabla \mathcal{E}$.

Proof. Suppose

$$\det(A) = \bigoplus_{\sigma}\left(\operatorname{sign}(\sigma) \otimes_{i=1}^{n} a_{i\sigma(i)}\right).$$

For $k = 1,2,\dots,n!$, let r_k is the product of $\otimes_{i=1}^{n} a_{i\sigma(i)}$ where $r_1^{\oplus}, r_2^{\oplus}, \dots, r_{\frac{n!}{2}}^{\oplus}$ and $r_1^{\ominus}, r_2^{\ominus}, \dots, r_{\frac{n!}{2}}^{\ominus}$ are the positive and the negative signed products, respectively. Let

$$r_1^{\oplus} \oplus r_2^{\oplus} \oplus \dots \oplus r_{\frac{n!}{2}}^{\oplus} = r^{\oplus}$$

$$r_1^{\ominus} \oplus r_2^{\ominus} \oplus \dots \oplus r_{\frac{n!}{2}}^{\ominus} = r^{\ominus}$$

then $r^{\oplus} \oplus r^{\ominus} \not\nabla \mathcal{E}$.

Suppose a_{ij} corresponds to $\tilde{a}_{ij}$, $\tilde{r}_k$ is the signed product of $\otimes_{i=1}^{n} \tilde{a}_{i\sigma(i)}$ in $\det(\tilde{A})$, for $k = 1,2,\dots,n!$. If $\tilde{r}_1^{\oplus}, \tilde{r}_2^{\oplus}, \dots, \tilde{r}_{\frac{n!}{2}}^{\oplus}$ are the positive signed product and $\tilde{r}_1^{\ominus}, \tilde{r}_2^{\ominus}, \dots, \tilde{r}_{\frac{n!}{2}}^{\ominus}$ are the negative signed product in $\det(\tilde{A})$ then $\det\left(\tilde{A}(s)\right) \sim e^{r^{\oplus}s \oplus r^{\ominus}s} = e^{(r^{\oplus}\oplus r^{\ominus})s} \neq 0, s \to \infty$. Consequently,

$$\lim_{s\to\infty} \frac{\det\left(\tilde{A}(s)\right)}{e^{(r^{\oplus}\oplus r^{\ominus})s}} = 1$$

where $e^{(r^{\oplus}\oplus r^{\ominus})s} \neq 0$ and $\det\left(\tilde{A}(s)\right) \neq 0$. Therefore, if $\det(A)\,\not\nabla \mathcal{E}$ then $\det(\tilde{A}) \neq 0$. ∎

The following theorem shows the existence of balanced inverse of matrix over symmetrized max-plus algebra.

Theorem 12

Let $A \in \mathbb{S}^{n\times n}$. If $\det(A) \not\nabla \mathcal{E}$ then there is $A_\nabla^{-1} \in (\mathbb{S}^{\vee})^{n\times n}$ such that $A \otimes A_\nabla^{-1} \nabla\, I_n$ and $A_\nabla^{-1} \otimes A \nabla I_n$.

Proof. If there exists a non-signed element in $A \in \mathbb{S}^{n\times n}$, then $\hat{A} \in (\mathbb{S}^{\vee})^{n\times n}$ is defined as:

$$\hat{a}_{ij} = \begin{cases} a_{ij} & , a_{ij} \text{ is signed element} \\ |a_{ij}|_{\oplus} & , a_{ij} \text{ is non signed element} \end{cases}$$

for all i,j. Since $\hat{a}_{ij} \nabla a_{ij}$ for all i,j, then $\hat{A} \nabla A$. Furthermore, if $\hat{A}\nabla A$, then $\hat{A} \otimes A_\nabla^{-1} \nabla\, I_n$ and $A_\nabla^{-1} \otimes \hat{A} \nabla\, I_n$. This shows that it is sufficient to prove the case of a signed matrix A. Then, assuming A is a signed matrix.

Let $\tilde{A} = [\tilde{a}_{ij}]$ be a matrix in conventional algebra that corresponds to $A = [a_{ij}] \in \mathbb{S}^{n\times n}$ by mapping in Definition 7. According to Theorem 11, since $\det(A) \not\nabla \mathcal{E}$, it follows that $\det(\tilde{A}) \neq 0$.

Suppose $\operatorname{cof}\left(\tilde{A}(s)\right)^T$ is transpose of cofactor matrix in $\tilde{A}(s)$, then

$$\frac{\tilde{A}(s).\operatorname{cof}\left(\tilde{A}(s)\right)^T}{\det\left(\tilde{A}(s)\right)} \sim \tilde{I}_n$$

and

$$\frac{\operatorname{cof}\left(\tilde{A}(s)\right)^T.\tilde{A}(s)}{\det\left(\tilde{A}(s)\right)} \sim \tilde{I}_n$$

for $s \to \infty$. Let

$$\tilde{\tilde{A}}(s) \stackrel{\text{def}}{=} \frac{\operatorname{cof}\left(\tilde{A}(s)\right)^T}{\det\left(\tilde{A}(s)\right)}$$

then $\tilde{A}(s)$ and $\tilde{\tilde{A}}(s)$ satisfy $\tilde{A}(s)\tilde{\tilde{A}}(s) \sim \tilde{I}_n$ and $\tilde{\tilde{A}}(s)\tilde{A}(s) \sim \tilde{I}_n$, respectively. Consequently, it follows that $A \otimes A_\nabla^{-1} \nabla I_n$ and $A_\nabla^{-1} \otimes A \nabla I_n$. This implies that the balanced inverse $A_\nabla^{-1} \in (\mathbb{S}^{\vee})^{n\times n}$ exists for a signed matrix A where $A \otimes A_\nabla^{-1} \nabla I_n$ and $A_\nabla^{-1} \otimes A \nabla I_n$. By the weak substitution properties, the existence of $A_\nabla^{-1} \in (\mathbb{S}^{\vee})^{n\times n}$ is also valid for a non-signed matrix A. ∎

According to [3], $A \otimes \operatorname{cof}(A)^T \nabla \det(A) \otimes I_n$. If $\det(A) \not\nabla \mathcal{E}$, then

$$A \otimes (\det(A)^{-1} \otimes \operatorname{cof}(A)^T)\ \nabla\ I_n.$$

This implies that $(\det(A)^{-1} \otimes \operatorname{cof}(A)^T)$ is also the balanced inverse form of A. The balanced inverse A_∇^{-1} offers more benefits than $(\det(A)^{-1} \otimes \operatorname{cof}(A)^T)$ as described in [3]. Since $A_\nabla^{-1} \in (\mathbb{S}^{\vee})^{n\times n}$, the weak substitution property can be used to obtain another matrix in $\mathbb{S}^{n\times n}$ which balances to A_∇^{-1}. Meanwhile, this is not necessarily true for $(\det(A)^{-1} \otimes \operatorname{cof}(A)^T)$, as the property may not be a signed matrix.

Corollary 13

Let $A_\nabla^{-1} \in (\mathbb{S}^{\vee})^{n\times n}$ is the balanced inverse of A. For any $A_\nabla^{-1\prime} \in \mathbb{S}^{n\times n}$, if $A_\nabla^{-1\prime} \nabla A_\nabla^{-1}$ then it satisfies $A \otimes A_\nabla^{-1\prime} \nabla\, I_n$ and $A_\nabla^{-1\prime} \otimes A\ \nabla\ I_n$.

Proof. Since $A_\nabla^{-1\prime} \nabla A_\nabla^{-1}$, $A \otimes A_\nabla^{-1} \nabla I_n$ and $A_\nabla^{-1} \otimes A \nabla I_n$ where $A_\nabla^{-1} \in (\mathbb{S}^{\vee})^{n\times n}$, then according to weak substitution properties, it follows that $A \otimes A_\nabla^{-1\prime} \nabla I_n$ and $A_\nabla^{-1\prime} \otimes A \nabla I_n$, respectively. ∎

Let $A = \begin{bmatrix} 1 & 0^{\bullet} \\ -1 & \ominus 2 \end{bmatrix}$, then $\det(A) = \ominus 3 \not\nabla \mathcal{E}$ and

$$\tilde{A}(s) = \begin{bmatrix} e^{s} & 1 \\ e^{-s} & -e^{2s} \end{bmatrix}$$

is a conventional matrix that corresponds to A. Since $\det\left(\tilde{A}(s)\right) = -e^{3s} - e^{-s} \sim -e^{3s}$ then

$$\tilde{A}(s)^{-1} = \frac{1}{-e^{3s}} \begin{bmatrix} -e^{2s} & -1 \\ -e^{-s} & e^{s} \end{bmatrix} \sim \begin{bmatrix} e^{-s} & e^{-3s} \\ e^{-4s} & -e^{-2s} \end{bmatrix}.$$

This corresponds with $A_\nabla^{-1} = \begin{bmatrix} -1 & -3 \\ -4 & \ominus(-2) \end{bmatrix}$. Note that

$$A \otimes A_{\nabla}^{-1} = \begin{bmatrix} 0 & \ominus (-2)^{\bullet} \\ \ominus (-2)^{\bullet} & 0 \end{bmatrix} \nabla\ I_2$$

and

$$A_{\nabla}^{-1} \otimes A = \begin{bmatrix} 0 & (-1)^{\bullet} \\ \ominus (-2)^{\bullet} & 0 \end{bmatrix} \nabla\ I_2.$$

Meanwhile,

$$\det(A)^{-1} \otimes \operatorname{cof}(A)^T = \begin{bmatrix} -1 & (-3)^{\bullet} \\ -4 & \ominus (-2) \end{bmatrix}$$

This also satisfies $A \otimes (\det(A)^{-1} \otimes \operatorname{cof}(A)^T) \nabla I_2$ and $(\det(A)^{-1} \otimes \operatorname{cof}(A)^T) \nabla I_2$. In this context, the matrix $\det(A)^{-1} \otimes \operatorname{cof}(A)^T$ is not a signed matrix, since there is a balance entry $(-1)^{\bullet}$. Furthermore, $\det(A)^{-1} \otimes \operatorname{cof}(A)^T$ is one of matrix that balances with A_{∇}^{-1} and satisfy the balance inverse of A.

IV. The Solution of System Of Linear Balances

In this section, the solution of system of linear balances $A \otimes x \nabla b$ for arbitrary coefficient matrix A is discussed. The solution is determined using minor rank of a square submatrix of coefficient matrix A. The solution of system of linear balances $A \otimes x \nabla b$, is characterized for cases where A has full-row rank, full-column rank, or neither. The identification of minor rank of A is performed in order to achieve a partition of A as described in [4]. Subsequently, the balanced inverse of the square submatrix of A corresponding to minor rank is used to construct the solution of system of linear balances.

Definition 14 [7]
Let $A \in \mathbb{S}^{m\times n}$. Max-algebraic minor rank of A is the dimension of the largest square submatrix of max-algebraic determinant of A which is not balanced.

Theorem 15 [3]
Let $A \otimes x\nabla b$ be system of linear balances where $A \in \mathbb{S}^{n\times n}$, $\det(A) \in \mathbb{S}_*^{\vee}$, $b \in \mathbb{S}^n$ and $\operatorname{cof}(A)^T \otimes b \in (\mathbb{S}^{\vee})^n$. Then, there exists a unique solution of $A \otimes x\nabla b$ and it satisfies

$$x \nabla (\operatorname{cof}(A)^T \otimes b)^T \otimes \det(A)^{-1}.$$

Theorem 15 is called Cramer's rule in symmetrized max-plus algebra. In addition, it shows determination of the solution of system of linear balances $A \otimes x\nabla b$ for matrix A of size $n \times n$ and $\det(A) \in \mathbb{S}_*^{\vee}$. This study extends system of linear balances $A \otimes x\nabla b$ to an arbitrary coefficient matrix of size $m \times n$. The following properties present the partition of A based on minor rank and permutation matrix.

Lemma 16
Let $A \in \mathbb{S}^{r\times n}$ and minor rank of A is r. Then, there exists a permutation matrix $Q \in \mathbb{S}^{n\times n}$ such that $A = [A_1 \quad A_2] \otimes Q^T$, where $A_1 \in \mathbb{S}^{r\times r}$ is a submatrix of A which corresponds to minor rank of A.

Proof. Since minor rank of $A \in \mathbb{S}^{r\times n}$ is r, there exists an $r \times r$ submatrix whose determinant is not balanced with $\mathcal{E}$. Let the columns corresponding to minor rank of A be $k_1, k_2, \dots, k_r$. Subsequently, a column swap is performed to position $k_1, k_2, \dots, k_r$ in the $1,2,\dots,r^{th}$ columns, respectively. After the column changes, a matrix is obtained where the entries in the first r column are $q_{ij} = e$ for $ij = k_1 1, k_2 2, \dots, k_r r$ and $q_{ij} = \mathcal{E}$ for others. This matrix is a permutation matrix $Q \in \mathbb{S}^{n\times n}$. Consequently, it follows that $A \otimes Q$ is a matrix where the first r column corresponds to minor rank of A.

Let A_1 be a matrix whose columns correspond to the r columns associated with minor rank of A, and let A_2 be a matrix whose columns represent the remaining $(n-r)$ columns that do not correspond to minor rank of A. Therefore, the matrix A can be partitioned as

$$A \otimes Q = [A_1 \quad A_2].$$

Since Q is permutation matrix, there exists $Q^T \in (\mathbb{S}^{\vee})^n$ such that $Q \otimes Q^T = I_n$ and $Q^T \otimes Q = I_n$. Consequently, $A = [A_1 \quad A_2] \otimes Q^T$. ■

Lemma 17
Let $A \in \mathbb{S}^{m\times r}$ minor rank of A is r. Then, there exists a permutation matrix $P \in \mathbb{S}^{m\times m}$ such that $A = P^T \otimes \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ where $A_1 \in \mathbb{S}^{r\times r}$ is a submatrix of A which corresponds to minor rank of A.

Proof. Since minor rank of $A \in \mathbb{S}^{m\times r}$ is r, there exists an $r \times r$ submatrix whose determinant is not balanced with $\mathcal{E}$. Let the rows corresponding to minor rank A be denoted as $b_1, b_2, \dots, b_r$. A row swap is then performed such that the positions $b_1, b_2, \dots, b_r$ are placed in the $1,2,\dots,r^{th}$ rows, respectively. After the rows swap, a matrix is obtained where entries in the first r rows are $p_{ij} = e$ for $ij = 1b_1, 2b_2, \dots, rb_r$ and $p_{ij} = \mathcal{E}$ for others. This matrix is a permuation matrix $P \in \mathbb{S}^{m\times m}$, and it follows that $P \otimes A$ is a matrix where the first r rows correspond to minor rank of A.

Let A_1 be a matrix whose rows correspond to r associated with minor rank of A, and let A_2 be a matrix whose rows are the remaining $(m-r)$ rows that do not correspond to minor rank of A. Therefore, A can be partitioned as $P \otimes A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. Since P is permutation matrix, there exists $P^T \in (\mathbb{S}^{\vee})^n$ such that $P \otimes P^T = I_m$ and $P^T \otimes P = I_m$. Consequently, $A = P^T \otimes \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. ■

Corollary 18
Let $A \in \mathbb{S}^{m\times n}$ and minor rank of A is $< m$, $r < n$. Then, there exists a permutation matrix $P \in \mathbb{S}^{m\times m}$ and $Q \in \mathbb{S}^{n\times n}$ such that

$$A = P^T \otimes \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \otimes Q^T$$

where $A_1 \in \mathbb{S}^{r\times r}$ is submatrix of A corresponding to minor rank of A.

Proof. Analogous to Lemma 16 and 17. ■

Lemma 16 and 17 are used to analyze the coefficient matrix of system of linear balances $A \otimes x \nabla b$ in order to create partition matrix A. Furthermore, the solution of the system of linear balances $A \otimes x \nabla b$ is determined based on minor rank of A. The existence of balanced inverse of the submatrix of A corresponding to minor rank of A is used to construct a solution of system of linear balances.

Let $A \otimes x\nabla b$ be a system of linear balances, $A \in \mathbb{S}^{r\times n}$ and minor rank of A is r. According to Lemma 16, the system of linear balances is formulated as follows:

$$[A_1 \quad A_2] \otimes Q^T \otimes x\nabla b \tag{1}$$

where $Q \in \mathbb{S}^{n\times n}$ is a permutation matrix and $A_1 \in \mathbb{S}^{r\times r}$ is a submatrix of A corresponding to minor rank of A. Since $\det(A_1) \not\nabla \mathcal{E}$, the existence of balanced inverse of A is guaranteed. Let $A_{1\nabla}^{-1}$ be the balanced inverse of A, then

$$[A_1 \quad A_2]_{r\times n} \otimes \begin{bmatrix} A_{1\nabla}^{-1} \\ \mathcal{E} \end{bmatrix}_{n\times r}$$

$$= [A_1 \quad A_2]_{r\times n} \otimes Q^T \otimes Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \\ \mathcal{E} \end{bmatrix}_{n\times r} \nabla\, I_r. \qquad (2)$$

If constant vector in (1) is $b = \mathcal{E}$, then system of linear balances is as follows:

$$[A_1 \quad A_2] \otimes Q^T \otimes x \nabla \mathcal{E}. \qquad (3)$$

Let $F = A_{1\nabla}^{-1} \otimes A_2$. If both equality is multiplied by A_1 then

$$A_1 \otimes F = A_1 \otimes A_{1\nabla}^{-1} \otimes A_2 \nabla A_2.$$

If A_2 is a signed matrix, then by the weak substitution property, it is obtained that:

$$A_1 \otimes [I_r \quad F] \otimes Q^T \otimes x \nabla \mathcal{E}. \qquad (4)$$

Since $[I_r \quad F]_{r\times n} \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} = F^{\bullet}$ then

$$[I_r \quad F]_{r\times n} \otimes Q^T \otimes Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} = F^{\bullet}.$$

Therefore,

$$[I_r \quad F]_{r\times n} \otimes Q^T \otimes Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \nabla\, \mathcal{E}. \qquad (5)$$

According to (2) and (5) the following theorems are considered, respectively.

Theorem 19

Let $A \otimes x \nabla b$ be a system of linear balances, $A \in \mathbb{S}^{r\times n}$ and $b \in \mathbb{S}^r$. If minor rank of A is r, $A = [A_1 \quad A_2] \otimes Q^T$ where $A_1 \in \mathbb{S}^{r\times r}$ is a submatrix of A which corresponds to minor rank of A and Q is an $n \times n$ permutation matrix then

$$x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix}$$

is solution of system of linear balances. Furthermore, if

$$Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \in (\mathbb{S}^{\vee})^n$$

then all of x which

$$x \nabla Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix}$$

is also the solution of system of linear balances.

Proof. If there exists a non-signed element in $A \in \mathbb{S}^{r\times n}$, then $\hat{A} \in (\mathbb{S}^{\vee})^{r\times n}$ is defined as

$$\hat{a}_{ij} = \begin{cases} a_{ij} & , a_{ij} \text{ is signed element} \\ |a_{ij}|_{\oplus} & , a_{ij} \text{ is non signed element} \end{cases}$$

for all i,j. Since $\hat{a}_{ij} \nabla a_{ij}$ for all i,j, it follows that $\hat{A} \nabla A$. If $\hat{A} \nabla A$ and $\hat{A} \otimes x \nabla b$, then $A \otimes x \nabla b$. Therefore, it is sufficient to prove a signed matrix A.

Let A be a signed matrix. Since $A = [A_1 \quad A_2] \otimes Q^T$, it follows that

$$\begin{aligned} & A \otimes \left(Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \right) \\ & = [A_1 \quad A_2] \otimes Q^T \otimes Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \\ & \nabla\, I_r \otimes b = b. \end{aligned}$$

Furthermore, $x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix}$ satisfies $A \otimes x \nabla b$, and it is a solution of $A \otimes x \nabla b$. If $Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \in (\mathbb{S}^{\vee})^n$, then by the weak substitution, it follows that all of x where $x \nabla Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix}$ also solutions to the system of linear balances $A \otimes x \nabla b$. ■

Theorem 20

Let $A \otimes x \nabla \mathcal{E}$ be a system of linear balances, $A \in \mathbb{S}^{r\times n}$. If minor rank of A is r, $A = [A_1 \quad A_2] \otimes Q^T$ with $A_1 \in \mathbb{S}^{r\times r}$ is the submatrix corresponding to minor rank of A, Q is an $n \times n$ permutation matrix and $F = A_{1\nabla}^{-1} \otimes A_2$ then

$$x = Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

for $y \in \mathbb{S}^{n-r}$, is solution of the system of linear balances. If

$$Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \in (\mathbb{S}^{\vee})^n$$

then all of x which

$$x \nabla Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

for $y \in \mathbb{S}^{n-r}$, is also the solution to the system of linear balances.

Proof. If there exists a non-signed element in $A \in \mathbb{S}^{r\times n}$, then $\hat{A} \in (\mathbb{S}^{\vee})^{r\times n}$ is defined as follows:

$$\hat{a}_{ij} = \begin{cases} a_{ij} & , a_{ij} \text{ is signed element} \\ |a_{ij}|_{\oplus} & , a_{ij} \text{ is non signed element} \end{cases}$$

for all i,j. Since $\hat{a}_{ij} \nabla a_{ij}$ for all i,j, it follows that $\hat{A} \nabla A$. If $\hat{A} \nabla A$ and $\hat{A} \otimes x \nabla b$, then $A \otimes x \nabla b$. This shows that it is sufficient to prove a signed matrix A.

Let A be a signed matrix. Since $A = [A_1 \quad A_2] \otimes Q^T$ then $[A_1 \quad A_2] \otimes Q^T x \nabla \mathcal{E}$. Since A_2 is a signed matrix, then

$$\begin{aligned} & [A_1 \quad A_1 \otimes F] \otimes Q^T \otimes \left(Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \right) \\ & = [A_1 \quad A_1 \otimes F] \otimes Q^T \otimes Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \\ & = A_1 \otimes [I_r \quad F] \otimes I_n \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \\ & = A_1 \otimes (\ominus F \oplus F) \otimes y \\ & = A_1 \otimes F^{\bullet} \otimes y \nabla \mathcal{E}, \end{aligned}$$

for $y \in \mathbb{S}^{n-r}$. The expression $x = Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$ where $y \in \mathbb{S}^{n-r}$ satisfies the balance linear systems $A \otimes x \nabla \mathcal{E}$, and it is solution to $A \otimes x \nabla \mathcal{E}$.

If $Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \in (\mathbb{S}^{\vee})^n$, then by applying the weak substitution property, it follows that all of x where

$$x \nabla Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y,\ y \in \mathbb{S}^{n-r}$$

also solution to the balanced linear systems. ■

Corollary 21

Let $A \otimes x \nabla \mathcal{E}$ be a system of linear balances, $A \in \mathbb{S}^{r\times n}$ and $b \in \mathbb{S}^r$. If minor rank of A is r, $A = [A_1 \quad A_2] \otimes Q^T$ with $A_1 \in \mathbb{S}^{r\times r}$ is the submatrix corresponding to minor rank of A, Q is an $n \times n$ permutation matrix and $F = A_{1\nabla}^{-1} \otimes A_2$ then

$$x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

for $y \in \mathbb{S}^{n-r}$, is solution of system of linear balances. If

$$Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \in (\mathbb{S}^{\vee})^n$$

then all of x which

$$x \nabla Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

for $y \in \mathbb{S}^{n-r}$, is also solution to the system of linear balances.

Proof. Since

$$\begin{aligned} & A \otimes \left(Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \right) \\ & = A \otimes \left(Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \right) \oplus A \otimes \left(Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \right) \end{aligned}$$

$\nabla b \oplus \mathcal{E} = b$,
then

$$x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

for $y \in \mathbb{S}^{n-r}$, satisfies the system of linear balances. Therefore, it is solution to $A \otimes x \nabla b$.

If $Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \in (\mathbb{S}^{\vee})^n$ then according to the weak substitution property, it is obtained that all of x where

$$x \nabla Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

for $y \in \mathbb{S}^{n-r}$, also solution of the balance linear systems. ∎

Let $A \otimes x \nabla b$ be the system of linear balances, $A \in \mathbb{S}^{m\times r}$ and minor rank of A is r. According to theorem 17, it is obtained that $P^T \otimes \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \otimes x \nabla b$. If P is partitioned into $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$, then $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^T \otimes \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \otimes x \nabla b$, with $P \in \mathbb{S}^{m\times m}$ is a permutation matrix and $A_1 \in \mathbb{S}^{r\times r}$ is a submatrix corresponding to minor rank of A. Since $\det(A_1) \not\nabla \mathcal{E}$, then the existence of the balanced inverse of A is guaranteed.

Let $G = A_2 \otimes A_{1\nabla}^{-1} \in \mathbb{S}^{(m-r)\times r}$ then $\otimes A_1 \nabla A_2$. If A_2 is a signed matrix then

$$\begin{bmatrix} I_r \\ G \end{bmatrix} \otimes A_1 \otimes x \nabla \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \otimes b.$$

Consequently, two sub-balance linear systems are obtained as follows:

$$A_1 \otimes x \nabla P_1 \otimes b \tag{6}$$

$$G \otimes A_1 \otimes x \nabla P_2 \otimes b. \tag{7}$$

The value of x that satisfies $A \otimes x \nabla b$ need to also satisfy equations (6) and (7). According to Theorem 15, the expression $x = A_{1\nabla}^{-1} \otimes P_1 \otimes b$ satisfies equation (6) as well as equation (7), expressed below.

$$\begin{aligned} &G \otimes A_1 \otimes x \\ &= G \otimes A_1 \otimes A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ &\;\nabla\, G \otimes P_1 \otimes b. \end{aligned}$$

Therefore, $G \otimes P_1 \otimes b \nabla P_2 \otimes b$.

Theorem 22

Let $A \otimes x \nabla b$ be a system of linear balances, $A \in \mathbb{S}^{m\times r}$ and $b \in \mathbb{S}^r$. If minor rank of A is r, $A = P^T \otimes \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ where $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ is an $m \times m$ permutation matrix, A_1 is an $r \times r$ submatrix of A which corresponds to minor rank of A, $G = A_2 \otimes A_{1\nabla}^{-1}$ and $P_2 \otimes b \nabla G \otimes P_1 \otimes b$, then

$$x = A_{1\nabla}^{-1} \otimes P_1 \otimes b$$

is solution to the system of linear balances.

If $A_{1\nabla}^{-1} \otimes P_1 \otimes b \in (\mathbb{S}^{\vee})^r$ then $x \nabla A_{1\nabla}^{-1} \otimes P_1 \otimes b$ is also the solution to the system of linear balances.

Proof. If there exists a non-signed element in $A \in \mathbb{S}^{r\times n}$, then $\hat{A} \in (\mathbb{S}^{\vee})^{r\times n}$ is defined as follows:

$$\hat{a}_{ij} = \begin{cases} a_{ij} & , a_{ij} \text{ is signed element} \\ |a_{ij}|_{\oplus} & , a_{ij} \text{ is non signed element} \end{cases}$$

for all i,j. Since $\hat{a}_{ij} \nabla a_{ij}$ for all i,j, it follows that $\hat{A} \nabla A$. If $\hat{A} \nabla A$ and $\hat{A} \otimes x \nabla b$, then $A \otimes x \nabla b$. This implies that the theorem is sufficient to prove a signed matrix A.

Let A be a signed matrix. Since $G = A_2 \otimes {A_1}^{-1}$ then $A_2 \nabla G \otimes A_1$ for A_2 is a signed matrix. Let A is partitioned into $A = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^T \otimes \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, then $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \otimes x \nabla \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \otimes b$, would lead to two sub-systems of linear balances as follows:

$$A_1 \otimes x \nabla P_1 \otimes b \tag{8}$$

$$A_2 \otimes x \nabla P_2 \otimes b \tag{9}$$

Provided that x satisfies both Equations (8) and (9).

For $x = A_{1\nabla}^{-1} \otimes P_1 \otimes b$, the solution is derived as follows:

$$\begin{aligned} &A_1 \otimes x \\ &= A_1 \otimes {A_1}^{-1} \otimes P_1 \otimes b \\ &\nabla\, I_r \otimes P_1 \otimes b = P_1 \otimes b \end{aligned}$$

and $x = A_{1\nabla}^{-1} \otimes P_1 \otimes b$, satisfies equation (8). Since

$$\begin{aligned} &A_2 \otimes x \\ &= A_2 \otimes {A_1}^{-1} \otimes P_1 \otimes b \\ &= G \otimes P_1 \otimes b \nabla P_2 \otimes b \end{aligned}$$

then $x = A_{1\nabla}^{-1} \otimes P_1 \otimes b$ also satisfies equation (9). Therefore, $x = A_{1\nabla}^{-1} \otimes P_1 \otimes b$ satisfies $A \otimes x \nabla b$. If

$$A_{1\nabla}^{-1} \otimes P_1 \otimes b \in (\mathbb{S}^{\vee})^r$$

then all of x that satisfies $x \nabla A_{1\nabla}^{-1} \otimes P_1 \otimes b$ also solution to $A \otimes x \nabla b$. ∎

According to Corollary 21 and Theorem 22, a solution to system of linear balances $A \otimes x \nabla b$, where A is not full-row rank and full-column rank, can be constructed as seen in the subsequent theorem.

Theorem 23

Let $A \otimes x \nabla b$ be a system of linear balances, $A \in \mathbb{S}^{m\times n}$ and $b \in \mathbb{S}^m$. If minor rank of A is $r < \min\{m,n\}$ and

$$A = P^T \otimes \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \otimes Q^T$$

with $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ is an $m \times m$ permutation matrix, Q is an $n \times n$ permutation matrix, A_1 is an $r \times r$ submatrix of A which corresponds to minor rank, $F = A_{1\nabla}^{-1} \otimes A_2$, $G = A_3 \otimes A_{1\nabla}^{-1}$, and $P_2 \otimes b \nabla G \otimes P_1 \otimes b$ then

$$x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y,$$

for $y \in \mathbb{S}^{n-r}$, is solution of the system of linear balances.

If $Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \in (\mathbb{S}^{\vee})^n$ then all of x which

$$x \nabla Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y,$$

for $y \in \mathbb{S}^{n-r}$, is also solution to the system of linear balances.

Proof. If there exists a non-signed element in $A \in \mathbb{S}^{r\times n}$, then $\hat{A} \in (\mathbb{S}^{\vee})^{r\times n}$ is defined as:

$$\hat{a}_{ij} = \begin{cases} a_{ij} & , a_{ij} \text{ is signed element} \\ |a_{ij}|_{\oplus} & , a_{ij} \text{ is non signed element} \end{cases}$$

for all i,j. Since $\hat{a}_{ij} \nabla a_{ij}$ for all i,j, it follows that $\hat{A} \nabla A$. If $\hat{A} \nabla A$ and $\hat{A} \otimes x \nabla b$, then $A \otimes x \nabla b$. Therefore, it is sufficient to prove a signed matrix A.

Since the property holds for signed matrix, the verification can proceed accordingly,

$$A = P^T \otimes \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \otimes Q^T, P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

and

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \otimes Q^T \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \nabla \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \otimes b.$$

Since $F = A_{1\nabla}^{-1} \otimes A_2$ and $G = A_3 \otimes A_{1\nabla}^{-1}$, then $A_2 \nabla A_1 \otimes F$ and $A_3 \nabla G \otimes A_1$, respectively. Consequently

$$\begin{bmatrix} I_r \\ G \end{bmatrix} \otimes A_1 \otimes [I_r \quad F] \otimes Q^T \otimes x \nabla \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \otimes b \qquad (10)$$

and there are two sub-systems of linear balances, i.e

$$A_1 \otimes [I_r \quad F] \otimes Q^T \otimes x \nabla P_1 \otimes b \qquad (11)$$

$$G \otimes A_1 \otimes [I_r \quad F] \otimes Q^T \otimes x \nabla P_2 \otimes b. \qquad (12)$$

If $x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix}$ and $x = Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$ are substituted to (11) then

$$A_1 \otimes [I_r \quad F] \otimes Q^T \otimes \left(Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \right)$$

$$= A_1 \otimes [I_r \quad F] \otimes \begin{bmatrix} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \nabla P_1 \otimes b$$

and

$$A_1 \otimes [I_r \quad F] \otimes Q^T \otimes Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

$$= \left(\ominus (A_1 \otimes F) \oplus (A_1 \otimes F) \right) \otimes y \nabla \mathcal{E}$$

for $y \in \mathbb{S}^{n-r}$, respectively. Therefore

$$x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

for $y \in \mathbb{S}^{n-r}$ satisfies (11).

Furthermore, if we substitute $x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix}$ and $x = Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$ to (12) then we have

$$G \otimes A_1 \otimes [I_r \quad F] \otimes Q^T \otimes Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix}$$

$$= G \otimes [A_1 \quad A_1 \otimes F] \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix}$$

$$= G \otimes P_1 \otimes b \nabla P_2 \otimes b$$

and

$$G \otimes A_1 \otimes [I_r \quad F] \otimes Q^T \otimes Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

$$= G \otimes [A_1 \quad A_1 \otimes F] \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

$$= G \otimes (A_1 \otimes F)^{\bullet} \otimes y \nabla \mathcal{E},$$

for $y \in \mathbb{S}^{n-r}$. Therefore, we have

$$x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

for $y \in \mathbb{S}^{n-r}$ satisfies (12). Since

$$x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y$$

for $y \in \mathbb{S}^{n-r}$ satisfies (11) and (12), it satisfies $A \otimes x \nabla b$. If

$$Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y \in (\mathbb{S}^{\vee})^n$$

then according to the weak substitution property, it is obtained that all of x which

$$x \nabla Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \mathcal{E} \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y,$$

for $y \in \mathbb{S}^{n-r}$, is also solution to $A \otimes x \nabla b$. ■

Theorem 15 in [3] discusses solution of system of linear balances $A \otimes x \nabla b$ where A is square matrix. This theorem can be viewed using Corollary 21 and Theorem 22 for $r = m = n$ as in the following corollary.

Corollary 24

Let $A \otimes x \nabla b$ be the system of linear balances, $A \in \mathbb{S}^{n \times n}$ with minor rank $r = n$, then $x = A_{1\nabla}^{-1} \otimes b$ is solution of the system of linear balances. Furthermore, if $A_{1\nabla}^{-1} \otimes b \in (\mathbb{S}^{\vee})^n$ then all of x which $x \nabla A_{1\nabla}^{-1} \otimes b$ is also solution of system of linear balances.

Let the system of linear balance

$$\begin{aligned} 0 \otimes x_1 \oplus 1 \otimes x_2 \nabla 3 \\ 0 \otimes x_1 \ominus 0 \otimes x_2 \nabla 1. \end{aligned} \qquad (13)$$

then, consider system of linear balances $A \otimes x \nabla b$, for

$$A = \begin{bmatrix} 0 & 1 \\ 0 & \ominus 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Minor rank of A is 2, and the submatrix of A_1 which corresponds to minor rank A is expressed as $A_1 = A$, and

$$A_{1\nabla}^{-1} = \begin{bmatrix} -1 & 0 \\ -1 & \ominus(-1) \end{bmatrix}.$$

According to Corollary 24, it follows that

$$x = A_{1\nabla}^{-1} \otimes b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

is the solution of system of linear balances. This vector represents the unique solution of equation (13) in $\mathbb{S}^{\vee}$. However, the balance solution in $\mathbb{S}^{\bullet}$ can also be determined using the weak substitution property of balance relations. Since x is a signed vector, the weak substitution property guarantees that any vector x' satisfying $x' \nabla \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is also a solution of system of linear balances.

The solution to system of linear balances can be interpretated as a geometric interpretation of the solutions for the two-dimensional case in the plane of signed coordinates $\mathbb{S}^{\vee} \times \mathbb{S}^{\vee}$. It can be interpreted as coordinates (x_1, x_2), which is the intersection of two linear balances. Fig. 1 shows the geometric interpretation of the signed solution of system in equation (13). The signed vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ as the intersection point at coordinates $(x_1, x_2) = (2,2)$ in the $\mathbb{S}^{\vee} \times \mathbb{S}^{\vee}$ plane.

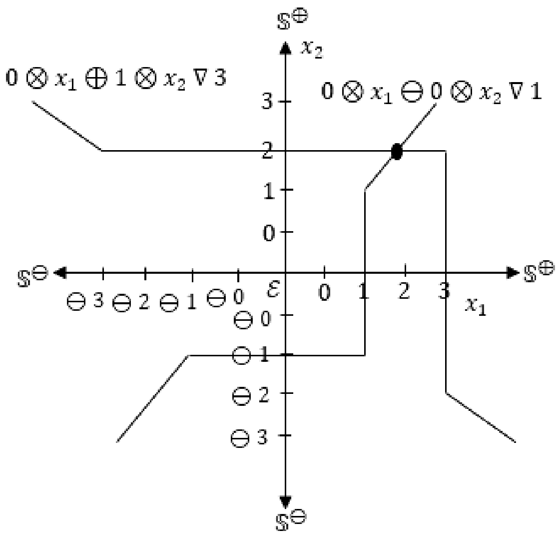

Fig. 1. Geometric interpretation of solution of system of linear balances (13) in two-dimensional plane $\mathbb{S}^{\vee} \times \mathbb{S}^{\vee}$.

If the second linear balance in (13) is replaced with its "parallel" given by $0 \otimes x_1 \ominus 0 \otimes x_2 \,\nabla\, 3$ then a new system of linear balances is obtained as follows:

$$\begin{gathered} 0 \otimes x_1 \oplus 1 \otimes x_2 \,\nabla\, 3 \\ 0 \otimes x_1 \ominus 0 \otimes x_2 \,\nabla\, 3. \end{gathered} \tag{14}$$

The vector $x = \begin{bmatrix} 3 \\ 2^\bullet \end{bmatrix}$ is a solution of system of linear balances in (14). However, determining a balanced solution in $\mathbb{S}^\bullet$ directly using the weak substitution property in balance relation is not feasible because x is not a signed vector.

Fig. 2 shows a geometric interpretation of the signed solutions for the system of linear balances in (14). The vector $\begin{bmatrix} 3 \\ 2^\bullet \end{bmatrix}$ is represented as the intersection point at coordinates $(x_1, x_2) = (3, 2^\bullet)$ in the $\mathbb{S}^\vee \times \mathbb{S}^\vee$ plane.

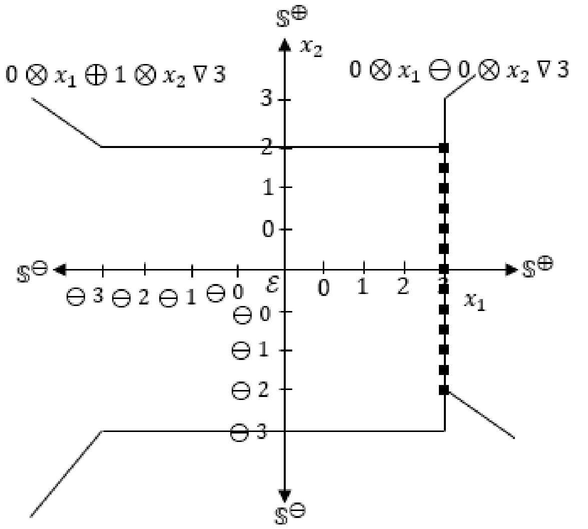

Fig. 2. Geometric interpretation of solution of system of linear balances (14) in two-dimensional plane $\mathbb{S}^\vee \times \mathbb{S}^\vee$.

In this context, $x_2 = 2^\bullet$ is a balanced element and includes all element $x_2' \in \mathbb{S}^\vee$ that are balanced with $2^\bullet$ specifically ranging from $x_2' = \ominus 2$ until $x_2' = 2$. Furthermore, a point $(3, 2^\bullet)$ is represented by all points where $x_1 = 3$ and $x_2 = 2^\bullet$ that is from $\ominus$ 2 until 2.

V. Application of System of Linear Balances

The solution of system of linear balances in symmetrized max-plus algebra can be applied to the reachability analysis of linear system over max-plus algebra $\mathbb{R}_{\max}$. Given an initial state $X(0) \in \mathbb{R}_{\max}^n$ and a state X, the tasks include determining an input vector U_q that drives the system state from $X(0)$ to $X(q) = X$. This is mathematically equivalent to determining an input vector U_q which satisfies the following relation:

$$X = A^q \otimes X(0) \oplus \mathcal{J}_q \otimes U_q \tag{15}$$

where $\mathcal{J}_q$ represents the reachability matrix of linear system. In case of max-plus algebra,

$$A^q \otimes X(0) \oplus \mathcal{J}_q \otimes U_q$$

cannot be equal to states that are less than the unforced terminal state $A^q \otimes X(0)$ due to the properties of max operation. Additionally, the framework is not possible to independently control all components except for a small class of systems.

Since $\mathbb{R}_{\max}$ is a special case of $\mathbb{S}$, the problem described in equation (15) can be viewed as a system of linear balances in $\mathbb{S}$. Let the balance relation be expressed as

$$X \,\nabla\, A^q \otimes X(0) \oplus \mathcal{J}_q \otimes U_q$$

then it is obtained

$$\mathcal{J}_q \otimes U_q \,\nabla\, X \ominus \left(A^q \otimes X(0)\right). \tag{16}$$

The problem of determining the input vector U_q that satisfies equation (16) is similar to solving system of linear balances, where the coefficient matrix is $\mathcal{J}_q$ and the constant vector is $X \ominus \left(A^q \otimes X(0)\right)$. By identifying minor rank of the coefficient matrix $\mathcal{J}_q$, the solution for U_q can be obtained. Consequently, an input vector U_q is derived, which drives the system state from $X(0)$ to $X(q) \,\nabla\, X$. If all entries of Equation (14) consist of signed elements, the reduction of balance properties can be used to simplify "balance" relation into an "equal" sense.

VI. Conclusion

In conclusion, the solution of the system of linear balances $A \otimes x \nabla b$ can be determined by analyzing minor rank of A. Given minor rank of A as r, and A_1 representing an $r \times r$ matrix corresponding to this minor rank, then A can be characterized into partition matrix according its minor rank. Specifically, the partition is $A = [A_1 \quad A_2] \otimes Q^T$, when A is full-row rank, $A = P^T \otimes \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ when A is full-column rank, and $A = P^T \otimes \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \otimes Q^T$ when it is neither. Then, the existence of balanced inverse of A_1 is used to construct solution of system of linear balances $A \otimes x \nabla b$. For a full-row rank A, the solution is given by

$$x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes b \\ \varepsilon \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y.$$

In the case of a full-column rank A the solution simplifies to $x = A_{1\nabla}^{-1} \otimes P_1 \otimes b$. When A is neither full-column rank or full-row rank, the solution becomes

$$x = Q \otimes \begin{bmatrix} A_{1\nabla}^{-1} \otimes P_1 \otimes b \\ \varepsilon \end{bmatrix} \oplus Q \otimes \begin{bmatrix} \ominus F \\ I_{n-r} \end{bmatrix} \otimes y.$$

If the solutions are signed vector, then all vectors balanced with the solution also qualify. Geometrically, the solution vector is interpreted as the intersection of all linear balances in system of linear balances.

Further studies can be carried out on the application of system of linear balances in reachability and observability of linear system over symmetrized max-plus algebra. Additionally, the system of linear balances can be extended to complex sets over symmetrized max-plus algebra.

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