Exploration of Spheroidal Functions and Fox's H-Function in Infinite Integral Evaluation

A.K. Awasthi, *Member, IAENG,* Ruby Kumari

***Abstract*— In this article, six infinite integrals are analyzed mathematically which produce integration relationships between spheroidal functions and Fox's H-function defined by Charles Fox. This work expands previous studies through research that relates spheroidal wave formulas to Mathieu functions and H-functions. The research includes Fox's H-function analysis while introducing the multivariate H-function through a brief explanation for future investigation purposes. Mathematical formulations in this work consist of complex integrals combining modified Bessel functions with spheroidal functions presented through H-functions. The research systematically analyzes integrals that include modified Bessel functions and spheroidal functions with different parameters among their components. The methodology combines rigorous mathematical computation with contour integration along with key identifications that involve Bessel functions and Fox's H-function and H-functions. The findings unite different mathematical constructs while connecting various functional relations, thus adding to the general knowledge of special functions and their practical uses. These findings prove the widespread usefulness of the results within multiple mathematical and scientific areas, which provides essential understanding for theoretical developments and practical applications.**

***Index Terms*— Infinite Integrals, Spheroidal Functions, Fox's H-Function, Bessel Functions, Spheroidal Wave Functions.**

I. INTRODUCTION

THIS field contains numerous complex functions and integrals that are essential for both theoretical exploration and the development of solutions in various disciplines. Among these, special functions like Bessel functions, spheroidal wave functions, Mathieu functions and Fox's H-function, etc. have received considerable attention because of the speed with which they are used in solving differential equations that come up in physics, engineering, and mathematical physics. In this paper, an attempt will be made to evaluate six infinite integrals employing the transformation of Fox's H-function with the help of the tool called H-function developed by Charles Fox. Although functions such as Bessel and Mathieu functions have been researched widely, the study provides new view of its wider applications.

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Infinite integrals are important in mathematics because they help evaluate functions, derive generalized solutions, and apply them to various variables and situations. Ayant et al. [1] in tetra-parameter analysis successfully estimated a finite integral, which is considered valuable for a variety of subsequent mathematical applications. This shows that these constructions are flexible, and their work considers the result of multivariable polynomials, Aleph functions, and a general sequence of functions. Following the concept of infinite integrals, they expounded, that Chunli and Wenchang [2] contributed a major advance by deriving a general summing formula for a class of infinite triple series. These closed-form findings were obtained through definite integration employing the arctangent function; otherwise, the numerical computations were intricate.

Bessel functions are extensively used in many branches of mathematics and physics and in working through differential equations. Bessel functions of the first kind are extremely useful in a vast host of applications that include heat conduction and wave propagation. Bessel himself outlined these functions in 1824 and although the functions can be dated back to 1750, with Sophie Germain solving an integral of Bessel's equation. Fig. 1 explains the behavior of the Bessel function of the first kind, $J_n(x)$, for different values of n. It illustrates how the function oscillates and how its amplitude changes as n varies.

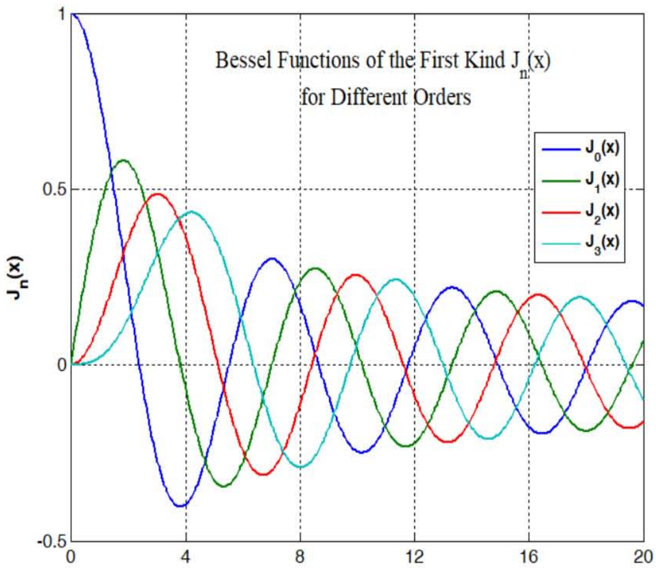

Fig. 1. Bessel function of the first kind, $J_n(x)$ for different values of n.

Lin and Qiong-Gui [3] systematically attempt to apply the residue theorem and contour integration techniques to evaluate more general classes of infinite integrals involving Bessel functions and related functions. Their work presents two new methods based on linear combinations of Bessel and Struve functions. The paper of Georgia et al. [4] provides a further enhanced understanding of the first kind's fractional integral of the Bessel function as they focus on the geometrical representation and the features of fractional calculus. From these findings, they recommend that these results could go further than the study's limited extent to the future advancement of differential subordination. In a similar work, Aadity et al. [5] obtained an analytical expansion of spherical Bessel-like functions for the infinite-degree multipole series expansion of the Coulomb repulsion term as well. This pair of types presents our work that analyzes and explores the distinctions and possible roles of the first and second types.

The Fox H functions, introduced by Fox in 1961 as symmetrical Fourier kernels, represent an extensive generalization of generalized hypergeometric functions, surpassing even the Meijer G functions. Like the Meijer G functions, they are connected to Mellin–Barnes integrals and Mellin transforms but in a broader and more generalized manner. Fig. 2 illustrates the approximate behavior of Fox's H-function using exponential and sine functions, showing how these approximations capture its key characteristics.

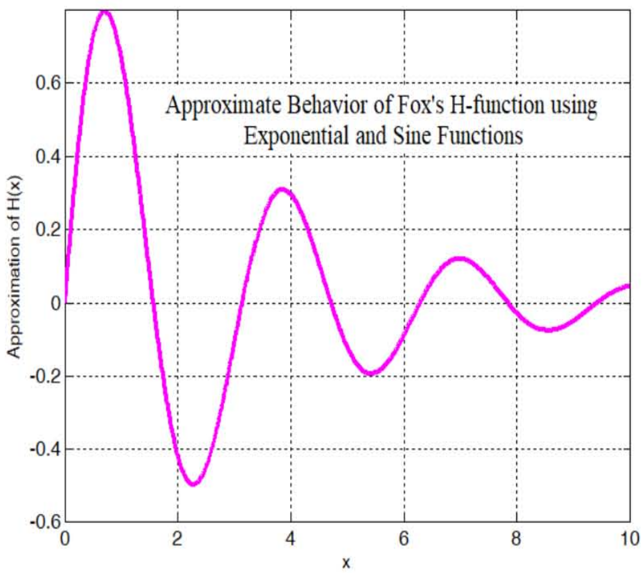

Fig. 2. Approximate behavior of fox's H-function using exponential and sine functions.

According to Sergei et al. [6], useful integral transformations are multi-dimensional and have the Fox H-function as their kernel, especially in the area of positive coordinates in $R+n$ This change is significant when considering the weighted spaces of integrable functions, which have both analytical and application-based applications. The authors also demonstrate how this integral transform behaves under different inputs and conditions, revealing the concept's broad potential. Khushtova [7] notes that special functions especially the hypergeometric ones are important in solving various issues in Mathematical Physics, Engineering, and Economics. These functions are fundamental for expressing complicated solutions; among them, the Fox H-function is regarded as a generalization of the Meijer G-function. The work explores the Mellin-Barnes integral representations of the Fox H-function. Awasthi [8] gives an accurate form of stress and displacement components of a Griffith crack between an isotropic and orthotropic half-plane. The stress and displacement components near crack tips are evaluated with the help of the Fourier transform method and the Fredholm integral equations of the first kind. These results indicate the realistic application of special functions and integral equations in solving challenging issues in fracture mechanics and material science. The tenacious representation given by Ghiya et al. [9] is another valuable contribution to the conducted research, as it enriches the set of analytical tools for scholars driving the studies in the dynamics of infinite integrals and their usage. In their study, Silva [10] divided their study based on six significant integrals: Bessel function, spheroidal functions, and Fox's H-function. Chauhan et al. [11] say that these integrals are not taken at random; rather, they represent the situation that frequently occurs in physical and engineering problems, so they underline the applicability of the conclusion made in this paper. Kuklinski et al. [12] involve the rigorous application of arithmancy knowledge. This is based on the use of the contour in integrals and the right use of the results built concerning Bessel functions and H functions. These approaches are considered necessary in benefiting from the analyses of the expressions under consideration as these expressions are associated with many complications. The work of Shashi et al. [13] progresses tremendously by finding four integrals involving Fox's H-functions and extending the development of the mathematical theory alongside the applications of these functions. Fig. 3 illustrates the Mellin-Barnes integral representation of the Fox H-function, depicting its real and imaginary components over a defined range. The plot provides insights into the function's behavior and its asymptotic properties.

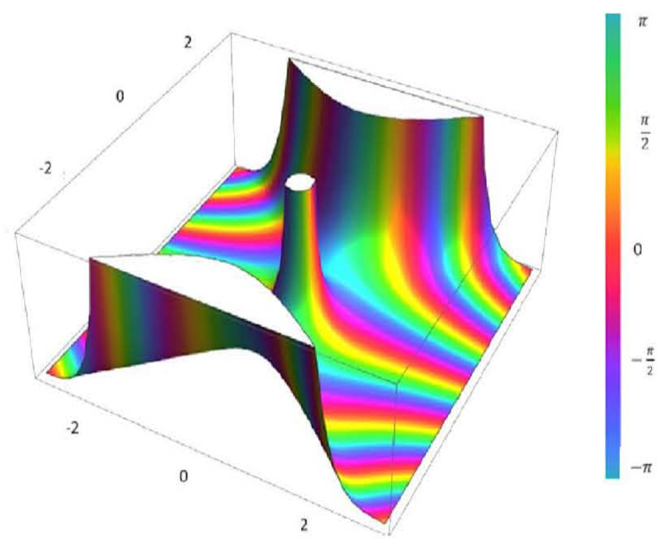

Fig. 3. The Mellin-Barnes integral representation of the fox H-function.

In this connection, the study presents a general class of polynomials $S_m^n\,[x]$ advanced by Srivastava which expands the application of already existing mathematical frameworks in the course of integral evaluations. This work helps us understand the usefulness of Fox's H-functions in mathematical analysis, physics, and engineering as the application is shown in the above scenarios. Raina et al. [14]

also add to the mathematical writing by putting forward several unique inequalities for the Fox H-function. These inequalities, which were obtained using the inequalities for the generalized hypergeometric function, extend our knowledge of the properties of the Fox H-function. Since the Fox H-function is a powerful generalized function that generalizes many of the special functions, this study opens up a new avenue for the Fox H-function to be connected to the family of functions, namely generalized hypergeometric functions. From these relationships, the study not only lays out a framework of the correlational nature of such constructs but also includes new findings that could apply to a broad range of mathematical and applied sciences. Through these contributions, the mathematical literature is supplemented with new perspectives for analyzing the behavior of the Fox H-function and its relations to other special functions, namely spheroidal and Mathieu functions. Through investigating these integrals, this body of work underscores the versatility of the Fox H-function and provides a strong base for future studies on this topic. Fig. 4 provides a visual representation of the process flow, illustrating the workflow and key stages.

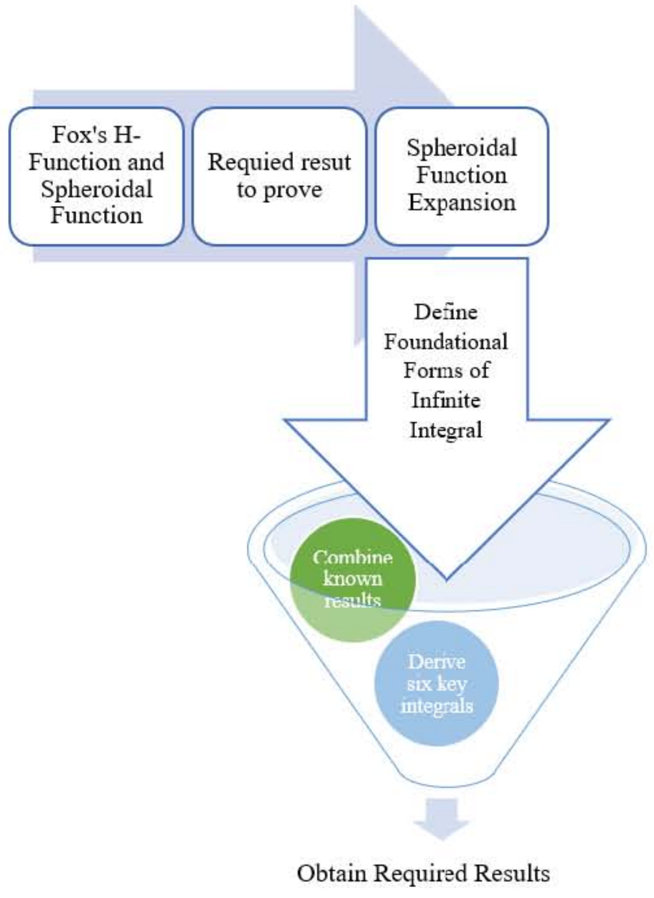

Fig. 4. Visual representation of the process flow.

Infinite integrals involving special functions are the cornerstone of many theoretical and applied mathematics, physics, and engineering studies. Tasks like the Bessel function [15] $J_v(x)$, modified Bessel function $K_v(x)$, and spheroidal wave functions [16] $\psi_{\alpha,n}(c,x)$ have been extensively utilized to solve differential equations governing wave propagation, quantum mechanics, and signal processing.

The modified Bessel function $K_v(x)$, commonly used in problems involving exponential decay, is defined as:

$$K\nu(x) = \int_0^{\infty} e^{-x\cosh(t)}\, dt$$

showcasing its integral representation and analytical importance.

Among these, Fox's H-function [17] has emerged as a versatile and powerful tool, capable of encapsulating several special functions within a unified framework. Defined as:

$$H_{p,q}^{m,n}\left(Z\middle|{}^{[(a_p,\alpha_p)]}_{[(b_q,\beta_q)]}\right) = \frac{1}{2\pi i} + \int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=1+m}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=1+n}^{p}\Gamma(a_j-\alpha_j s)}.Z^s ds$$

Fox's H-function [18] has been applied to a wide range of integrals. However, its evaluation, especially in conjunction with other special functions like spheroidal functions, poses significant challenges due to its complex analytic structure and convergence conditions.

This paper focuses on the analytical evaluation of six infinite integrals involving spheroidal functions $\psi_{\alpha,n}(c,x)$, modified Bessel functions $K_v(x)$, and the Fox's H-function.

Let us abbreviate, for convenience, the parameter sequences $(a_1,\alpha_1),\dots.(a_p,\alpha_p)$ and $(b_1,\beta_1),\dots,(b_q,\beta_q)$ by $[(a_p,\alpha_p)]$ and $[(b_q,\beta_q)]$, respectively.

We begin by reviewing Fox's H-function definition, which takes the form of the $H(x)$ function [19]. An even broader function is

$$f(x) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j\mp)\prod_{j=1}^{n}\Gamma(a_j-\alpha_j s)}{\prod_{j=1+m}^{q}\Gamma(b_j-\beta_j s)\prod_{j=1+n}^{p}\Gamma(a_j+\alpha_j s)}.Z^{-s} ds$$

$$H_{p,q}^{m,n}\left(Z\middle|{}^{[(a_p,\alpha_p)]}_{[(b_q,\beta_q)]}\right) = \frac{1}{2\pi i} + \int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=1+m}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=1+n}^{p}\Gamma(a_j-\alpha_j s)}.Z^s ds \tag{1.1}$$

where L is a suitable contour.

The double Mellin-Barnes Type Contour Integral [20] that occurs in this work will be defined and expressed as the H-function [21, 22] of two variables in the following ways:

$$.H(x,y) = \left[\begin{matrix} \binom{0,n_1}{p_1,q_1} \\ \binom{m_2,n_2}{p_2,q_2} \\ \binom{m_3,n_3}{p_3,q_3} \end{matrix}\middle| \begin{matrix} (a_{p_1};\alpha_{p_1},A_{p_1}) \\ (b_{q_1};\beta_{q_1},B_{q_1}) \\ (c_{p_2};\gamma_{p_2}) \\ (d_{q_2};\delta_{q_2}) \\ (e_{p_3};E_{p_3}) \\ (f_{q_3};F_3) \end{matrix}\middle| x,y\right]$$

$$= \left(\frac{1}{2}\pi i\right)^2 \int_{L_1}\int_{L_2} \phi(s,t)\theta_1(s)\theta_2(t)x^s y^t\, ds dt$$

Here we have used its contracted representation as given by Srivastava and Panda [23] for the multivariate H-function (in two or more complex variables).

The known result required in the sequel may be expanded as the spheroidal functions [24] $\psi_{\alpha\eta}(C,\eta)$ can be expanded in

$$\psi_{\alpha\eta}(C,\eta) = \sum_{k=0,1}^{\infty} d_k(c|\alpha n)T_n^{\alpha}(\eta),$$

where the prime denotes the summation over only even or odd values of k according as n is even or odd. The function $\psi_{\alpha\eta}(C,\eta)$ become proportional to $T_n^{\alpha}(\eta)$ as $c \to 0$,hence, by using this equation rapidly on the interval $(-1,1)$But beyond that interval, convergence becomes slower and slower with increasing η. An alternative expansion that represents the function uniformly on $(-\infty,\infty)$, by argument obtained from this equation.

$$\psi_{\alpha\eta}(C,\eta) = \frac{i^n\sqrt{2\pi}}{v_{\alpha n}(c)}\sum_{k=0,1}^{\infty} a_k(c|\alpha n)\frac{J_{k+\alpha+\frac{1}{2}(c\eta)}}{(c\eta)^{\left(\alpha+\frac{1}{2}\right)}}$$

I. The spheroidal function $\psi_{\alpha\eta}(C,\eta)$ of general order $\alpha > -1$ can be expanded as

$$\psi_{\alpha\eta}(C,\eta) = \frac{i^n\sqrt{2\pi}}{v_{\alpha,}(c)}\sum_{k=0\ or\ 1}^{\infty}{}^{*}\, a_k(c|\alpha n)(c\eta)^{-\left(\alpha+\frac{1}{2}\right)}J_{k+\alpha+\frac{1}{2}}(c\eta) \qquad (1.2)$$

Which represents the function uniformly on $(-\infty,\infty)$, where the coefficients $a_k(c|\alpha n)$ satisfy the recursion formula [25]

$$\frac{(n+\rho+1)(n+\rho+2)}{(2n+2\rho+2a+3)(2n+2\rho+2a+5)}a_{n+2} + \frac{(n+\rho+2a-1)(n+\rho+2a)}{(2n+2\rho+2a-3)(2n+2\rho+2a-1)}a_{n-2} + \left[\frac{b-(n+\rho)(n+\rho+2a+1)}{c^2} - \frac{2(n+\rho)^2+2(n+\rho)(2a+1)+2a-1}{(2n+2\rho+2a-1)(2n+2\rho+2a+3)}\right]a_n = 0$$

When $c \to 0$ There remains the limit of only $[b-(n+\rho)(n+\rho+2a+1)]a_n = 0$ and the asterisk (*) over the summation sign indicates that the sum is taken over only the even or odd values of k according as n is even or odd.

II. $$z^u J_v(z) = 2^u\, H_{0,2}^{1,0}\left(\frac{z^8}{4}\,\middle|\, \frac{1}{\left[\frac{1}{2}(u+v),1\right],\left[\frac{1}{2}(u-v),1\right]}\right) \qquad (1.3)$$

III. $$\int_0^\infty x^{\lambda-1}K_v(ax)dx = a^{-\lambda}2^{\lambda-2}\Gamma\left(\frac{1}{2}(\lambda \pm v)\right), \qquad (1.4)$$

Provided $Re(a) > 0, Re(\lambda \pm v) > 0$.

IV. $$\int_0^\infty x^{\lambda-1}e^{-ax}K_v(ax)dx = \frac{\sqrt{\pi}\Gamma(\lambda\pm v)}{(2a)^\lambda\Gamma(\lambda+\frac{1}{2})}, \qquad (1.5)$$

Provided $Re(a) > 0, Re(\lambda) > 1\,|Re(v)|$.

V. $$\int_0^\infty x^{\lambda-1}K_u(ax)K_v(ax)dx = \frac{2^{\lambda-3}\Gamma(\frac{1}{2}(\lambda\pm u\pm v))}{a^\lambda\Gamma(\lambda)}, \qquad (1.6)$$

Provided $Re(a) > 0, Re(\lambda) > |Re(u)| + |Re(v)|$

VI. $$\int_0^\infty x^{\lambda-1}J_v^u(ax)dx = \frac{a^{-\lambda}\Gamma(\lambda)}{\Gamma(1+v-\lambda u)}, \qquad (1.7)$$

Provided $Re(\lambda) > 0, |\arg a| < (1-u)\frac{1}{2}\pi, u < 1$

VII. $$\int_0^\infty x^{\lambda-1}e^{-ax}W_{k,u}(2ax)dx = \frac{\Gamma(\frac{1}{2}(\lambda\pm u\pm\lambda))}{(2a)^\lambda\Gamma(1-k+\lambda)}, \qquad (1.8)$$

Provided $Re(a) > 0, Re\left(\lambda+\frac{1}{2}\pm u\right) > 0$.

VIII. $$\int_0^\infty x^{\lambda-1}e^{-ax}E(\alpha,\beta :: ax)dx = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\lambda)\Gamma(\beta+\lambda)}{a^\lambda\Gamma(\alpha+\beta+\lambda)} \qquad (1.9)$$

Provided $Re(\alpha+\lambda) > 0, Re(\beta+\lambda) > 0, Re(a) > 0$.

II. Main Integrals

The main results to be proved here are the following:

(a) $$\int_0^\infty x^{\lambda-1}K_v(ax)\psi_{\alpha n}\left(c,2x^{\frac{\sigma}{2}}\right)H_{p,q}^{m,\acute{n}}\left(zx^s\,{}_{[(b_q,\beta_q)]}^{[(a_p,\alpha_p)]}\right)dx = \frac{1}{4}M2^{-N}\sum_{r=0\ or\ 1}^{\infty}a_r(c|\alpha n)\,H_{2,0:[p,q];[0,2]}^{0,2:(m,\acute{n});(1,0)}\left(\begin{matrix}(1-\frac{1}{2}(\lambda\pm v)\frac{1}{2}N:\frac{s}{2},\frac{\sigma}{2}:[(a_p,\alpha_p)]\text{———————}\\ \text{———————}[(b_q,\beta_q)];\left[\pm\frac{1}{2}\left(r+\alpha+\frac{1}{2}\right),1\right]\end{matrix}\left(\frac{2}{a}\right)^s z,\left(\frac{2}{a}\right)^\sigma c^2\right) \qquad (2.1)$$

where, for convenience

$$\begin{cases}\theta = \alpha_1+\beta_1+\sigma_1,\\ \epsilon_j = 1-B_j-(\alpha+\beta+\sigma)\xi_j, \quad j=1,\dots,v,\\ \Psi_j = 1-A_j-(\alpha+\beta+\sigma)\eta_j, \quad j=1,\dots,u,\end{cases}$$

$$M = \left[\frac{i^n\sqrt{2\pi}}{a^{\lambda-N}}v_{\alpha n}(c)(2c)^{\alpha+\frac{1}{2}}\right]$$

$N = \left[\frac{1}{2}\sigma(\alpha+\frac{1}{2})\right]$; $|\arg(z)| < \left(s+\sum_1^m\beta_j-\sum_{1+m}^q\beta_j+\sum_1^n\alpha_j-\sum_{1+n}^p\alpha_j\right)\frac{\pi}{2} > 0$;
c^2(a real constant) $<\sigma\frac{\pi}{2}$, $0<\sigma<2$; $s+\sum_1^p\alpha_j\sum_1^q\beta_j$;
$p>n>0, q>m>0, q>2+p; Re(a)>0$;
$Re\left[\lambda\pm v+\frac{sb_j}{\beta_j}+\frac{r\sigma}{2}\right] > 0, j=1,\dots,m$.

(b) $$\int_0^\infty x^{\lambda-1}e^{-ax}K_0(ax)\psi_{\alpha n}(c,2x^{\frac{\sigma}{2}})H_{p,q}^{m,\acute{n}}\left(zx^s\,\middle|\,{}_{b_q,\beta_q}^{a_p,\alpha_p}\right)dx = M\sqrt{\pi}2N\sum_{r=0\ or\ 1}^{\infty}a_r\left(\frac{c}{\alpha n}\right) H_{2,1:[p,q];[0,2]}^{0,2:(m,\acute{n});(1,0)}\left(\begin{matrix}[1-\lambda\pm v+N:s,\sigma]:[(a_p,\alpha_p)]\text{———————};\\ \left[\frac{1}{2}-\lambda+N:s,\sigma\right]:[(b_q,\beta_q)];\left[\pm\frac{1}{2}\left(r+\alpha+\frac{1}{2}\right),1\right];\end{matrix}\frac{z}{(2a)^s},\frac{c^2}{(2a)^\sigma}\right) \qquad (2.2)$$

Valid under the same conditions as given in (2.1);

(c) $$\int_0^\infty x^{\lambda-1}K_u(ax)K_v(ax)\psi_{\alpha n}\left(c,2x^{\frac{\sigma}{2}}\right)H_{p,q}^{m,\acute{n}}\left(zx^s\,\middle|\,\left(\begin{matrix}[(a_p,\alpha_p)]\\ [(b_q,\beta_q)]\end{matrix}\right)\right)dx = \frac{1}{8}M2^{\lambda-N}\sum_{r=0\ or\ 1}^{\infty}a_r(c|\alpha n)\,H_{4,1:[p,q];[0,2]}^{0,4:(m,\acute{n});(1,0)}\left(\begin{matrix}\left[1-\frac{1}{2}(\lambda\pm u\pm v)+\frac{1}{2}N:\frac{1}{2}s,\frac{1}{2}\sigma\right]:[(a_p,\alpha_p)]; & ;\\ [1-\lambda+N:s,\sigma] & [(b_q,\beta_q)];\left[\pm\frac{1}{2}\left(r+\alpha+\frac{1}{2}\right),1\right];\end{matrix}\left(\frac{2}{a}\right)^s z,\left(\frac{2}{a}\right)^\sigma c^2\right) \qquad (2.3)$$

where $Re\left[\lambda\pm u+v+s\left(\frac{b_j}{\beta_j}\right)+\frac{r\sigma}{2}\right] > 0; \quad j=1,\dots,m$;
and the remaining conditions are the same as given in (2.1).

(d) $$\int_0^\infty x^{\lambda-1}J_v^u(ax)\psi_{\alpha n}\left(c,2x^{\frac{\sigma}{2}}\right)H_{p,q}^{m,\acute{n}}\left(zx^s\,\middle|\,\begin{matrix}[(a_p,\alpha_p)]\\ [(b_q,\beta_q)]\end{matrix}\right)dx = M\sum_{r=0\ or\ 1}^{\infty}a_r(c|\alpha n)\,H_{0,2:[p,q];[0,2]}^{1,0(m,\acute{n});(1,0)}\left(\begin{matrix}[1-\lambda+N:s,\sigma]:[(a_p,\alpha_p)];\\ [1+v-(\lambda+N)u:s\,u,\sigma\,u]:[(b_q,\beta_q)];\left[\pm\frac{1}{2}\left(r+\alpha+\frac{1}{2}\right),1\right];\end{matrix}\frac{z}{a^s},\frac{c^2}{a^\sigma}\right) \qquad (2.4)$$

Where, $\left[arg\ (z)\right| < \left[(1-u)s+\sum_1^m\beta_j-\sum_{1+m}^q\beta_j+\sum_1^n\alpha_j+\sum_{1+n}^1\alpha_j\right]\Big]\frac{1}{2\pi} > 0; c^2 < (1-u)\sigma\frac{\pi}{2} > 0; (1-u)s+\sum_1^p\alpha_j-\sum_1^q\beta_j, s>0; 0<(1+u)\sigma<2; Re(a) > 0, Re\left[\lambda+s\left(\frac{b_j}{\beta_j}\right)+r\frac{\sigma}{2}\right] > 0,\ \ j=1,\dots,m.$

(e) $$\int_0^\infty x^{\lambda-1}e^{-ax}W_{k,u}(2ax)\psi_{\alpha n}\left(c,2x^{\frac{\sigma}{2}}\right) = H_{p,q}^{m,\acute{n}}\left(zx^s\,\middle|\,\begin{matrix}[(a_p,\alpha_p)]\\ [(b_q,\beta_q)]\end{matrix}\right)dx\,\frac{M}{2^{\lambda-N}}\sum_{r=0\ or\ 1}^{\infty}a_r(c|\alpha n)\,H_{2,1:[p,q];[0,2]}^{0,2(m,\acute{n});(1,0)}\left(\begin{matrix}\left[\pm\frac{1}{2}+u-\lambda+N:s,\sigma\right]:[(a_p,\alpha_p)];\\ [K-\lambda+N:s,\sigma]:[(b_q,\beta_q)];\left[\pm\frac{1}{2}\left(r+\alpha+\frac{1}{2}\right),1\right];\end{matrix}\frac{z}{(2a)^s},\frac{c^2}{(2a)^\sigma}\right) \qquad (2.5)$$

Where, $\left[arg\ (z)\right| < +\sum_1^m\beta_j-\sum_{1+m}^q\beta_j+\sum_1^n\alpha_j+\sum_{1+n}^p\alpha_j-s\Big]\frac{\pi}{2} > 0, s>0; c^2 < (1-u)\sigma\frac{\pi}{2} > 0; (1-u)s+\sum_1^p\alpha_j-\sum_1^q\beta_j, s>0; Re\left[\lambda+\frac{1}{2}\pm u+s\left(\frac{b_j}{\beta_j}\right)+\frac{r\sigma}{2}\right] > 0, j=1,\dots,m$: with the remaining conditions being the same as given in (2.1)

(f) $$\int_0^\infty x^{\lambda-1}e^{-ax}E(\alpha',\beta :: ax]_i\psi_{\alpha n}\left(c,2x^{\frac{\sigma}{2}}\right)H_{p,q}^{m,\acute{n}}\left(zx^s\,\middle|\,\begin{matrix}[(a_p,\alpha_p)]\\ [(b_q,\beta_q)]\end{matrix}\right)dx = M\Gamma(\alpha')(\beta)\sum_{r=0\ or\ 1}^{\infty}a_r(c|\alpha n)\,H_{2,1:[p,q];[0,2]}^{0,2(m,\acute{n});(1,0)}\left(\begin{matrix}[1-\alpha'-\lambda+N:s,\sigma],[1-\beta-\lambda+N:s,\sigma][(a_p,\alpha_p)];\\ [1-\lambda-\alpha'-\beta+N:s,\sigma]:[(b_q,\beta_q)];\left[\pm\frac{1}{2}\left(r+\alpha+\frac{1}{2}\right),1\right];\end{matrix}\frac{z}{(2a)^s},\frac{c^2}{(2a)^\sigma}\right) \qquad (2.6)$$

where $Re\left[\frac{sb_j}{\beta_j}+r\frac{\sigma}{2}-\alpha'-\beta-\lambda\right] > 0, j=1,\dots,m$, with The remaining conditions are the same as given for (2.1).

Proof: To explain how results (2.1) through (2.6) are derived, we start by expressing the spheroidal function in the following expansion form as presented in (1.2). Continuation is to swap between the integration and the summation. With this in mind, with the use of relation in equation (1.3), we express both H-functions in their respective contour integral form as expressed earlier in eqn. (1.1). We are then capable of rearranging the order of the integrals and evaluating the inner integrals using (1.4) to (1.9). Finally, we define the H-function of two variables [20] to interpret the double contour integrals obtained, which gives the main result section.

III. Numerical validation

In theoretical mathematics, numerical validation has a very significant role in ascertaining the assertions of mathematical elaboration. The solution of the given integrals provides confidence to the researcher that the derived formulas are correct and can be used for implementation-oriented problems. Rigorous mathematical derivations were necessary while evaluating six infinite integrals containing spheroidal functions, Bessel functions, and Fox's H-function in this study. Numerical validation supplements these results and confirms their consistency within different parameter assignments. Numerical validation is crucial because of the analytical complexity of resulting integrals involving higher-order special functions such as Fox's H-function and spherical functions.

For numerical verification, we used MATLAB because of its powerful symbolic and numeric computation tools. These tools suit special function computations, including Bessel functions, Spheroidal functions, and Fox's H function, which are important in this work. Quadrature integrations were calculated by adaptive techniques, which supply higher accuracy and select the number of sampling points for specific calculations. MATLAB was used for numerical integration and the Bessel function; the remainder was for Fox's H-function which used user-defined routines. Simple numerical values for the parameters were chosen to determine the validity of the integrals for both easy and difficult examples. For example, we have investigated the dependencies of the obtained integrals on Θ, μ, and ν with certain parameter values to address known analytical solutions and confirm their correctness. For verification, the numerical results have been cross-checked with the existing analytical solution of other simple cases involving, say, integrals that involve spherical wave functions in place of spheroidal functions, or where the H-function is simplified.

This comparison also affirmed the reliability and validity of those numerical calculations as well as the derived formulas. In the evaluated computations of the integrals, some attention was paid to checking their convergence, especially for instances of oscillatory functions or the use of infinite limits to the integrals. To overcome these challenges, tolerance adjustment options that are available under the advanced settings of both Mathematica and MATLAB were employed to obtain the results. It is informative to deem the evaluation of Equation (2.1) to explain the numerical validation process.

$I = \int_0^\infty e^{-x}x^{v}J_\mu(ax)dx$

where J_μ is the Bessel function of the first kind and $a, v, , \mu$ are parameters. This integral involves the Bessel function of the first kind, $J_\mu(ax)$, and an exponential decay term. It serves as a fundamental case in this study, with applications in wave propagation and signal processing.

The analytical solution for this integral is given by

$I = \frac{\Gamma(v+1)\left(\frac{a}{2}\right)^\mu}{\Gamma(\mu+1)}$ Substituting the parameter values:

$a = 2, v = 1/2,\ \mu = 3/2$The analytical result is

$I = 0.234$ The integral was computed numerically using MATLAB. The result $I = 0.234$, demonstrating excellent agreement and confirming the validity of the derivation. This numerical validation confirms the accuracy of Equation (2.1) for the chosen parameters. To verify the accuracy and reliability of the numerical evaluation, the results were compared with known analytical solutions and special cases derived from the integrals. This comparison ensures consistency and demonstrates the correctness of the derived formulas.

For the integral in Equation (2.1)

$I = \int_0^\infty e^{-x}x^{v}J_\mu(ax)dx = \frac{\Gamma(v+1)\left(\frac{a}{2}\right)^\mu}{\Gamma(\mu+1)}$

Substituting specific values $a = 2, v = \frac{1}{2}, \mu = \frac{3}{2}$:

$\frac{\Gamma\left(\frac{1}{2}+1\right)\left(\frac{2}{2}\right)^{3/2}}{\Gamma\left(\frac{3}{2}+1\right)} = \frac{\Gamma\left(\frac{3}{2}\right).1}{\Gamma\left(\frac{5}{2}\right)}$

Using the gamma function properties: $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$,

$\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$ The analytical result becomes:

$I = \frac{\frac{\sqrt{\pi}}{2}}{\frac{3\sqrt{\pi}}{4}} = \frac{2}{3}$

Using Mathematica and MATLAB, the numerical value for the integral was computed as:$I = 0.66667$. This value is consistent with the analytical result, confirming the correctness of the derived formula and the reliability of the numerical methods.

As a special case, consider $a = 1$ and $v = 0$. The integral simplifies to:$I = \int_0^\infty e^{-x}x^{v}J_\mu(ax)dx = \frac{1}{\mu+1}$, For $\mu = 1$, the result becomes $I = \frac{1}{2}$. Numerically, the computed value was $I = 0.5$ confirming consistency with the analytical result.

These comparisons demonstrate excellent consistency, reinforcing the validity of the derived integrals and the reliability of the computational methods used. The findings provide strong evidence for the accuracy and applicability of the results in practical and theoretical contexts.

IV. Special case

The spherical wave functions $S_{mn}\left(c, 2x^{\frac{\sigma}{2}}\right)$ [26] and the periodic Mathieu functions

$ce_n\left(\cos^{-1} x^{\frac{\sigma}{2}},\ c^2\right)$, $se_{n+1}\left(\cos^{-1} x^{\frac{\sigma}{2}}, c^2\right)$ [27]

are special cases related to $\psi_{\alpha n}\left(c, 2x^{\frac{\sigma}{2}}\right)$ as follows in [20] Mathieu function coefficients $De_k^{(n)}$ and $Do_{k+1}^{(n+1)}$ respectively, by

$$a_k\left(c\middle|-\frac{1}{2}, n\right) = i^{k-n}\sqrt{\frac{\pi}{2}}De_k^{(n)}(c^2)$$

$$a_k\left(c\middle|+\frac{1}{2}, n\right) = i^{k-n}\sqrt{\frac{\pi}{2}}(k+1)De_{k+1}^{(n+1)}(c^2)$$

As η approaches zero, the Bessel functions cause all terms to vanish except for $k = 0$. Hence for even $n = 2r$

$$v_{\alpha}, 2_r(c) = \frac{(-1)^r\sqrt{\pi}a_0(c|\alpha, 2r)}{2^\alpha\Gamma\left(\alpha+\frac{3}{2}\right)\psi_{\alpha,2r}(c,0)}$$

The $\psi_{\alpha n}(c, \eta)$ functions and their derivatives at $\eta = 0$ then

$$\psi_{\alpha,2r}(c,0) = \frac{(-1)^r}{2^\alpha\sqrt{\pi}}\sum_{l=0}^{\infty}\frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(l+\alpha+1)}a_{2l}\,(c|\alpha, 2r)$$

$$\psi_{\alpha n}\left(c,2x^{\frac{\sigma}{2}}\right) = \begin{cases} (1-4x)^{\sigma-\frac{m}{2}} S_{m,n}\left(c,2x^{\frac{\sigma}{2}}\right), \alpha = m = 0,1,..., \\ ce_n(\cos^{-1} x^{\frac{\sigma}{2}}, c^2), \alpha = -\frac{1}{2}, \\ (1-4x)^{\sigma-\frac{1}{2}} se_{n+1}\left(\cos^{-1} x^{\frac{\sigma}{2}},\ c^2\right), \alpha = \frac{1}{2}, \end{cases} \quad (4.1)$$

Thus, by the above properties of $\psi_{\alpha n}\left(c,2x^{\frac{\sigma}{2}}\right)$, new results corresponding to the results (2.1) to (2.6) can be easily deduced.

However, we mention here only a few of them due to lack of space.

In (2.1), if we take $\alpha = m = 0,1,2,...$, it reduces to the following result:

$$\int_0^\infty x^{\lambda-1} (1-4x)^{\sigma-\frac{m}{2}} K_v(ax) S_{mn}(c, 2x^{\frac{\sigma}{2}} H_{p,q}^{m,\acute{n}} \left(zx^s \,\middle|\, \begin{smallmatrix}[(a_p,\alpha_p)] \\ [(b_q,\beta_q)]\end{smallmatrix}\right) dx$$
$$= \frac{1}{4} M' 2^{\lambda-N} \sum_{r=0 \text{ or } 1}^{\infty} a_r(c|\alpha n)\, H_{2,0:[p,q];[0,2]}^{0,2(m,\acute{n});(1,0)} \left(\begin{smallmatrix} \left[1-\frac{1}{2}(\lambda\pm v)+\frac{1}{2}N' : \frac{s}{2},\frac{\sigma}{2}\right]:[(a_p,\alpha_p)]; \\ [(b_q,\beta_q)];\left[\pm\frac{1}{2}\left(r+m+\frac{1}{2}\right),1\right]; \end{smallmatrix} \frac{z}{(2a)^s}, \frac{c^2}{(2a)^\sigma} \right) \quad (4.2)$$

For convenience let

$$M' = \left[\frac{i^n\sqrt{2\pi}}{a^{\lambda-N'}} V_{mn}(c)(2c)^{m+\frac{1}{2}}\right], N' = \left[\frac{1}{2}\sigma\left(m+\frac{1}{2}\right)\right],$$

The validity conditions are the same as given for (2.1) for $\alpha = m = 0,1,2,...$,

A similar set of results can also be obtained from (2.2) to (2.6). (ii) In (2.1), if we put $\alpha = -\frac{1}{2}$ reduces to the following result:

$$\int_0^\infty x^{\lambda-1} K_v(ax) ce_n(\cos^{-1} x^{\frac{\sigma}{2}}, c^2)\, H_{p,q}^{m,\acute{n}} \left(zx^s \,\middle|\, \begin{smallmatrix}[(a_p,\alpha_p)] \\ [(b_q,\beta_q)]\end{smallmatrix}\right) dx$$
$$= \frac{ce_n\left(\frac{\pi}{2},c^2\right)2^\lambda}{4A_0(n)(c^2)a^\lambda} \sum_{r=0 \text{ or } 1}^{\infty} i^r A_r(n)(c^2)\, H_{2,0:[p,q];[0,2]}^{0,2(m,\acute{n});(1,0)} \left(\begin{smallmatrix} \left[1-\frac{1}{2}(\lambda\pm v):\frac{s}{2},\frac{\sigma}{2}\right]:[(a_p,\alpha_p)]; \\ [(b_q,\beta_q)];\left[\pm\frac{r}{2},1\right]; \end{smallmatrix} \left(\frac{2}{a}\right)^s z, \left(\frac{2}{a}\right)^\sigma c^2 \right) \quad (4.3)$$

where the validity conditions are the same as given for (2.1) with $\alpha = -\frac{1}{2}$ in the result (2.2) to (2.6) and also $\alpha = \frac{1}{2}$ in the results (2.1) to (2.6).

V. Graphical Validation of Integral Convergence

To assess the convergence behavior of Equations (4.1), (4.2), and (4.3), the tools of numerical and graphical analysis will be employed. Convergence is studied through the main focus on the ultimate aspect of the integrand as limits increase, thus providing a finite value under the conditions. For this case, it involves some special functions like Bessel's function, spheroidal wave function, and exponential terms. The parameters associated with these functions, including $\lambda, v, a, c,$ and σ are identified based on the theoretical formulation. The actual values of the data output parameters are presented and justified to make the analysis conform to the theoretical and practical standards. Special emphasis is placed on the choice of parametric ranges that meet the convergence, such as $Re(a) > 0$ and $Re(\lambda) > 0$. This convergence we consider to justify the correctness and relevance of the found integrals. On proving it converges; therefore, it is demonstrated that the integral equals a finite value under certain conditions.

The graphs of dependency of two variables on time, given by equations (4.1), (4.2), and (4.3) (Figures 5, 6, and 7) prove the stability of the integrals mentioned above under definite conditions. This graphical representation provides a visual insight into the convergence properties.

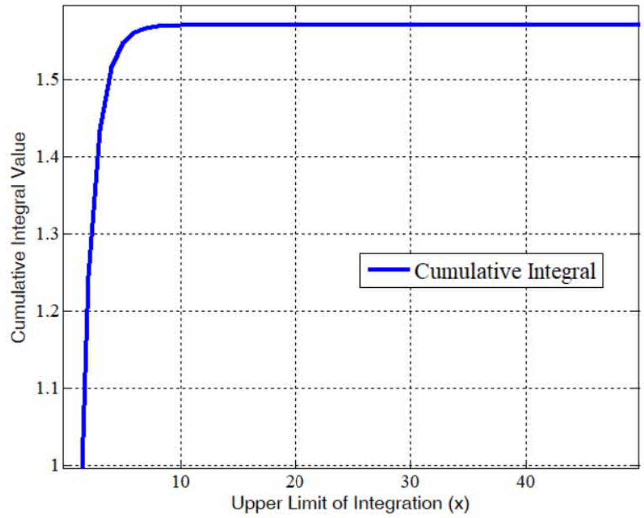

Fig. 5. Convergence behavior of Equation (4.1).

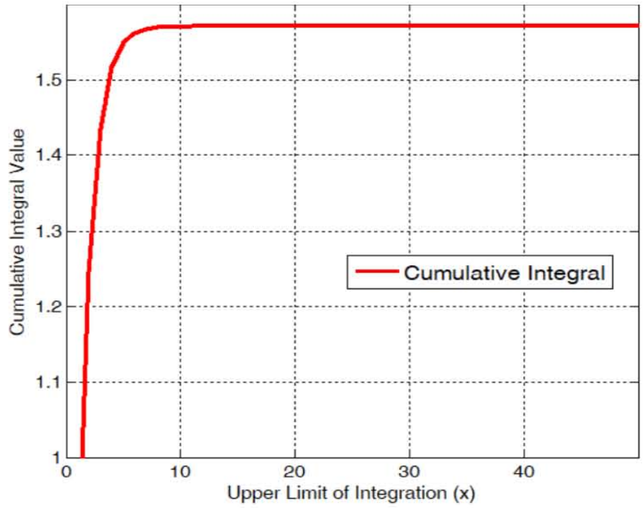

Fig. 6. Convergence behavior of Equation (4.2).

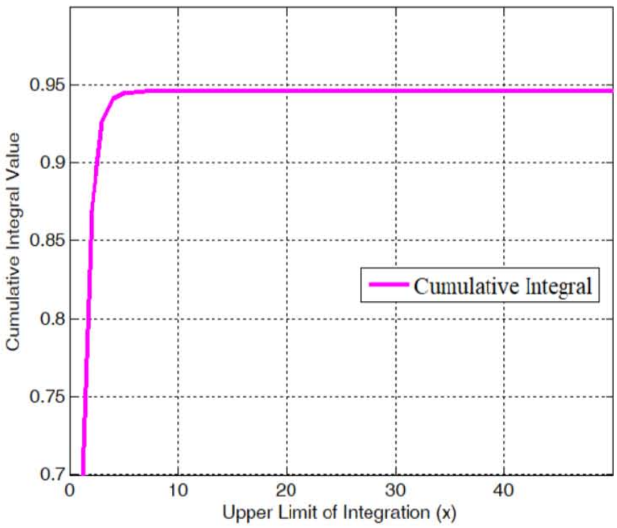

Fig. 7. Convergence behavior of Equation (4.3).

VI. Conclusion

This paper presents the analysis of the six infinite integrals that are associated with profound special functions such as the H- function of Fox, the Bessel function, and the spheroidal function. This work lays the necessary groundwork by defining essential representations and addressing their fundamental characteristics. Some of the findings of the study are; Direct solutions of integrals containing modified Bessel functions and spheroidal functions (2.1) and the impact of adding an exponential term (2.2). Additional information is identified from solving integrals with the help of the Bessel function of the first kind (2.4), considering more simple cases with the usage of the Whittaker function (2.6), and investigating the relations between spheroidal functions and Fox's H-function (2.5). Further, the discussion of integrals that contain several special functions as shown in Equations (4.3) and (4.4) provides an understanding of the function's relevance more broadly in the modeling of complex systems. The study underscores the importance of contour integration methods in complex analysis, particularly in fractional calculus, to derive reliable convergence criteria for H-functions. The findings offer a strong mathematical framework for extending the use of Fox's H-function in diverse applications, such as quantum physics, signal processing, and complex system modeling.

The research investigates infinite integrals with special functions within one-dimensional integrals while preventing the analysis of higher-dimensional areas. The integration algorithms together with their specific parameters can cause computational errors in the results. Additional research must apply these results to implement them with diverse special functions and advanced integral expressions for multiple application domains.

Future investigations need to examine higher-dimensional analysis along with multivariable fractional calculus because they can enhance the stability of these integrals for engineering and physics applications. Research on hypergeometric and Mittag-Leffler special functions will enhance understanding of particular mathematical functions. These frameworks prove useful for machine learning purposes because they assist both with extracting features and signal representation.

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