Dynamic Analysis and Control of a Rumor Propagation Model with Saturation Incidence in Online Social Networks

Yujiao Ding, and Ahmadjan Muhammadhaji

***Abstract*—Online social networks (OSNs) have significantly improved the speed and universality of communication, inadvertently fostering the propagation of rumors. However, the spread of rumors not only harms individuals but also threatens social stability and security, disrupting the social and economic order. To explore the dynamics of rumor propagation on OSNs and formulate the corresponding control strategies, this paper presents a rumor-propagation model with a saturated propagation rate, both with and without delay. First, the basic regeneration number $\mathcal{R}_0$ is calculated using the next-generation matrix method. The theoretical analysis focuses on the local and global asymptotic stabilities of both the rumor-free and rumor-prevailing equilibrium points, denoted as E^0 and E^*, respectively. The critical condition of the Hopf branch was obtained by selecting time delay as a branch parameter. Pontryagin's maximum principle was used to determine the optimal control for minimizing the frequency of rumor suppression. Finally, several numerical simulations were conducted to verify the accuracy of the theoretical results.**

***Index Terms*—rumor propagation; stability analysis; time delay; Hopf bifurcation; optimal control.**

I. Introduction

INTERNET rumors refer to the dissemination of unsubstantiated information through online media. These rumors may involve emergencies, public health, food and drug safety, political figures, social phenomena, natural disasters, and campus safety. Traditionally, rumors were spread through word-of-mouth, newspapers, magazines, radio, or television. However, with the rapid development of science and technology, network information technology has made significant breakthroughs, and the internet and mass media have penetrated millions of households. In March 2024, the China Internet Network Information Center released the 53rd Statistical Report on Internet Development e Internet in China. The data in the report, shown in Figure 1, indicate that by December 2023, China's Internet users had reached 1.092 billion, an increase of 24.8 million from December 2022. The Internet penetration rate was 77.5%. Among these users, 1.091 billion used mobile Internet, with the proportion of Internet users accessing the Internet through mobile phones as high as 99.9% [1]. Thus, mobile terminal users have become key participants in OSNs. The Internet provides a virtual social platform [2], and its virtual nature and anonymity have lowered netizens' sense of responsibility and legal awareness. The immediacy and openness of the Internet have accelerated the spread of rumors and violence. As Voltaire stated, "no snowflake is innocent during an avalanche." Netizens' blind or even malicious comments and posts on the Internet may contribute to online rumors and cyber violence. Recently, frequent cyberviolence incidents have led to many suicides. Long-term practice and research have shown that understanding only the laws of scientific communication can enable the development of effective defense and intervention to mitigate the spread of rumors. Therefore, it is crucial to investigate the propagation law of rumors to develop effective methods for preventing their spread.

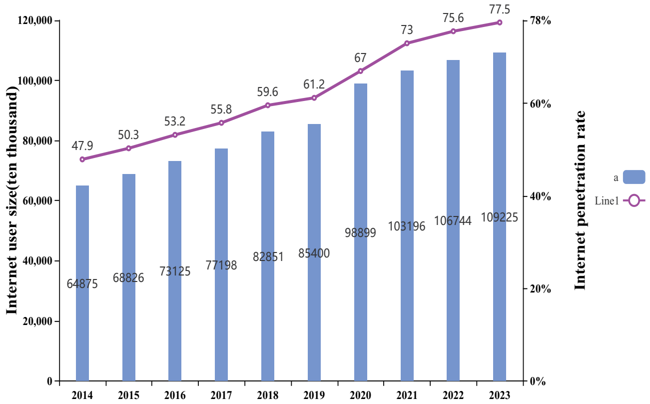

Figure 1. Size of Internet users and Internet penetration rate.

In this study, a rumor-propagation model was first established and analyzed based on an infectious disease model. Using the susceptible-infectious-recovered disease transmission model as an example, the population on the social network was divided into three groups: those susceptible to rumors, those who spread rumors, and those who recovered from rumors. Such a model can simply and intuitively describe the rumor-spreading process on social networks, thereby predicting the development trend and peak value of rumors through parameter analysis. In 1965, Daley and Kendall highlighted the differences between the spread of infectious diseases and rumors by proposing the classic Daley–Kendal (DK) rumor spread model, which includes detailed state-variable transfer rules [3], [4]. In 1973, Maki and Thomson modified the DK model to create the Maki-Thomson (MK) model, and proposed that rumors can spread through two-way contact between spreaders and others in the population. This model states that if the probability of one spreader contacting another is $y(y-1)$, then only the

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first spreader becomes the stifler [5]. With the development of social networks, many scholars have started using their topological characteristics to study rumor propagation across different environments. In 2001, Zanette [6], [7] was the first to study rumor propagation within small-world networks, highlighting the significant influence of the network topology on rumor spread. In 2008, Kawachi [8] extended the classical DK model to establish global behavioral patterns for rumor-propagation models. Through continuous revisions and improvements by scholars, the dynamic model of rumor-propagation model has gradually matured and become more comprehensive.

In recent years, most studies on internet rumor propagation models have focused on ordinary differential equations in the time dimension. These studies expand and develop the traditional SIR rumor propagation model by adding realistic constraints to better simulate and describe the rumor propagation process in real life. Zhao et al. [9] studied the SIHR rumor propagation model, which incorporates forgetting and memory mechanisms as hibernators (H) to the SIR model within a homogeneous network. Wang et al. [10] proposed two media reporting models to study the influence of different media on rumor propagation. Considering real-world scenarios, Yu et al. [11] investigated the dynamic behavior of the multilingual 2I2SR rumor propagation model, which examines rumor propagation with and without time delay in a multilingual environment. Based on people's different attitudes towards rumors, Hu et al. [12] proposed the susceptible-hospitalized-asymptomatic-recovered (SHAR) model. Tong et al. [13] studied deterministic and random class-age structure rumor propagation models that consider media coverage and age-related education, respectively. Additionally, considering the heterogeneity of network users and the random disturbances in the network environment, a random ignorants-forwarders-collectors-deleters rumor propagation model was proposed for heterogeneous networks. This model explores the dynamic behavior of rumor propagation on social networks, paving the way for effective control of rumor spread and providing positive guidance for managing network public opinion.

Morbidity plays a crucial role in the transmission of diseases and the spread of rumors. Initial studies on rumor propagation models focused on rumor spread with bilinear incidence. For example, in [14], [15], [16], a bilinear rumor propagation model was proposed. However, further in-depth research found that bilinear incidence does not closely reflect reality , and that nonlinear incidence is more accurate. Using both bilinear and standard incidence models is reasonable for studying rumor propagation in a limited number of closed chambers. Considering the public opinion system network, the total population is very large, and an individuals contact ability is limited. This prompted the study of the saturation incidence of rumor propagation[17]. For example, Chen et al. [18] proposed a novel susceptible-exposed-infected-recovered delayed rumor propagation model with saturation incidence in heterogeneous networks . Additionally, considering the cooling-off period, a modified innocents-spreaders-calmness-removes model was introduced, incorporating saturated incidence and time delay in a scale-free network[19].

Fake news and harmful statements spread on OSNs in a manner similar to an epidemic, characterized by the sudden dissemination of a large amount of malicious information in a short time. When an emergency occurs and malicious comments are forwarded rapidly, official platforms often ban users [20] or blocks their accounts, while relevant departments quickly release authoritative information to refute rumors, This approach helps ensure the authenticity of news and provides the public with a healthier and cleaner network environment. However, in a public opinion system network with a very large total population, the saturation propagation rate is more realistic than bilinear propagation rate. Therefore, we selected the saturation propagation rate to study the mechanism of the rumor propagation model.

Based on the relevant background, we constructed time-lagged and non-time-lagged rumor propagation models that include saturation incidence and studied their properties in depth. Through the next generation matrix method, we accurately calculate the basic propagation number. Using linear stability theory, we clarify the local stability conditions of the system and conclude the global stability using Lypanov function theory. To cope with the rumors, we designed the optimal control strategy using the Pontrygin maximum principle and verified its feasibility through numerical simulations. In addition, we study the existence conditions of Hopf branching and determine the critical parameters and time lag values. Finally, by fitting the numerical simulation to a real case, we verify the accuracy of the theoretical analysis and the effectiveness of the practical application.

The structure of this paper is as follows. In Section 2, we consider the mechanism of rumor propagation and model customization. In Section 3, the local and asymptotic stability of the rumor-free equilibrium point and the rumor-prevailing equilibrium point are analyzed by calculating the basic regeneration number. In Section 4, the optimal control problem is studied by using Pontryagin's maximum principle. In Section 5, the dynamical properties of the rumor propagation model with time delay are studied, and the critical conditions of Hopf bifurcation are obtained by taking time delay as the branch parameter. In Section 6, the correctness of the theoretical results is verified by numerical simulation. This article is summarized in Section 7.

II. The model formulation and preliminaries

In [12], three common attitudes towards rumors among the general public were assumed: liking rumor spreading, disliking rumor spreading, and being hesitant or neutral toward rumor spreading. Based on these assumptions, a SHAR model was established to incorporate the different attitudes of individuals toward rumor spreading. The model is formulated as follows:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = B - \alpha SI - \mu S, \\ \dfrac{\mathrm{d}H(t)}{\mathrm{d}t} = \theta_1 \alpha SI - \eta H - \mu H, \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \theta_2 \alpha SI + \varphi \eta H - \varepsilon I - \mu I, \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = (1-\theta_1-\theta_2)\alpha SI + (1-\varphi)\eta H + \varepsilon I - \mu R. \end{cases} \tag{1}$$

The basic reproduction number of the model was calculated, and the local and global stabilities of the rumor-free and prevailing equilibria were analyzed. This helped demonstrate how parameter changes influence rumor propagation through numerical simulations.

In [21], [22], [23], the influence of media refutation of rumors on rumor spreading was considered following emergencies. Additionally, in [24], [25], a rumor propagation model with a forced silence function was proposed for social networks.

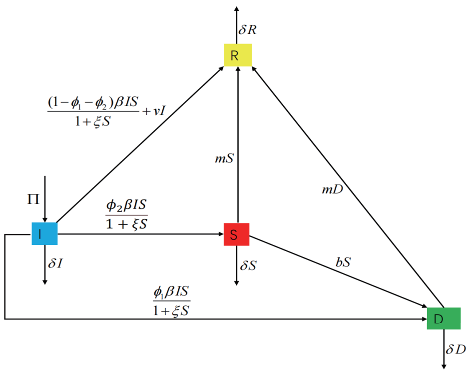


Figure 2. The dynamic transitions between different groups.

Inspired by the literature reviewed above, we propose a rumor propagation model tailored for online social networks, incorporating a saturation propagation rate. The rumor spreading process is illustrated in Figure 2. The transmission dynamics equations of the model are formulated as follows:

$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} =\Pi - \dfrac{\beta I(t)S(t)}{1+\xi S(t)} - vI(t) - \delta I(t), \\ \dfrac{\mathrm{d}D(t)}{\mathrm{d}t} =\phi_1 \dfrac{\beta I(t)S(t)}{1+\xi S(t)} - mD(t) + bS(t) - \delta D(t), \\ \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} =\phi_2 \dfrac{\beta I(t)S(t)}{1+\xi S(t)} - mS(t) - bS(t) - \delta S(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} =(1-\phi_1-\phi_2) \dfrac{\beta I(t)S(t)}{1+\xi S(t)} + mD(t) \\ \qquad\qquad + mS(t) + vI(t) - \delta R(t). \end{cases} \tag{2}$$

In our study, the total number of individuals in online social networks is denoted as $N(t)$, which changes over time. Individuals in the online social network can be classified into four categories: $I(t)$ (ignorants), who are susceptible to rumors; $D(t)$ (discussants), who are exposed to rumors but do not spread them; $S(t)$ (spreaders), who actively spread rumors; and $R(t)$ (recoverers), who are exposed to rumors but do not believe them.

The following assumptions were made in developing the rumor spreading model on OSN.

(1) On the social networking platform, assuming that the number of individuals entering each unit of time is constant, with an immigration rate denoted as Π. Each category experiences the same immigration rate, represented as δ.

(2) After an emergency, when a rumor-prone individual comes into contact with a rumor-spreading individual, the susceptible individual undergoes a type change. The transmission rate is denoted as β; ϕ_1 represents the probability that a rumor-spreading individual spreads a rumor to a rumor-susceptible individual, who then becomes a believer of the rumor but does not spread it. ϕ_2 represents the probability that the rumor-susceptible individual believes and spreads the rumor upon compact. $1-\phi_1-\phi_2$ represents the probability that the rumor-susceptible individual does not believe the rumor and thus does not spread it.

(3) Because each individual possesses the ability of self-judgment, $D(t)$ and $S(t)$ can transition to rumor-recovery individuals $R(t)$ at a rate m, known as the self-recovery rate. Upon identifying the rumor source, relevant authorities promptly release authoritative information to refute the rumor, causing susceptible individuals to transition to rumor-recovery status at a rate of v. When rumors spread, the ratio b represents the proportion of government and social media platforms that ban or block users engaged in malicious rumor spreading.

System (2) with the initial conditions

$$I(0) \geq 0, D(0) \geq 0, S(0) > 0, R(0) \geq 0.$$

It is already known that the size of the entire set of individuals within online social networks is $N(t)$, that is

$$N(t) = I(t) + D(t) + S(t) + R(t).$$

It is easy to know that $\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \Pi - (\delta+v)N$. Thus $N(t) = \left(N_0 - \frac{\Pi}{\delta+v}\right) e^{-(\delta+v)t} + \frac{\Pi}{(\delta+v)}$ and $N_0 = I(0) + D(0) + S(0) + R(0)$. Then we have $\lim_{t\to+\infty} N(t) = \frac{\Pi}{\delta+v}$. Thus, the positive variant set of system (2) is

$$\Gamma = \left\{ (I,D,S,R) \in R_4^+ : I + D + S + R \leq \frac{\Pi}{\delta+v} \right\}.$$

III. Analysis of the Model

3.1 Rumor-free equilibrium and the basic reproduction number

In this section, we will apply the next - generation matrix method proposed in [26] to derive the basic reproduction number $\mathcal{R}_0$ for the purpose of exploring the characteristics of rumor propagation. Subsequently, a detailed investigation into the dynamics of system (2) will be carried out.

Evidently, it can be observed that $D(t)$ and $R(t)$ are independent of the system of differential equations governing $I(t)$ and $S(t)$. Consequently, we are able to solve the system of ordinary differential equations for $I(t)$ and $S(t)$ without taking $D(t)$ and $R(t)$ into account. For the sake of convenience, system (2) can be simplified to the following equivalent form:

$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \Pi - \dfrac{\beta I(t)S(t)}{1+\xi S(t)} - vI(t) - \delta I(t), \\ \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \phi_2 \dfrac{\beta I(t)S(t)}{1+\xi S(t)} - mS(t) - bS(t) - \delta S(t), \end{cases} \tag{3}$$

with the initial conditions $I(0) \geq 0$, $S(0) > 0$.

Let $X = (I,S)^T$, then model yields $\frac{\mathrm{d}X}{\mathrm{d}t} = \mathcal{F} - \mathcal{V}$, where $\mathcal{F}, \mathcal{V}$ are define as

$$\mathcal{F} = \phi_2 \frac{\beta I(t)S(t)}{1+\xi S(t)}, \ \mathcal{V} = (m+\delta+b)S(t).$$

The Jacobian of system (3) around $E^0 = (\frac{\Pi}{\delta+v}, 0)$, one gets

$$F = \phi_2 \beta I_0, \ V = m + \delta + b,$$

then $\mathcal{R}_0$ is the spectral radius of FV^{-1} , which takes the following form

$$\mathcal{R}_0 = \frac{\phi_2 \beta \Pi}{(\delta+v)(m+\delta+b)}.$$

In order to conduct a local and global qualitative analysis of the proposed model in the vicinity of the rumor - free equilibrium, we make use of the following results.

Theorem 1. The rumor-free equilibrium $E^0 = (\frac{\Pi}{\delta+v}, 0)$ of system (2) is locally asymptotically stable when $\mathcal{R}_0 < 1$, while it is unstable when $\mathcal{R}_0 > 1$.

Proof: The Jacobian matrix of system (3) at the rumor-free equilibrium E^0 is

$$J(E^0) = \begin{pmatrix} -(\delta+v) & -\frac{\beta\Pi}{\delta+v} \\ 0 & \frac{\phi_2\beta\Pi}{\delta+v} - (m+\delta+b) \end{pmatrix}. \quad (4)$$

Accordingly, the characteristic equation can be readily obtained from matrix (4) as

$$(\lambda+\delta+v)\left(\lambda - \frac{\phi_2\beta\Pi}{\delta+v} + (m+\delta+b)\right) = 0. \quad (5)$$

The eigenvalues of $J(E^0)$ are given by

$$\begin{aligned} \lambda_1 &= -(\delta+v), \\ \lambda_2 &= \frac{\phi_2\beta\Pi}{\delta+v} - (m+\delta+b) \\ &= -(m+\delta+b)(1-\mathcal{R}_0). \end{aligned}$$

Since λ_1 is negative and λ_2 depend on the value of $\mathcal{R}_0$, which mean that all eigenvalues of $J(E^0)$ are negative if and only if $\mathcal{R}_0 < 1$. Consequently, system (2) is locally asymptotically stable at the rumor-free equilibrium E^0. Now, when $\mathcal{R}_0 > 1$, the Jacobian matrix $J(E^0)$ has both positive and negative eigenvalues. Specifically, λ_1 is negative and λ_2 is positive. This indicates that system (2) is unstable, and the rumor - free equilibrium E^0 is an unstable saddle point. ■

3.2 Global stability of the rumor-free equilibrium

Theorem 2. The rumor-free equilibrium $(I_0, 0, 0, 0)$ is globally asymptotically stable, if $\beta\Pi \leq \delta(\delta+v)$.

Proof: We consider the Lyapunov function:

$$V(t) = D(t) + S(t) + R(t).$$

Calculating the derivative of $V(t)$ along positive solutions of system (2) yields

$$\begin{aligned} V'(t) =& \phi_1\frac{\beta I(t)S(t)}{1+\xi S(t)} - mD(t) + bS(t) - \delta D(t) \\ &+ \phi_2\frac{\beta I(t)S(t)}{1+\xi S(t)} - mS(t) - bS(t) - \delta S(t) \\ &+ (1-\phi_1-\phi_2)\frac{\beta I(t)S(t)}{1+\xi S(t)} \\ &+ mD(t) + mS(t) + vI(t) - \delta R(t) \\ =& \frac{\beta I(t)S(t)}{1+\xi S(t)} - \delta(D(t)+S(t)+R(t)) \\ =& \left(\frac{\beta I(t)}{1+\xi S(t)} - \delta\right)S(t) - \delta(D(t)+R(t)). \end{aligned}$$

Since $I \leq \frac{\Pi}{\delta+v}$, we have

$$\begin{aligned} V'(t) &= \left(\frac{\beta I(t)}{1+\xi S(t)} - \delta\right)S(t) - \delta(D(t)+R(t)) \\ &\leq (\beta I(t) - \delta)S(t) - \delta(D(t)+R(t)) \\ &\leq (-\delta + \frac{\beta\Pi}{\delta+v})S(t) - \delta(D(t)+R(t)) \leq 0, \\ &\text{if } \frac{\beta\Pi}{\delta+v} \leq \delta. \end{aligned}$$

Moreover, since $\delta > 0$, it follows that $V'(t) \leq 0$, if $\beta\Pi \leq \delta(\delta+v)$.

Furthermore, $V'(t) \leq 0$ holds if and only if $D = S = R = 0$. The only solution of system (2) within the invariant region Γ satisfying $V'(t) = 0$ is the rumor-free equilibrium E^0. By LaSalle's Invariance Principle [28], every solution of system (2) converges to E^0 as $t \to \infty$. Therefore, the rumor-free equilibrium E^0 is globally asymptotically stable in the region Γ. ■

3.3 The existence of the rumor-existence equilibrium

Assuming that $E^*(I^*, D^*, S^*, R^*)$ is the steady state of system (2), then we obtain the following equations:

$$\begin{aligned} I^* &= \frac{(m+\delta+b)(1+\xi S^*)}{\beta\phi_2}, \\ D^* &= \frac{1}{\delta+m}\left(\frac{\phi_1\beta I^* S^*}{1+\xi S^*} + bS^*\right), \\ S^* &= \frac{\delta+v}{\beta+\xi(\delta+v)}(\mathcal{R}_0 - 1), \\ R^* &= \frac{(1-\phi_1-\phi_2)\beta I^* S^*}{\delta(1+\xi S^*)} + m\delta(D^* + S^*). \end{aligned}$$

Clearly the rumor-existence equilibrium exists, if $\mathcal{R}_0 > 1$. Thus we state the following lemma

Lemma 1. If $\mathcal{R}_0 > 1$, the rumor-existence equilibrium exists, otherwise, it does not.

To conduct a local qualitative analysis in the vicinity of the rumor - existence equilibrium, we utilize the following result.

Theorem 3. When $\mathcal{R}_0 > 1$, the rumor - existence equilibrium state E^* of the proposed system (2) exhibits local asymptotic stability.

Proof: The Jacobian matrix of system (2) at rumor equilibrium E^* can be expressed as

$$\begin{aligned} &J(E^*) \\ &= \begin{pmatrix} \frac{-\beta S^*}{1+\xi S^*} - k_1 & 0 & \frac{-\beta I^*}{(1+\xi S^*)^2} & 0 \\ \frac{\phi_1\beta S^*}{1+\xi S^*} & -(m+\delta) & \frac{\phi_1\beta I^*}{(1+\xi S^*)^2} + b & 0 \\ \frac{\phi_2\beta S^*}{1+\xi S^*} & 0 & \frac{\phi_2\beta I^*}{(1+\xi S^*)^2} - k_2 & 0 \\ \frac{(1-\phi_1\phi_2)\beta S^*}{1+\xi S^*} & m & \frac{(1-\phi_1\phi_2)\beta I^*}{(1+\xi S^*)^2} + m & -\delta \end{pmatrix}, \end{aligned} \quad (6)$$

where $k_1 = \delta + v$ and $k_2 = m + \delta + b$. Evidently, among the eigenvalues of $J(E^*)$, $\lambda_1 = -\delta$, $\lambda_2 = -(m+\delta)$ possess negative real parts. For the remaining eigenvalues, we analyze the following matrix:

$$A = \begin{pmatrix} \frac{-\beta S^*}{1+\xi S^*} - k_1 & \frac{-\beta I^*}{(1+\xi S^*)^2} \\ \frac{\phi_2\beta S^*}{1+\xi S^*} & \frac{\phi_2\beta I^*}{(1+\xi S^*)^2} - k_2 \end{pmatrix}. \quad (7)$$

The eigenvalues of the above matrix (7) are negative if the Hurwitz criterion [30], [31] (H_1): $tr(A) < 0$ and $det(A) > 0$ is satisfied. Therefore, the $tr(A)$ of matrix A is as follows:

$$\begin{aligned}
tr(A) =& -\frac{\beta S^*}{1+\xi S^*} + \frac{\phi_2 \beta I^*}{(1+\xi S^*)^2} - k_1 - k_2 \\
=& \frac{1}{(1+\xi S^*)^2}\big[-\beta S^*(1+\xi S^*) + \phi_2\beta I^* \\
& - k_1(1+\xi S^*)^2 - k_2(1+\xi S^*)^2\big] \\
=& \frac{1}{(1+\xi S^*)^2}\big[-\beta S^*(1+\xi S^*) + k_2(1+\xi S^*) \\
& - k_1(1+\xi S^*)^2 - k_2(1+\xi S^*)^2\big] \\
=& \frac{1}{(1+\xi S^*)^2}\big[-\beta S^*(1+\xi S^*) - k_1(1+\xi S^*)^2 \\
& - k_2\xi S^* - k_2\xi^2 S^{*2}\big] \\
=& \frac{1}{(1+\xi S^*)^2}\big[-\beta S^*(1+\xi S^*) - k_1(1+\xi S^*)^2 \\
& - k_2\xi S^*(1+\xi S^*)\big] \\
<& 0.
\end{aligned} \tag{8}$$

Similarly determinant of A becomes

$$\begin{aligned}
det(A) =& \left(-\frac{\beta S^*}{1+\xi S^*} - k_1\right)\left(\frac{\phi_2\beta I^*}{(1+\xi S^*)^2} - k_2\right) \\
& + \frac{\beta I^*}{(1+\xi S^*)^2}\frac{\phi_2\beta S^*}{(1+\xi S^*)} \\
=& \left(-\frac{\beta S^*}{1+\xi S^*}\right)\frac{\phi_2\beta I^*}{(1+\xi S^*)^2} + \frac{k_2\beta S^*}{1+\xi S^*} \\
& - \frac{k_1\phi_2\beta I^*}{(1+\xi S^*)^2} + k_1k_2 + \frac{\beta I^*}{(1+\xi S^*)^2}\frac{\phi_2\beta S^*}{(1+\xi S^*)} \\
=& k_2\beta S^*(1+\xi S^*) - k_1\phi_2\beta I^* + k_1k_2(1+\xi S^*)^2 \\
=& k_2\beta S^*(1+\xi S^*) - k_1k_2(1+\xi S^*) \\
& + k_1k_2(1+\xi S^*)^2 \\
=& k_2\beta S^*(1+\xi S^*) - k_1k_2 - k_1k_2\xi S^* \\
& + k_1k_2(1+2\xi S^* + \xi^2 S^{*2}) \\
=& k_2\beta S^*(1+\xi S^*) + k_1k_2\xi S^*(1+\xi S^*) \\
>& 0.
\end{aligned} \tag{9}$$

It can be observed from equations (8) and (9) that the Hurwitz criterion (H_1) holds if and only if $\mathcal{R}_0 > 1$. Specifically, this implies that all eigenvalues of the matrix have negative real parts if and only if $\mathcal{R}_0 > 1$. ■

3.4 Global stability of the rumor-existence equilibrium

Theorem 4. If $\mathcal{R}_0 > 1$, then the rumor-existence equilibrium state $E^*(I^*, D^*, S^*, R^*)$ of the proposed system (2) is globally asymptotically stable.

Proof: Consider the Lyapunov function:

$$\begin{aligned}
W(t) =& \big[(I(t)-I^*) + (D(t)-D^*) \\
& + (S(t)-S^*) + (R(t)-R^*)\big]^2.
\end{aligned}$$

Calculating the derivative of $W(t)$ along positive solutions of system (2) yields

$$\begin{aligned}
W'(t) =& 2[(I(t)-I^*) + (D(t)-D^*) + (S(t)-S^*) \\
& + (R(t)-R^*)][I'(t) + D'(t) + S'(t) + R'(t)] \\
=& 2[(I(t)-I^*) + (D(t)-D^*) + (S(t)-S^*) \\
& + (R(t)-R^*)]\big[\Pi - \delta I(t) - \delta D(t) \\
& - \delta S(t) - \delta R(t)\big].
\end{aligned}$$

It is easy to see that $\Pi = \delta I^* - \delta D^* - \delta S^* - \delta R^*$. Then, we can get

$$\begin{aligned}
W'(t) =& 2[(I(t)-I^*) + (D(t)-D^*) + (S(t)-S^*) \\
& + (R(t)-R^*)][I'(t) + D'(t) + S'(t) + R'(t)] \\
=& 2[(I(t)-I^*) + (D(t)-D^*) + (S(t)-S^*) \\
& + (R(t)-R^*)][\delta I^* + \delta D^* + \delta S^* + \delta R^* \\
& - \delta I - \delta D - \delta S - \delta R] \\
=& 2[(I(t)-I^*) + (D(t)-D^*) + (S(t)-S^*) \\
& + (R(t)-R^*)][\delta(I^* - I(t)) + \delta(D^* - D(t)) \\
& + \delta(S^* - S(t)) + \delta(R^* - R(t))] \\
=& 2\delta[(I^* - I(t)) + (D^* - D(t)) \\
& + (S^* - S(t)) + (R^* - R(t))]^2 \\
\leq& 0.
\end{aligned}$$

Therefore, according to LaSalle's Invariance Principle [28], the equilibrium point E^* is globally asymptotically stable. ■

IV. The optimal control

In cases where a rumor persists, effective measures must be implemented to suppress it. In this section, we propose an optimal control mechanism [32], [33], [34], [35] aimed at preventing rumor transmission. We utilize three control variables designed to restrain the spread of rumors over an expected period while minimizing costs.

(i) $\sigma_1(t)$ represents the intensity of popular science education as a control variable. By educating ignorants through popular science programs, their ability to discern and refute rumors is improved, making them immune to rumors.

(ii) $\sigma_2(t)$ represents the intensity of deleted posts in the network influenced by rumors as a control variable. The percentage of those affected by the rumors in the media is reduced by removing high-impact posts.

(iii) $\sigma_3(t)$ represents the punishment intensity for malicious spreaders as a control variable. Rumors are primarily spread through dissemination by these spreaders. Punishing malicious spreaders, banning or blocking them, and suppressing their effectivities are effective means of controlling the spread of rumors.

To reduce the negative impact of rumor propagation on social stability, productivity, and daily life, while also reducing control costs and maximizing social utility, this study aims to reduce the proportion of affected individuals in the media. By ensuring that those who have not been exposed to rumors can learn the truth without spreading false information, the goal is to maximize the number of individuals who recover and minimize the costs associated with controlling rumor propagation.

$$J(\sigma_1(t),\sigma_2(t),\sigma_3(t))=\int_0^T\Big[w_1S(t)+\frac{1}{2}\big(w_2\sigma_1^2(t)+w_3\sigma_2^2(t)+w_4\sigma_3^2(t)\big)\Big]dt, \tag{10}$$

subject to the control system

$$\begin{cases}\dfrac{\mathrm{d}I(t)}{\mathrm{d}t}=&\Pi-\dfrac{\beta I(t)S(t)}{1+\xi S(t)}(1-\sigma_1(t))-\sigma_3 I(t)-\delta I(t),\\ \dfrac{\mathrm{d}D(t)}{\mathrm{d}t}=&\phi_1\dfrac{\beta I(t)S(t)}{1+\xi S(t)}(1-\sigma_1(t))-mD(t)\\ &+\sigma_2S(t)-\delta D(t),\\ \dfrac{\mathrm{d}S(t)}{\mathrm{d}t}=&\phi_2\dfrac{\beta I(t)S(t)}{1+\xi S(t)}(1-\sigma_1(t))-mS(t)\\ &-\sigma_2S(t)-\delta S(t),\\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t}=&(1-\phi_1-\phi_2)\dfrac{\beta I(t)S(t)}{1+\xi S(t)}+mD(t)+mS(t)\\ &+\sigma_3I(t)-\delta R(t).\end{cases} \tag{11}$$

The initial data for system(2) satisfied $I(0)\geq 0, D(0)\geq 0, S(0)>0, R(0)\geq 0$.

The feasible region of $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_3(t)$ is

$$\Omega=\Big\{(\sigma_1(t),\sigma_2(t),\sigma_3(t))\mid 0\leqslant\sigma_1(t)\leqslant\sigma_1^{\max},0\leqslant\sigma_2(t)\leqslant\sigma_2^{\max},0\leqslant\sigma_3(t)\leqslant\sigma_3^{\max},t\in(0,T]\Big\},$$

where $\sigma_1^{\max}\leq 1$, $\sigma_2^{\max}\leq 1$ and $\sigma_3^{\max}\leq 1$ are the upper bound of $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_3(t)$, respectively. Optimal control σ_1^* , σ_2^* and σ_3^* satisfy

$$J(\sigma_1^*,\sigma_2^*,\sigma_3^*)=\min\big\{J(\sigma_1(t),\sigma_2(t),\sigma_3(t)):(\sigma_1(t),\sigma_2(t),\sigma_3(t))\in\Omega\big\}.$$

In order to obtain the optimal control, we construct the following Lagrangian function

$$L(S(t),\sigma_1(t),\sigma_2(t),\sigma_3(t))=w_1S(t)+\frac{1}{2}\big(w_2\sigma_1^2(t)+w_3\sigma_2^2(t)+w_4\sigma_3^2(t)\big).$$

The Hamiltonian function is defined as

$$\begin{aligned}&H(S(t),\sigma_i(t),\lambda_j(t))\\ =&w_1S(t)+\frac{1}{2}\left(w_2\sigma_1^2(t)+w_3\sigma_2^2(t)+w_4\sigma_3^2(t)\right)\\ &+\lambda_1\left[\Pi-\frac{\beta I(t)S(t)}{1+\xi S(t)}(1-\sigma_1(t))-\sigma_3I(t)-\delta I(t)\right]\\ &+\lambda_2\Big[\phi_1\frac{\beta I(t)S(t)}{1+\xi S(t)}(1-\sigma_1(t))-mD(t)\\ &+\sigma_2S(t)-\delta D(t)\Big]\\ &+\lambda_3\Big[\phi_2\frac{\beta I(t)S(t)}{1+\xi S(t)}(1-\sigma_1(t))-mS(t)\\ &-\sigma_2S(t)-\delta S(t)\Big]\\ &+\lambda_4\Big[(1-\phi_1-\phi_2)\frac{\beta I(t)S(t)}{1+\xi S(t)}+mD(t)\\ &+mS(t)+\sigma_3I(t)-\delta R(t)\Big],\end{aligned} \tag{12}$$

where $i=1,2,3$, and $j=1,2,3,4$. Using Pontryagins Maximum Principle[36], we can obtain the following theorem.

Theorem 5. Let I^*, D^*, S^* and R^* be optimal state solutions with an associated optimal control $(\sigma_1(t),\sigma_2(t),\sigma_3(t))$ for optimal control problem (11). Then, there exist adjoint variables $\lambda_1(t),\lambda_2(t),\lambda_3(t)$ and $\lambda_4(t)$ satisfying

$$\begin{cases}\dfrac{\mathrm{d}\lambda_1(t)}{\mathrm{d}t}=&(1-\sigma_1(t))\left(\lambda_1(t)-\phi_1\lambda_2(t)-\phi_2\lambda_3(t)\right)\dfrac{\beta S^*}{1+\xi S^*}\\ &-(1-\phi_1-\phi_2)\lambda_4(t)\dfrac{\beta S^*}{1+\xi S^*}\\ &+(\lambda_1(t)-\lambda_4(t))\sigma_3(t)+\lambda_1(t)\delta,\\ \dfrac{\mathrm{d}\lambda_2(t)}{\mathrm{d}t}=&(m+\delta)\lambda_2(t)-\lambda_4(t)m,\\ \dfrac{\mathrm{d}\lambda_3(t)}{\mathrm{d}t}=&-\omega_1+(1-\sigma_1(t))\left(\lambda_1(t)-\phi_1\lambda_2(t)-\phi_2\lambda_3(t)\right)\\ &\times\dfrac{\beta I^*}{(1+\xi S^*)^2}-\lambda_4(t)\dfrac{(1-\phi_1-\phi_2)\beta I^*}{(1+\xi S^*)^2}\\ &-\sigma_2(t)\lambda_2(t)+(\sigma_2(t)+\delta)\lambda_3(t)\\ &+m(\lambda_3(t)-\lambda_4(t)),\\ \dfrac{\mathrm{d}\lambda_4(t)}{\mathrm{d}t}=&\delta\lambda_4(t),\end{cases} \tag{13}$$

with the transversality conditions $\lambda_j(T)=0$, for $j=1,2,3,4$. The optimal control σ_1^*,σ_2^* and σ_3^* are given by

$$\begin{aligned}\sigma_1^*=&\min\left\{\max\left\{\frac{1}{w_2}\left[(\phi_1\lambda_2(t)-\phi_2\lambda_3(t)-\lambda_1(t))\times\frac{\beta I^*S^*}{1+\xi S^*}\right],0\right\},\sigma_1^{\max}\right\},\\ \sigma_2^*=&\min\left\{\max\left\{\frac{1}{w_3}\left[(\lambda_3(t)-\lambda_2(t))S^*\right],0\right\},\sigma_2^{\max}\right\},\\ \sigma_3^*=&\min\left\{\max\left\{\frac{1}{w_4}\left[(\lambda_1(t)-\lambda_4(t))I^*\right],0\right\},\sigma_3^{\max}\right\}.\end{aligned} \tag{14}$$

Proof: By Pontryagins Maximum Principle, and let $I(t)=I^*, D(t)=D^*, S(t)=S^*, R(t)=R^*$, we obtain following adjoint equation

$$\begin{cases}\dfrac{\mathrm{d}\lambda_1(t)}{\mathrm{d}t}=&(1-\sigma_1(t))\left(\lambda_1(t)-\phi_1\lambda_2(t)-\phi_2\lambda_3(t)\right)\dfrac{\beta S^*}{1+\xi S^*}\\ &-(1-\phi_1-\phi_2)\lambda_4(t)\dfrac{\beta S^*}{1+\xi S^*}\\ &+(\lambda_1(t)-\lambda_4(t))\sigma_3(t)+\lambda_1(t)\delta,\\ \dfrac{\mathrm{d}\lambda_2(t)}{\mathrm{d}t}=&-\dfrac{\partial H(t)}{\partial D(t)}=(m+\delta)\lambda_2(t)-\lambda_4m,\\ \dfrac{\mathrm{d}\lambda_3(t)}{\mathrm{d}t}=&-\omega_1+(1-\sigma_1(t))\left(\lambda_1(t)-\phi_1\lambda_2(t)-\phi_2\lambda_3(t)\right)\\ &\times\dfrac{\beta I^*}{(1+\xi S^*)^2}-(1-\phi_1-\phi_2)\lambda_4(t)\dfrac{\beta I^*}{(1+\xi S^*)^2}\\ &-\sigma_2(t)\lambda_2(t)+(\sigma_2(t)+\delta)\lambda_3(t)\\ &+m(\lambda_3(t)-\lambda_4(t)),\\ \dfrac{\mathrm{d}\lambda_4(t)}{\mathrm{d}t}=&-\dfrac{\partial H(t)}{\partial R(t)}=\lambda_4\delta.\end{cases} \tag{15}$$

Under the optimality condition, the derivative of (14) with respect to $\sigma_1(t), \sigma_2(t)$ and $\sigma_3(t)$ are as follows

$$\begin{aligned}
\frac{\partial H(t)}{\partial \sigma_1(t)} |_{\sigma_1(t)=\sigma_1^*} =& w_2\sigma_1^*(\lambda_1(t) - \phi_1\lambda_2(t) - \phi_2\lambda_3(t)) \\
& \times \frac{\beta I^* S*}{1+\xi S^*} = 0, \\
\frac{\partial H(t)}{\partial \sigma_2(t)} |_{\sigma_2(t)=\sigma_2^*} =& w_3\sigma_2^* + \lambda_2(t)S^* - \lambda_3(t)S^* = 0, \\
\frac{\partial H(t)}{\partial \sigma_3(t)} |_{\sigma_3(t)=\sigma_3^*} =& w_3\sigma_3^* + \lambda_4(t)S^* - \lambda_1(t)S^* = 0.
\end{aligned} \tag{16}$$

By combining the properties of bounded set Ω, the interval of $\sigma_1^*(t), \sigma_2^*(t)$ and $\sigma_3^*(t)$ are shown in the following form

$$\begin{aligned}
\sigma_1^* =& \min\left\{ \max\left\{ \frac{1}{w_2}\left[(\phi_1\lambda_2(t) - \phi_2\lambda_3(t) - \lambda_1(t)) \right.\right.\right. \\
& \left.\left.\left. \times \frac{\beta I^* S^*}{1+\xi S^*} \right], 0 \right\}, \sigma_1^{\max} \right\}, \\
\sigma_2^* =& \min\left\{ \max\left\{ \frac{1}{w_3} \left[(\lambda_3(t) - \lambda_2(t))S^* \right], 0 \right\}, \sigma_2^{\max} \right\}, \\
\sigma_3^* =& \min\left\{ \max\left\{ \frac{1}{w_4} \left[(\lambda_1(t) - \lambda_4(t))I^* \right], 0 \right\}, \sigma_3^{\max} \right\}.
\end{aligned} \tag{17}$$

■

V. Rumor Propagation with Time-Delay

During the process of rumor spreading in social networks, users cannot be immediately banned or blocked by the platform. In this study, we further incorporate a time delay effect associated with the implementation of control measures. Consequently, system (2) is redefined as follows:

$$\begin{cases}
\frac{\mathrm{d}I(t)}{\mathrm{d}t} =& \Pi - \frac{\beta I(t)S(t)}{1+\xi S(t)} - vI(t) - \delta I(t), \\
\frac{\mathrm{d}D(t)}{\mathrm{d}t} =& \phi_1 \frac{\beta I(t)S(t)}{1+\xi S(t)} - mD(t) + bS(t-\tau) - \delta D(t), \\
\frac{\mathrm{d}S(t)}{\mathrm{d}t} =& \phi_2 \frac{\beta I(t)S(t)}{1+\xi S(t)} - mS(t) - bS(t-\tau) - \delta S(t), \\
\frac{\mathrm{d}R(t)}{\mathrm{d}t} =& (1-\phi_1-\phi_2) \frac{\beta I(t)S(t)}{1+\xi S(t)} + mD(t) + mS(t) \\
& + vI(t) - \delta R(t),
\end{cases} \tag{18}$$

with the initial conditions

$$\begin{cases}
I(t) \geq 0, & t \in (-\tau, 0], \\
D(t) \geq 0, & t \in (-\tau, 0], \\
S(t) > 0, & t \in (-\tau, 0], \\
R(t) \geq 0, & t \in (-\tau, 0].
\end{cases}$$

Similarly, for the sake of simplicity, we focus solely on the following equivalent subsystem of system (18).

$$\begin{cases}
\frac{\mathrm{d}I(t)}{\mathrm{d}t} = \Pi - \frac{\beta I(t)S(t)}{1+\xi S(t)} - vI(t) - \delta I(t), \\
\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \phi_2 \frac{\beta I(t)S(t)}{1+\xi S(t)} - mS(t) - bS(t-\tau) - \delta S(t),
\end{cases} \tag{19}$$

with the initial conditions

$$\begin{cases}
I(t) \geq 0, & t \in (-\tau, 0], \\
S(t) > 0, & t \in (-\tau, 0].
\end{cases}$$

5.1 Hopf bifurcation of the rumor-free equilibrium

First, we discuss the stability and Hopf bifurcation of system (19) at rumor-free equilibrium $E^0 = (\frac{\Pi}{\delta+v}, 0)$.

Theorem 6. In system (19), the rumor-free equilibrium E^0 exhibits local asymptotic stability when $\mathcal{R}_0 < 1$, whereas it is unstable when $\mathcal{R}_0 > 1$.

Proof: The Jacobian matrix of system (19) at $E^0 = \left(\frac{\Pi}{\delta+v}, 0\right)$ is

$$J(E^0) = \begin{pmatrix} -(\delta+v) & -\frac{\beta\Pi}{\delta+v} \\ 0 & \frac{\phi_2\beta\Pi}{\delta+v} - (m+\delta) - be^{-\lambda\tau} \end{pmatrix}. \tag{20}$$

The characteristic equation is given by

$$\begin{aligned}
& \lambda^2 + \left[(m+\delta) - \frac{\phi_2\beta\Pi}{\delta+v} + (\delta+v) + be^{-\lambda\tau} \right] \lambda \\
& - (\delta+v) \left[\frac{\phi_2\beta\Pi}{\delta+v} - (m+\delta) - be^{-\lambda\tau} \right] = 0.
\end{aligned} \tag{21}$$

When $\tau = 0$, it is known that the rumor - free equilibrium E^0 is locally asymptotic stable when $\mathcal{R}_0 < 1$.

Next, we consider the effect of time delay on system stability. When $\tau > 0$, we assume that $\lambda = i\omega$ is a solution of (21), then we have

$$\begin{aligned}
& -\omega^2 + \left[(m+\delta) - \frac{\phi_2\beta\Pi}{\delta+v} + (\delta+v) + be^{-i\omega\tau} \right] i\omega \\
& - (\delta+v) \left[\frac{\phi_2\beta\Pi}{\delta+v} - (m+\delta) - be^{-i\omega\tau} \right] = 0.
\end{aligned} \tag{22}$$

After separating the real and imaginary parts of the above equation, it can be rewritten as follows:

$$\begin{aligned}
& -\omega^2 - \phi_2\beta\Pi + (\delta+v)(m+\delta) + b\big[\omega sin(\omega\tau) + (\delta+v) \\
& \times cos(\omega\tau)\big] + i\Big[(m+\delta)\omega - \frac{\omega\phi_2\beta\Pi}{\delta+v} + (\delta+v)\omega \\
& + b\big[\omega cos(\omega\tau) - (\delta+v)sin(\omega\tau)\big]\Big] = 0,
\end{aligned} \tag{23}$$

specifically

$$\begin{cases}
\omega sin(\omega\tau) + (\delta+v)cos(\omega\tau) =& \frac{1}{b}\big[\omega^2 + \phi_2\beta\Pi \\
& - (\delta+v)(m+\delta)\big] \\
=& \frac{1}{b}\left(\omega^2 + (\delta+v)M\right), \\
\omega cos(\omega\tau) - (\delta+v)sin(\omega\tau) =& \frac{1}{b}\Big[-(\delta+v)\omega + \frac{\omega\phi_2\beta\Pi}{\delta+v} \\
& - \omega(m+\delta)\Big] \\
=& \frac{1}{b}\omega\left(M - (\delta+v)\right),
\end{cases} \tag{24}$$

where $M = \frac{\phi_2\beta\Pi}{\delta+v} - (m+\delta)$. Square and add the two equations of (24), and let $z = \omega^2$, we obtain

$$z^2 + f_2 z + f_1 = 0, \tag{25}$$

where

$$f_1 = (\delta+v)^2(M^2 - b^2),\ f_2 = (\delta+v)^2 + M^2 - b^2. \tag{26}$$

It can be readily shown that if $f_1 \geq 0$, then $f_2 > 0$,conversely, if $f_2 \leq 0$, then $f_1 < 0$. In the case where

if $f_1 < 0$, equation (25) necessarily has at least one positive root. Hence,
(i) When $f_1 < 0$, there is a unique positive roots of (25), defined by $z_1 = \omega_1^2\,(\omega_1 > 0)$;
(ii) When $f_1 \geq 0$, there does not exist any positive root for (25).

If $f_1 < 0$, by (24), we obtain

$$\cos\omega_1\tau = \frac{M}{b}, \sin\omega_1\tau = \frac{\omega_1}{b}. \tag{27}$$

Thus

$$\tau_i = \frac{1}{\omega_1}\arccos\frac{M}{b} + \frac{2i\pi}{\omega_1}, \tag{28}$$

where $i = 0, 1, 2, \ldots$ and (21) possesses a pair of purely imaginary roots in the form of $\pm i\omega_1$ when $\tau = \tau_i$. Then from the derivative of (22) with respect to τ, one has

$$\begin{aligned} 2\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} &+ \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\left((\delta+v) - M + be^{-\lambda\tau}\right) \\ &- (\delta+v+\lambda)be^{-\lambda\tau}\left(\lambda + \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\tau\right) = 0. \end{aligned} \tag{29}$$

Then

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{1}{(\lambda+\delta+v)\,\lambda} + \frac{2\lambda + (\delta+v) - M}{be^{-\lambda\tau}(\lambda+\delta+v)\lambda} - \frac{\tau}{\lambda}. \tag{30}$$

According to (21), one gets

$$be^{-\lambda\tau}\,(\delta+v+\lambda) = -\left(\lambda^2 + (\delta+v-M)\lambda - (\delta+v)M\right). \tag{31}$$

Use (31) to simplify (30) it shows that

$$\begin{aligned} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} =& \frac{1}{(\lambda+\delta+v)\,\lambda} - \frac{\tau}{\lambda} \\ &- \frac{2\lambda + (\delta+v) - M}{\left[\lambda^2 + (\delta+v-M)\lambda - (\delta+v)M\right]\lambda}. \end{aligned} \tag{32}$$

After substituting $\lambda = i\omega_1$ and $\tau = \tau_1$ into (32), separate the real part and the imaginary part to obtain that

$$\begin{aligned} \left.\frac{\mathrm{d}(\operatorname{Re}\lambda)}{\mathrm{d}\tau}\right|_{\lambda=i\omega_1,\tau=\tau_1} &= \operatorname{Re}\left.\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\right|_{\lambda=i\omega_1,\tau=\tau_1} \\ &= \frac{1}{\omega_1^2 + M^2} > 0. \end{aligned} \tag{33}$$

■

In summary, we can get the following theorem.

Theorem 7. If $\mathcal{R}_0 < 1$ holds and $f_1 < 0$ is satisfied, then the following conclusions apply to system (19).

(1) When $\tau \in [0, \tau_1)$, the rumor-free equilibrium point E^0 of system (19) is locally asymptotically stable.

(2) When $\tau > \tau_1$, the rumor - free equilibrium point E^0 of system (19) becomes unstable. The value $\tau = \tau_1$ corresponds to a Hopf bifurcation point. Specifically, system (19) exhibits a branch of periodic solutions that bifurcate from the rumor - free equilibrium point E^0 in the vicinity of $\tau = \tau_1$.

5.2 Hopf bifurcation of the rumor-prevailing equilibrium

In this section, we analyze the stability and Hopf bifurcation of system (19) at the rumor - prevailing equilibrium point $E^* = (I^*, S^*)$. Specifically, the Jacobian matrix of system (19) evaluated at E^*

$$\begin{aligned} &J(E^*) \\ &= \begin{pmatrix} -\frac{\beta S^*}{1+\xi S^*} - (\delta+v) & -\frac{\beta I^*}{(1+\xi S^*)^2} \\ \frac{\phi_2\beta S^*}{1+\xi S^*} & \frac{\phi_2\beta I^*}{(1+\xi S^*)^2} - (m+\delta) - be^{-\lambda\tau} \end{pmatrix}. \end{aligned} \tag{34}$$

The characteristic equation is

$$\lambda^2 + (P+Q)\lambda + PQ + be^{-\lambda\tau}(\lambda+Q) + \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} = 0, \tag{35}$$

where

$$P = m + \delta - \frac{\phi_2\beta I^*}{(1+\xi S^*)^2},\ Q = \frac{\beta S^*}{1+\xi S^*} + \delta + v. \tag{36}$$

When $\tau = 0$, according to Theorem 3, the rumor-prevailing equilibrium point E^* of system (19) is locally asymptotically stable if (H_1) is satisfied.

Next, we consider the effect of the time delay on system stability. When $\tau > 0$, we assuming that $\lambda = i\omega$ is a solution of (35), then we obtain

$$\begin{aligned} -\omega^2 &+ (P+Q)\omega i + PQ \\ &+ be^{-i\omega\tau}(i\omega + Q) + \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} = 0. \end{aligned} \tag{37}$$

After separating the real and imaginary parts of the equation above, it can be rewritten as below

$$\begin{aligned} -\omega^2 + PQ &+ \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} + b\left(\omega\sin(\omega\tau) + Q\cos(\omega\tau)\right) \\ &+ i\left[(P+Q)\omega + b(\omega\cos(\omega\tau) - Q\sin(\omega\tau))\right] = 0, \end{aligned} \tag{38}$$

namely

$$\begin{cases} \omega\sin(\omega\tau) + Q\cos(\omega\tau) = \dfrac{1}{b}\left[\omega^2 - PQ - \dfrac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3}\right], \\ \omega\cos(\omega\tau) - Q\sin(\omega\tau) = -\dfrac{1}{b}(P+Q)\omega. \end{cases} \tag{39}$$

Respectively squaring and adding the two side of (39) and letting $z = \omega^2$ leads to

$$z^2 + f_4 z + f_3 = 0, \tag{40}$$

where

$$\begin{aligned} f_3 &= \left(PQ + \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3}\right)^2 - b^2Q^2, \\ f_4 &= P^2 + Q^2 - \frac{2\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} - b^2. \end{aligned} \tag{41}$$

Denote that

$$F(z) = z^2 + f_4 z + f_3. \tag{42}$$

If $f_3 < 0$, there must be a positive root of (42).
If $f_3 \geq 0, f_4 \geq 0$, there does not exist any positive root for (42).
If $f_3 \geq 0, f_4 < 0, F(-f_4/2) = -f_4^2/4 + f_3 > 0$, there does not exist any positive root for (42).
If $f_3 \geq 0, f_4 < 0, F(-f_4/2) = -f_4^2/4 + f_3 \leq 0$, there must be two positive roots of (42).

Hence, we have the following results.
(i) When $(G_1)\, f_3 < 0$ holds, there must be a positive root of (42), defined by $z_0 = \omega_0^2$.

(ii) When $(G_2)\, f_3 > 0$ and $f_4 < 0$ and $F(-f_4/2) = -f_4^2/4 + f_3 < 0$ holds, there must be two positive roots of (42), defined by $z_1 = \omega_1^2$ and $z_2 = \omega_2^2$.
(iii) When $(G_3)\, f_3 \geq 0, f_4 \geq 0$ or $f_3 \geq 0, f_4 < 0$ and $F(-f_4/2) = -f_4^2/4 + f_3 > 0$ holds, there does not exist any positive root for (42).

Without loss of generality, we assume that (42) exists the only positive root $z_0 = \omega_0^2$. By (42), we obtain

$$\cos\omega\tau = \frac{-PQ^2 - \frac{Q\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} - P\omega^2}{(\omega^2 + Q^2)\, b}, \tag{43}$$

and then

$$\tau_i = \frac{1}{\omega_0}\arccos\frac{-PQ^2 - \frac{Q\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} - P\omega_0^2}{(\omega_0^2 + Q^2)\, b} + \frac{2i\pi}{\omega_0}, \tag{44}$$

where $i = 0, 1, 2, \ldots$, and $\pm i\omega_0$ is a pair of purely imaginary roots of (35) with $\tau = \tau_i$. Then from the derivative of (35) with respect to τ, one has

$$\begin{aligned} &2\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}(P+Q) \\ &\quad + be^{-\lambda\tau}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} - \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\tau + \lambda\right)(\lambda + Q)\right] = 0. \end{aligned} \tag{45}$$

Then

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{1}{\lambda(\lambda+Q)} + \frac{2\lambda + P + Q}{\lambda(\lambda+Q)be^{-\lambda\tau}} - \frac{\tau}{\lambda}. \tag{46}$$

According to (35) , one gets

$$be^{-\lambda\tau}(\lambda+Q) = -\left[\lambda^2 + (P+Q)\lambda + PQ + \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3}\right]. \tag{47}$$

Thus

$$\begin{aligned} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} =& \frac{1}{(\lambda+Q)\lambda} - \frac{\tau}{\lambda} \\ &- \frac{2\lambda + P + Q}{\left[\lambda^2 + (P+Q)\lambda + PQ + \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3}\right]\lambda}. \end{aligned} \tag{48}$$

After substituting $\lambda = i\omega_0$ and $\tau = \tau_0$ into (49), separate the real part and the imaginary part to obtain that

$$\begin{aligned} &\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Bigg|_{\lambda=i\omega_0,\tau=\tau_0} = -\frac{1}{\omega_0^2 + Q^2} \\ &+ \frac{(P+Q)^2\omega_0^2 + 2\omega_0^2\left(-\omega_0^2 + PQ + \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3}\right)}{\omega_0^4(P+Q)^2 + \omega_0^2\left(-\omega_0^2 + PQ + \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3}\right)^2}. \end{aligned} \tag{49}$$

According to (37), squaring and adding both sides and substitute ω by ω_0, we have

$$\begin{aligned} b^2(\omega_0^2 + Q^2) = &\left(-\omega_0^2 + PQ + \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3}\right)^2 \\ &+ \omega_0^2(P+Q)^2. \end{aligned} \tag{50}$$

Then

$$\begin{aligned} &\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Bigg|_{\lambda=i\omega_0,\tau=\tau_0} \\ &= \frac{-\omega_0^2 b^2 + (P+Q)^2\omega_0^2 + 2\omega_0^2\left(-\omega_0^2 + PQ + \frac{\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3}\right)}{\omega_0^2 b^2(\omega_0^2 + Q^2)} \\ &= \frac{F'(\omega_0^2)}{\omega_0^2 b^2(\omega_0^2 + Q^2)}. \end{aligned} \tag{51}$$

When (G_1) and $F'(\omega_0^2) > 0$ hold, it can be readily verified that

$$\frac{\mathrm{d}(\operatorname{Re}\lambda)}{\mathrm{d}\tau}\Bigg|_{\lambda=i\omega_0,\tau=\tau_0} = \operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Bigg|_{\lambda=i\omega_0,\tau=\tau_0} > 0. \tag{52}$$

In the following, we continue to study the stability switches of the rumor-prevailing equilibrium point E^*. If (G_2) holds, there must be two positive roots of (41), defined by $z_1 = \omega_1^2$ and $z_2 = \omega_2^2$, where $z_1 > z_2, \omega_1 > \omega_2$ and $F'(\omega_2) < 0, F'(\omega_1) > 0$. As same as (44) and (52), we obtain that

$$\begin{aligned} \tau_1^p =& \frac{1}{\omega_1}\arccos\frac{-PQ^2 - \frac{Q\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} - P\omega_1^2}{(\omega_1^2 + Q^2)\, b} + \frac{2p\pi}{\omega_1}, \\ & p = 0, 1, 2, \cdots, \\ \tau_2^q =& \frac{1}{\omega_2}\arccos\frac{-PQ^2 - \frac{Q\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} - P\omega_2^2}{(\omega_2^2 + Q^2)\, b} + \frac{2q\pi}{\omega_2}, \\ & q = 0, 1, 2, \cdots, \end{aligned} \tag{53}$$

and assume that $\lambda_1 = v_1 + i\omega_1, \lambda_2 = v_2 + i\omega_2$, we get

$$\begin{aligned} &\operatorname{Re}\left(\frac{\mathrm{d}\lambda_1}{\mathrm{d}\tau}\right)^{-1}\Bigg|_{\lambda_1=i\omega_1,\tau=\tau_1^p} = \frac{F'(\omega_1^2)}{\omega_1^2 b^2(\omega_1^2 + Q^2)} > 0, \\ &\text{namely, } \frac{\mathrm{d}(\operatorname{Re}\lambda_1)}{\mathrm{d}\tau}\Bigg|_{\lambda_1=i\omega_1,\tau=\tau_1^p} > 0, p = 0, 1, 2, \cdots, \\ &\operatorname{Re}\left(\frac{\mathrm{d}\lambda_2}{\mathrm{d}\tau}\right)^{-1}\Bigg|_{\lambda_2=i\omega_2,\tau=\tau_2^q} = \frac{F'(\omega_2^2)}{\omega_2^2 b^2(\omega_2^2 + Q^2)} < 0, \\ &\text{namely, } \frac{\mathrm{d}(\operatorname{Re}\lambda_2)}{\mathrm{d}\tau}\Bigg|_{\lambda_2=i\omega_2,\tau=\tau_2^q} < 0, q = 0, 1, 2, \cdots. \end{aligned} \tag{54}$$

Lemma 2. If $\omega_1 > \omega_2$, then $\tau_1^i < \tau_2^i$.

Proof: Define

$$f(\omega) = \frac{-PQ^2 - \frac{Q\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} - P\omega^2}{(\omega^2 + Q^2)\, b},$$
$$g(\omega) = \arccos f(\omega), G(\omega) = \frac{1}{\omega}g(\omega).$$

Then

$$\begin{aligned} &f'(\omega) \\ =& \frac{-2P\omega(\omega^2 + Q^2)b + 2b\omega\left(PQ^2 + \frac{Q\beta^2\phi_2 I^* S^*}{(1+\xi S^*)^3} + P\omega^2\right)}{(\omega^2 + Q^2)^2 b^2} \\ =& \frac{2Q\frac{\beta^2\phi_2 S^* I^*}{(1+\xi I^*)^3}\omega}{(\omega^2 + Q^2)^2 b}, \end{aligned}$$

and we know that $Q > 0$ and $b > 0$,

$$f'(\omega) = \frac{2Q\frac{Q\beta^2\phi_2 I^* S^*}{(1+\xi I^*)^3}\omega}{\left(\omega^2+Q^2\right)^2 b} > 0.$$

Thus, $f(\omega)$ is a monotonically increasing function. It can be readily deduced that $g(\omega)$ is a monotonically decreasing function. Then

$$G'(\omega) = -\frac{1}{\omega^2}g(\omega) + \frac{1}{\omega}g'(\omega) < 0.$$

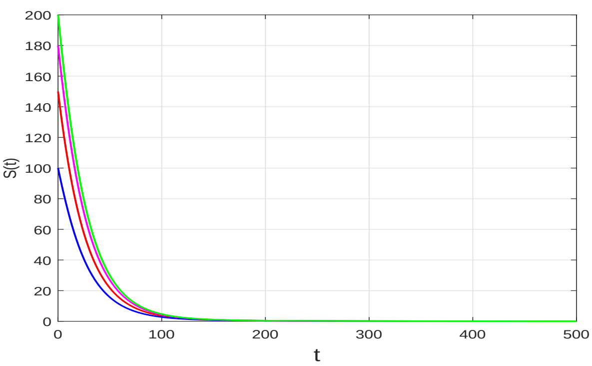
Therefore, $\tau_1^i < \tau_2^i$ when $\omega_1 > \omega_2$. This proves the lemma. ■

Theorem 8. If $\mathcal{R}_0 > 1$ holds, the following statements are true for system (19).
(1) If (G_3) holds, the rumor-prevailing equilibrium point E^* of system (19) is locally asymptotically stable for all $\tau \in [0, \infty)$.
(2) If condition (G_1) holds, then for $\tau \in [0, \tau_0)$, the rumor - prevailing equilibrium E^* of system (19) is locally asymptotically stable, whereas it becomes unstable when $\tau > \tau_0$. Specifically, system (19) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$.

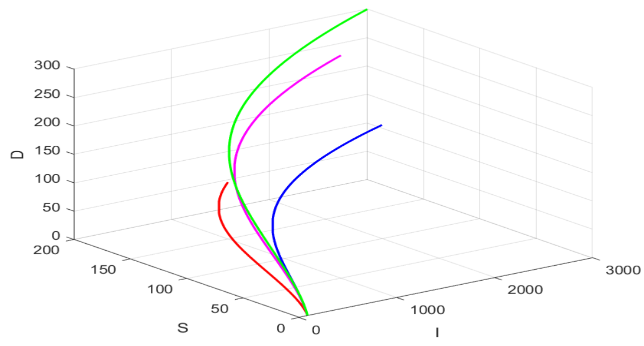
VI. Numerical Simulations

Example 1. The stability of rumor free equilibrium.

System (2) adopts the following parameters: $\Pi = 10, \beta = 0.002, \phi_1 = 0.1, \phi_2 = 0.2, \xi = 1, \delta = 0.01, m = 0.02, b = 0.01$, and $v = 0.1$. Through calculation, it can be shown that when $\mathcal{R}_0 = 0.9091 < 1$, system (2) has a unique rumor - free equilibrium point $E^0 = (\frac{\Pi}{\delta+v}, 0)$. Additionally, it can be shown that the rumor - free equilibrium point is globally asymptotically stable when $\mathcal{R}_0 = 0.9091 < 1$. To validate the global stability of the equilibrium point, we choose several parameter sets with distinct initial values. As depicted in Figure 3(a), all trajectories converge to the equilibrium, with $S(t)$ approaching zero. Figure 3(b) clearly illustrates that $I(t)$ asymptotically converges to $\frac{\Pi}{\delta+v}$, confirming the local asymptotic stability of system (2).



(a)

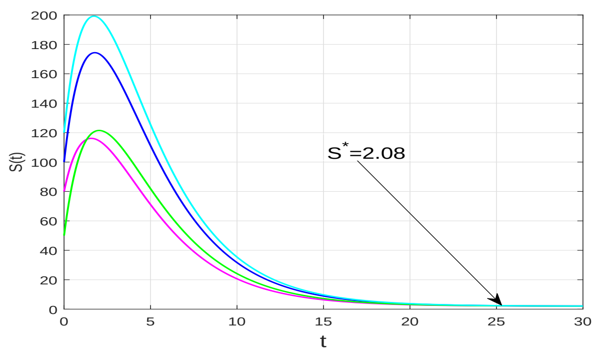


(b)

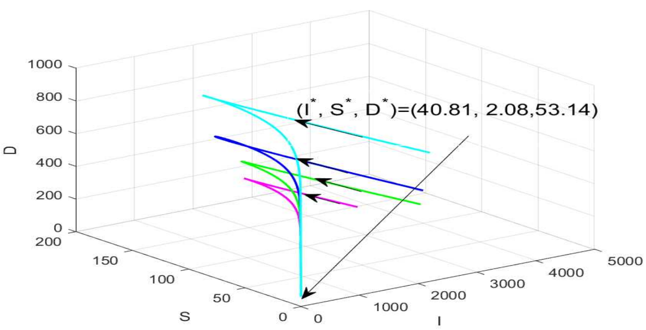
Figure 3. Dynamical behaviors of system (2) with $\mathcal{R}_0 < 1$.

Example 2. The stability of the rumor-existence equilibrium.

Similarly, considering the system with the following parameters $\Pi = 10, \beta = 0.2, \phi_1 = 0.1, \phi_2 = 0.2, \xi = 1, \delta = 0.01, m = 0.02, b = 0.5$, and $v = 0.1$. We can obtain that $\mathcal{R}_0 = 6.8611 > 1$. Then, according to Theorem 3, the system has a unique rumor - equilibrium point E^*.This equilibrium point E^* is locally asymptotically stable when $tr(A) < 0$ and $det(A) > 0$. As illustrated in Figure 4(a), all trajectories approach a stable state. Moreover, as depicted in Figure 4(b), all trajectories converge to the equilibrium point E^*. This observation confirms the local asymptotic stability of E^*.



(a)

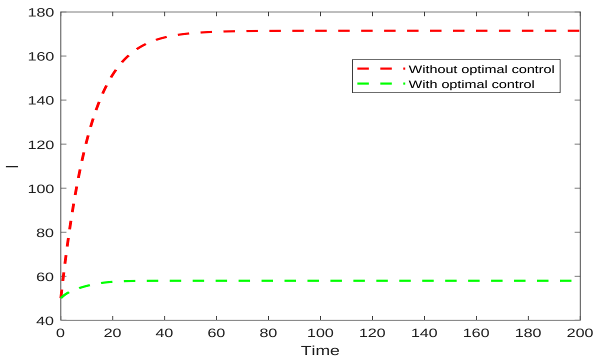


(b)

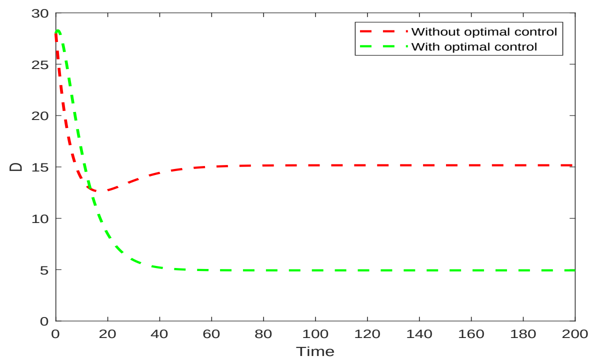
Figure 4. Dynamical behaviors of system (2) with $\mathcal{R}_0 > 1$.

Example 3.

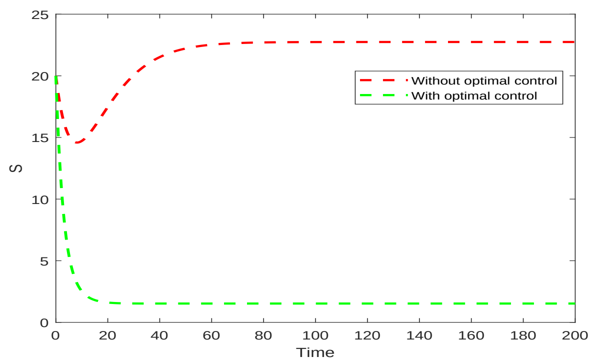
For widely spreading rumors, we propose an optimal control mechanism to prevent their spread. This mechanism involves intensifying popular science education, deleting influential posts affected by rumors to reduce the percentage of media influence, and intensifying the punishment for malicious disseminators. These control optimization measures aim to minimize the scale of rumors. Figure 5 clearly illustrates the control effect. Figure 5(a) represents the dynamics of susceptible individuals with control (green dotted) and without (optimal) control (red dotted). Similarly, Figures 5(b), (c), and (d) show the dynamics of individuals who come into contact with rumors but do not spread them, those who spread rumors, and those who do not believe them, both with control (green dotted) and without control (red dotted). This clearly indicates that after joining the control group, the number of people susceptible to rumors, those who believe rumors but do not spread them, and those who spread rumors are greatly reduced, while the number of people recovering from rumors increases. This is consistent with our goal of suppressing the spread of rumors.



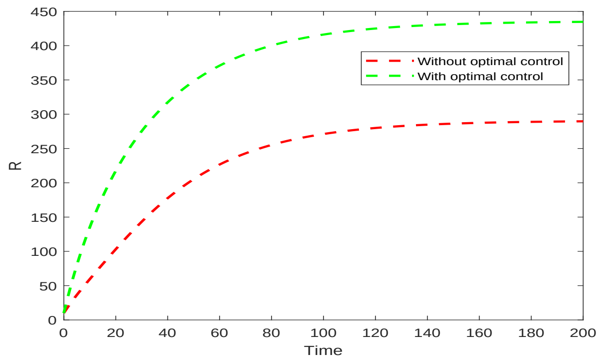
(a)



(b)



(c)



(d)

Figure 5. Dynamics of four population classes in controlled and uncontrolled.

Example 4.

Choose a set of parameters: $\Pi = 20, \beta = 0.1, \phi_1 = 0.1, \phi_2 = 0.2, \xi = 1, \delta = 0.2, m = 0.02, b = 0.5$, and $v = 0.5$. Then, system (18) can be expressed in the following specific form:

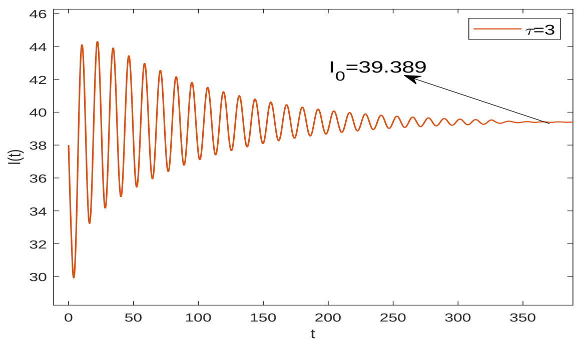
$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = 20 - \dfrac{0.1I(t)S(t)}{1+S(t)} - 0.5I(t) - 0.2I(t), \\ \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = 0.2\dfrac{0.1I(t)S(t)}{1+S(t)} - 0.02S(t) - 0.5S(t-\tau) - 0.2S(t), \end{cases}$$

Where $\tau = 3 < \tau_1 = 3.143$. By calculating, we can get

$$\mathcal{R}_0 = 0.6098 < 1,$$

$$E^0 \approx (39.389, 0).$$

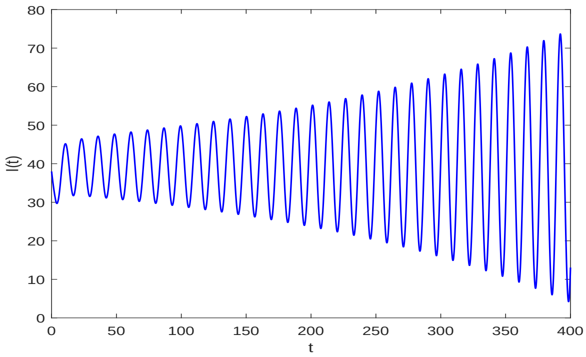
For model (19), the waveform diagrams are depicted in Figures 6 and 7. Analysis shows that the curves of each state node asymptotically converge to a straight line, confirming the local asymptotic stability of the model at E^0.



(a)

Figure 6. The waveform diagram of model (55) when $\tau = 3 < 3.143$.

However, when $\tau = 3.2 > \tau_1 = 3.143$, the waveform and phase diagrams of each node are shown respectively in Figure 9. Under this condition, the waveform diagram exhibits periodic oscillations, indicating that the model becomes unstable at E^0 and undergoes a Hopf bifurcation at this equilibrium point.



(b)

Figure 7. The waveform diagram of model (55) when $\tau = 3.2 > 3.143$.

Example 5.

Choose a set of parameters: $\Pi = 60, \beta = 0.3, \phi_1 = 0.1, \phi_2 = 0.2, \xi = 1, \delta = 0.2, m = 0.02, b = 0.6$, and $v = 0.3$. Then, system (18) can be expressed in the following specific form:

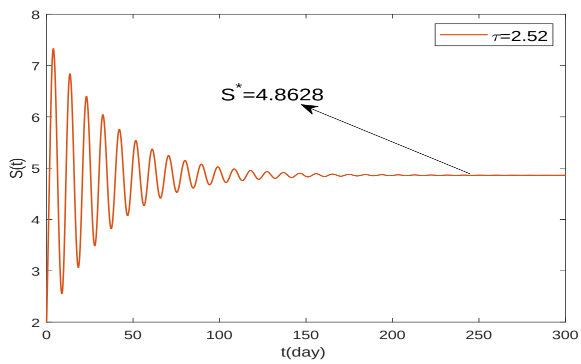
$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = 60 - \dfrac{0.3I(t)S(t)}{1+S(t)} - 0.3I(t) - 0.2I(t), \\ \dfrac{\mathrm{d}D(t)}{\mathrm{d}t} = 0.1\dfrac{0.3I(t)S(t)}{1+S(t)} - 0.02D(t) \\ \qquad + 0.6S(t-\tau) - 0.2D(t), \\ \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = 0.2\dfrac{0.3I(t)S(t)}{1+S(t)} - 0.02S(t) \\ \qquad - 0.6S(t-\tau) - 0.2S(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = (1-0.1-0.2)\dfrac{0.3I(t)S(t)}{1+S(t)} \\ \qquad + 0.02D(t) + 0.02S(t) + 0.3I(t) - 0.2R(t), \end{cases} \tag{55}$$

where $\tau = 2.52 < \tau_0 = 2.71$, By calculating, we can get

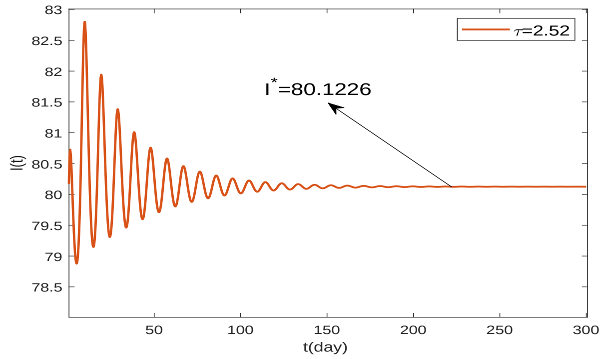
$$\mathcal{R}_0 = 8.7805 > 1,$$

$$E^* \approx (80.1226, 22.3248, 4.8628, 192.687).$$

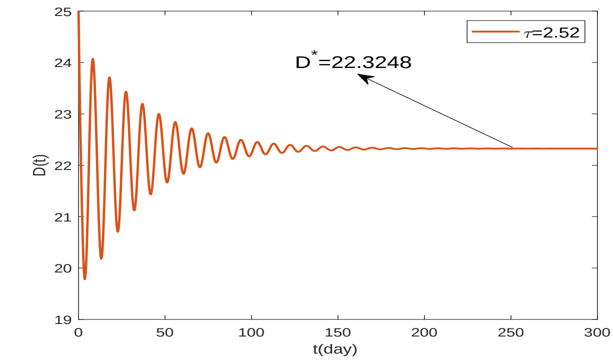
The waveform and phase diagrams of the nodes in model (18) are presented in Figures 8, 9, and 10. The waveform curve of each state node asymptotically converges to a straight line, and the phase diagram converges to the limit point. This observation indicates that the model is locally asymptotically stable at E^* .



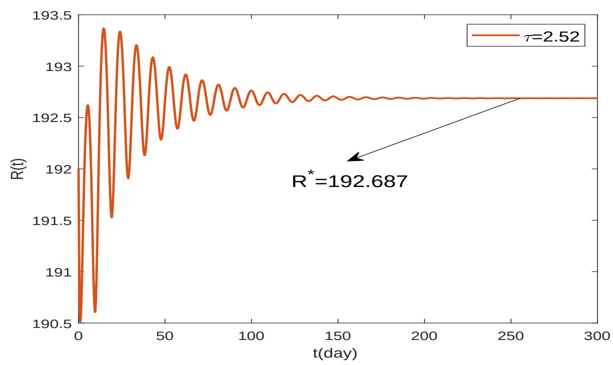
(a)



(b)

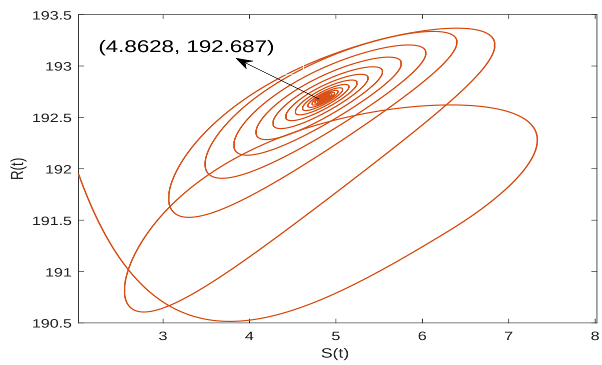


(c)

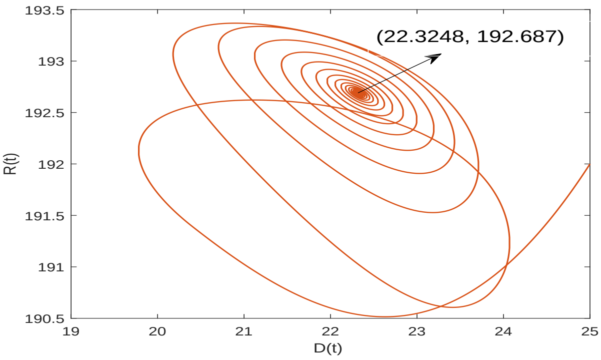


(d)

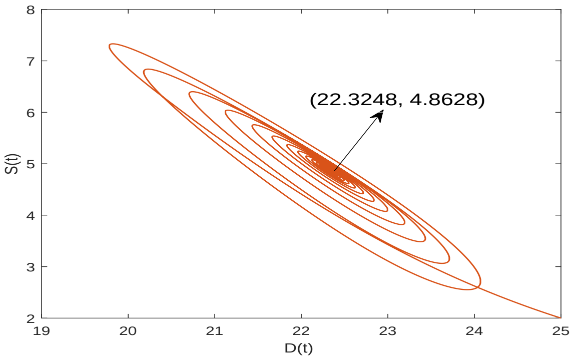
Figure 8. The waveform diagram of model (55) when $\tau = 2.52 < 2.71$.



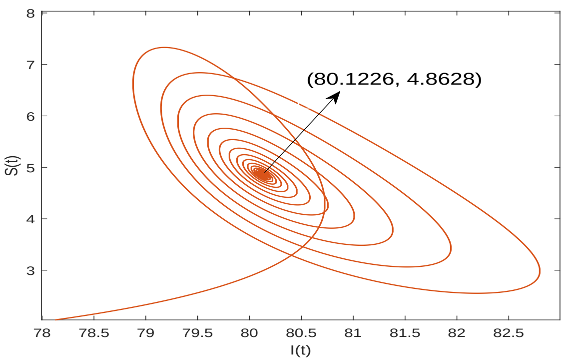
(a)



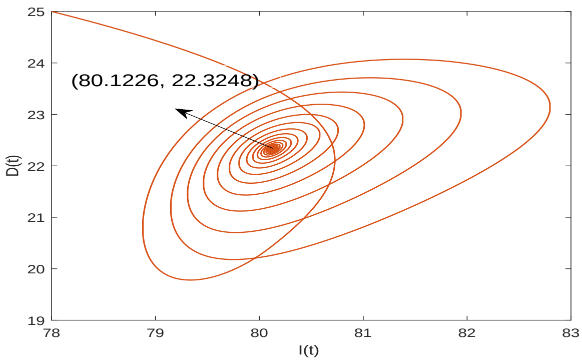
(b)



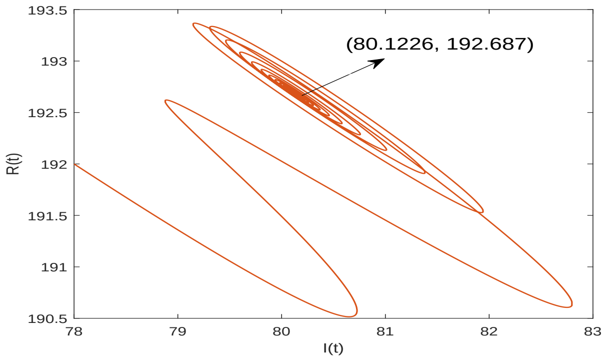
(c)



(d)

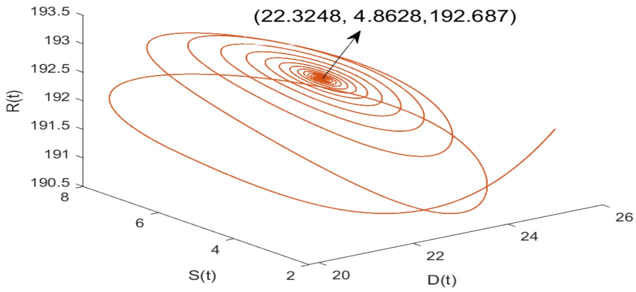


(e)

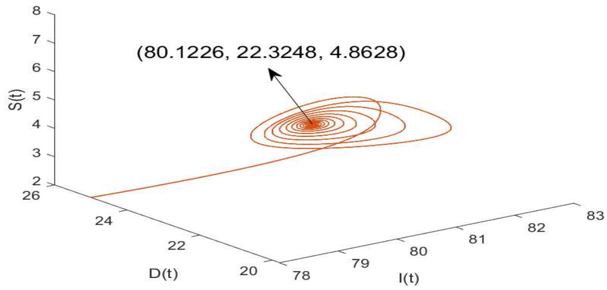


(f)

Figure 9. Two-dimensional phase diagram of model (55) when $\tau = 2.52 < \tau_0 = 2.71$.



(a)



(b)

Figure 10. 3D phase diagram of model (55) when $\tau = 2.52 < \tau_0 = 2.71$.

However, when $\tau = 2.73 > \tau_0 = 2.71$, the waveform and phase diagrams of each node are shown in Figures 11, 12, and 13 respectively. Under this condition, the waveform diagram exhibits periodic oscillations, and a limit cycle emerges on the phase - diagram curve. This indicates that the model becomes unstable at E^*, and a Hopf bifurcation occurs at this equilibrium point of the system.

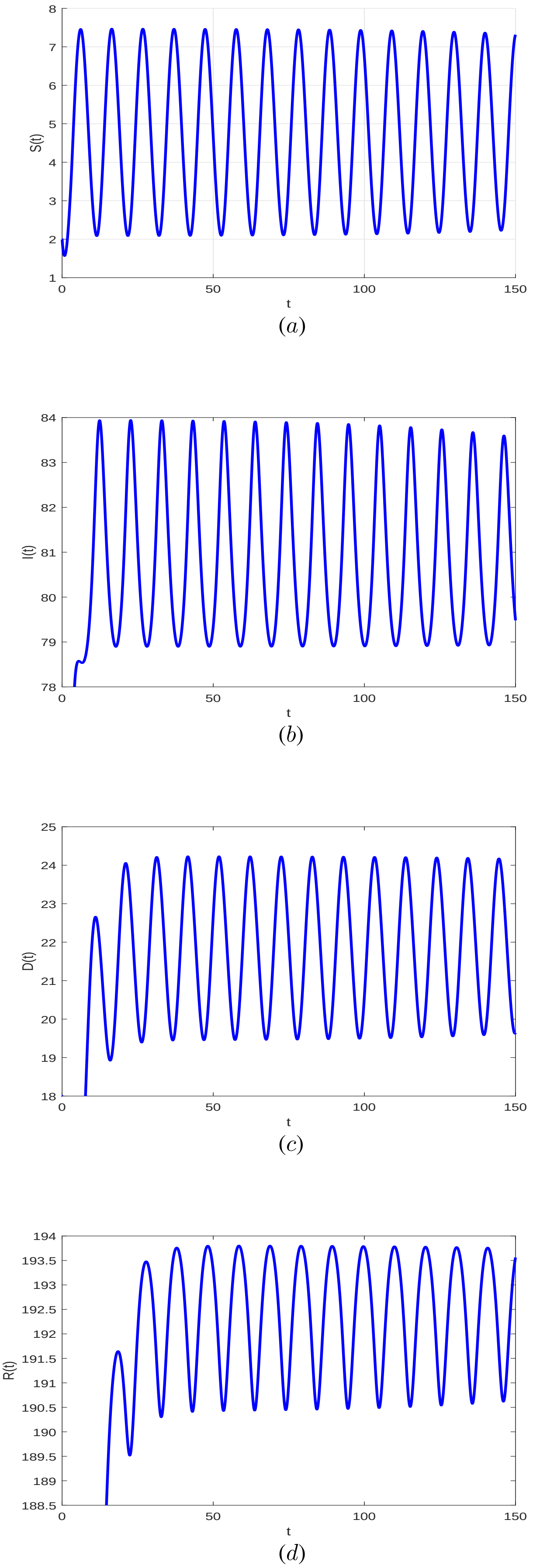
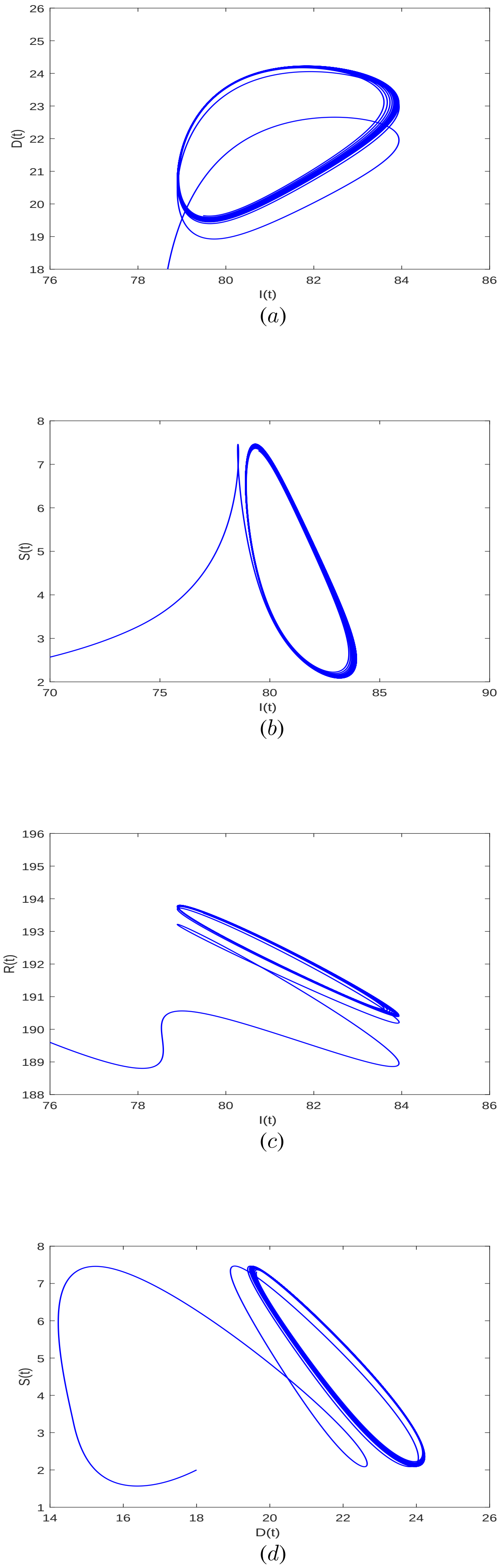
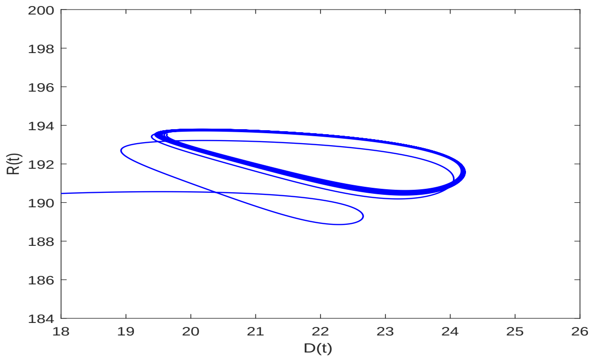
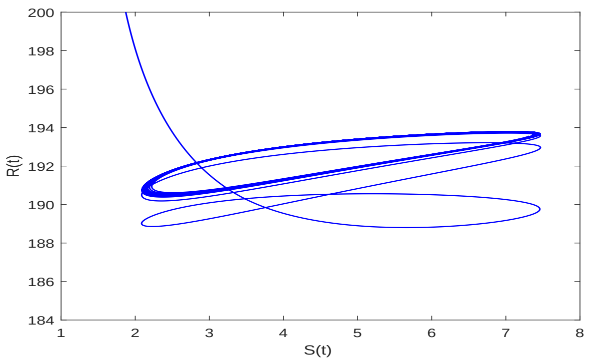


Figure 11. The waveform diagram of model (55) when $\tau = 2.73 > 2.71$.



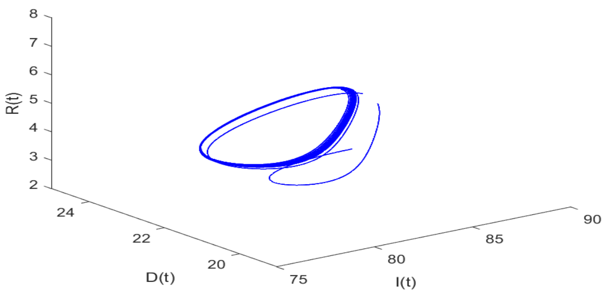


(e)

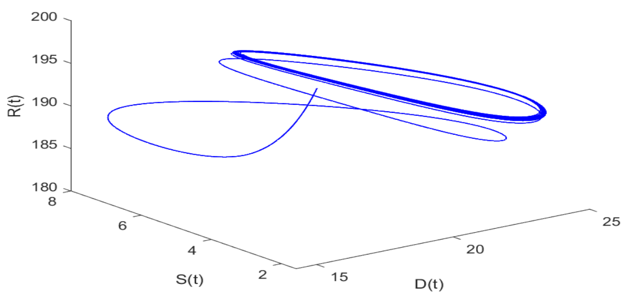


(f)

Figure 12. Two-dimensional phase diagram of model (55) when $\tau = 2.73 > \tau_0 = 2.71$.



(a)



(b)

Figure 13. 3D phase diagram of model (55) when $\tau = 2.73 > \tau_0 = 2.71$.

Example 6.

Considering the time delay for official media to deny rumors and for social platforms to ban accounts when unexpected events cause rumors to spread, we used $\tau = 2$, $\tau = 2.4$ and $\tau = 2.6$ to draw the $S(t)$ trajectory, as shown in Figure 14, to reflect the impact of time delay on rumor spread, keeping other parameters unchanged. Through careful observation, we found that the larger the delay, the greater the amplitude of the orbit oscillation and the longer it takes to reach a stable state, which is not conducive to rumor control. Therefore, minimizing the societal harm caused by rumor propagation requires official media to release rumor-refuting news and social platforms to ban user accounts spreading malicious rumors as quickly as possible.

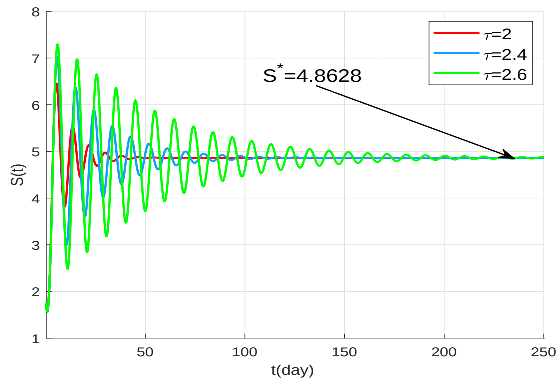
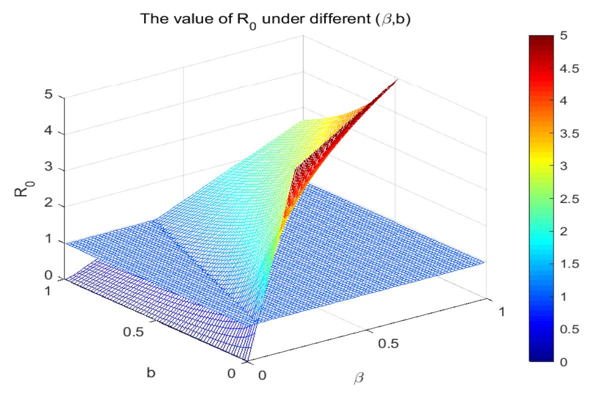


Figure 14. The trajectories of S(t) with $\mathcal{R}_0 > 1$; $\tau = 2, \tau = 2.4$ and $\tau = 2.6$.

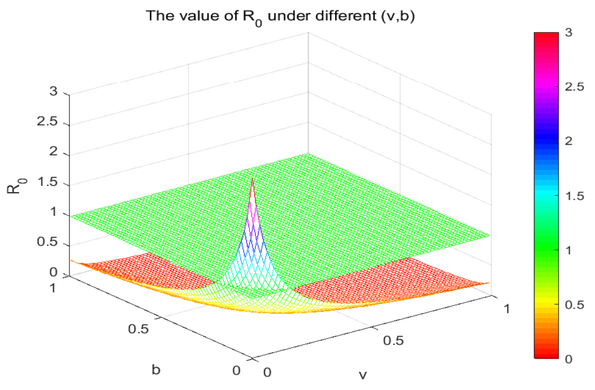
Example 7. The relationship between the basic reproduction number $\mathcal{R}_0$ and the parameters β, υ, and b.

The basic reproduction number $\mathcal{R}_0$ directly determines whether rumors will spread. In the context of rumors spreading due to unexpected events, the parameter υ represents the probability of official media refuting rumors, b represents the probability of social platforms banning accounts, and β represents the transmission rate from disseminators to susceptible individuals. Analyzing the relationship between $\mathcal{R}_0$, υ, b, and β is crucial for understanding the rumor system. As shown in Figure 15, R_0 is positively correlated with β and b. This indicates that in online networks, the timely banning of users harmful remarks by social media and a decrease in the transmission rate β from disseminators to susceptible individuals reduce $\mathcal{R}_0$, thereby inhibiting the spread of rumors. Furthermore, as shown in Figure 15, careful observation indicates that objective, open, and transparent rumor-refuting news released by official media over time is negatively correlated with $\mathcal{R}_0$. This figure also shows the effects of b and υ on $\mathcal{R}_0$, providing practical suggestions for curbing the spread of rumors.



(a)

Figure 15. The relation between b, β and $\mathcal{R}_0$.



(b)

Figure 16. The relation between b, v and $\mathcal{R}_0$.

Example 8. Model application.

There is a particularly attention-grabbing incident that illustrates this phenomenon. On December 22, 2017, shocking news appeared on Sina Weibo: tourists angered elephants by pranking them in Thailand, and a Chongqing tour guide was trampled to death while attempting to save them. The incident was widely reported, and immediately aroused heated discussions among all sectors of the community. However, the narrative was reversed on December 25, when the Thai police released an official message stating that there had been no such incident of tourists provoking elephants by dragging their tails. For simplicity, we refer to this news as the ”rumor of elephants trampling tourists.”

To verify the validity of the constructed model, we selected real data from the literature [11]. For the specific rumor of elephant trampling tourists, we conducted meticulous statistical work, focusing on recording the number of times the rumor was posted every after its outbreak. By analyzing these statistics and combining them with the corresponding graphs (as shown in Figure 17), we can clearly observed a remarkable phenomenon: the rumor spread effectively between the 8th and 22nd hours. At the 8th hour, the rumor entered a rapid spreading stage, likely owing to the initial exposure of the event and involvement of some key communication nodes. However, by the 22nd hour, its spreading effectiveness gradually diminished, perhaps because of to the gradual revelation of the truth, decline of public attention, or the emergence of other new hotspots.

TABLE I
THE PARAMETERS EMPLOYED IN EXAMPLE 7.

Parameters	Value	Source
Π	14429	[11]
β	1.02×10^{-4}	Fitted
ξ	0.0001	Fitted
δ	0.045	[11]
m	0.045	[11]
b	0.4	Fitted
ϕ_1	0.49	Estimated
ϕ_2	0.16	Estimated
v	0.5	Fitted

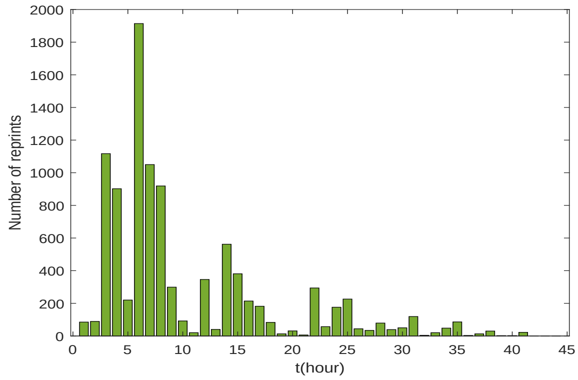


Figure 17. The number of reprints, date from [11].

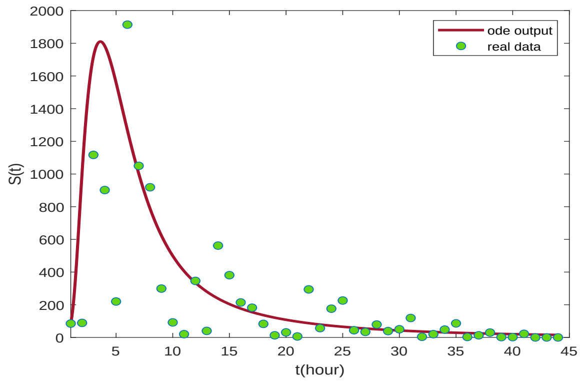


Figure 18. The evolution of rumor Spreaders and real data.

The key result of $\mathcal{R}_0 = 0.8818 < 1$ was computed. Based on Theorem 2, the rumor-free equilibrium exhibits global asymptotic stability. This characteristic is of great significance because it clearly indicates that rumors eventually disappear. From a realistic perspective, this is perfectly consistent with actual situations. Using the parameters set in Table 1 and the real data in Figure 17, the evolution of the specific event of “rumor of elephant trampling on tourists” is illustrated in Figure 18. Notably, the simulation results of the “rumor-free” equilibrium point show the global asymptotic stability. Notably, the simulation results showed excellent consistency with real data. This suggests that that the proposed model and analysis method can effectively simulate and predict the evolution of rumor spread. In this case, it accurately reflects the entire dynamic change of the ”elephant trampling on tourists rumor,” from its emergence to its gradual disappearance. This consistency not only verifies the scientific validity of our research methodology but also provides strong support and reference for further in-depth investigations of the rumor-propagation dynamics and the formulation of corresponding prevention and control strategies. With the emergence of new hotspots, the effectiveness of spreading rumors has gradually diminished.

VII. Conclusion

In this study, a rumor propagation model with a saturation propagation rate is considered. First, the next-generation matrix method is used to calculate the basic reproduction number $\mathcal{R}_0$ of network rumors, establishing the threshold for rumor spread in social networks. Second, the global stability of the model is proven by constructing the Lyapunov function, and the local asymptotic stability of the two equilibrium points is analyzed using the Routh-Hurwitz stability criterion and linearization technique. Based on this analysis, the rumor-free equilibrium point E^0 is found to be locally asymptotically stable when $\mathcal{R}_0 < 1$, globally asymptotically stable when $\beta\Pi \leq \delta(\delta + v)$, and locally and globally asymptotically stable when $\mathcal{R}_0 > 1$. By applying the Pontryagin maximum principle, real-time optimal control was achieved at the desired time, aiming to prevent rumors from spreading within the expected time at minimal cost in online social networks. This was accomplished through measures such as improving public media literacy, strengthening media supervision, enhancing rumor monitoring and response, and timely issuance of authoritative information to refute rumors. In addition, considering practical scenarios, network time delay τ was added to system (2), and the conditions for the existence of bifurcation were obtained by selecting the time delay as the Hopf bifurcation parameter. Finally, the numerical simulation and practical application have strongly verified the scientificity, accuracy and practicability of the relevant conclusions obtained, which provide a solid foundation for our further in-depth study of the rumor propagation phenomenon.

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