Tripolar Fuzzy Bi-Ideals in Semigroups

Pannawit Khamrot, Aiyared Iampan, Thiti Gaketem

***Abstract*—In 2018, Rao gave expanded the concept of fuzzy sets, bipolar fuzzy sets, and intuitionistic fuzzy sets to tripolar fuzzy sets. This paper presents the concept and examines the qualifications of tripolar fuzzy bi-ideals of semigroups. Finally, we characterized regular semigroups in terms of tripolar fuzzy bi-ideals.**

***Index Terms*—Regular, Intra-regular, Tripolar fuzzy sets, Tripolar fuzzy bi-ideals**

I. Introduction

THE THEORY of fuzzy sets is the most appropriate theory for dealing with uncertainty and was introduced by Zadeh [1] in 1965. After the concept of fuzzy sets, several researchers have generalized of the notions of fuzzy sets with huge applications in computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machines, graph theory, logic, operations research, and many branches of pure and applied mathematics. In 1979, N. Kuroki [2] investigated the properties of fuzzy ideals and other types of semigroups. In 1986, K. T. Attsnsov [3] gave the concept and studied the properties of intuitionistic fuzzy sets. The bipolar fuzzy sets are an extension of fuzzy sets whose memberely degree range is $[-1,1]$ studied by Zhang in 1994, [4] In 2000, K. M. Lee [5] developed knowledge of bipolar fuzzy sets extension to algebraic systems. In addition, Gaketem and Khamrot [6] studied bipolar weakly interior ideals in semigroups. Gaketem et al. [7] expand cubic bipolar fuzzy subsemigroups and ideals in semigroups. In 2018, M. M. K. Rao [8] was introduced to the concept of tripolar fuzzy set, which is a generalization of fuzzy sets, bipolar fuzzy sets, and intuitionistic fuzzy sets. In the same year, M. M. K. Rao and B. Venkateswarlu [9] studied tripolar fuzzy ideals Γ-semirings. In 2020, M. M. K. Rao and B. Venkateswarlu [10] studied tripolar fuzzy soft interior ideals Γ-semirings. In 2022, N. Wattansiripong et al. [11] present properties of tripolar fuzzy pure ideals in ordered semigroups. In the same year, N. Wattansiripong et al.[12] gave the concept of tripolar fuzzy interior ideals in ordered semigroups and characterized semisimple ordered semigroups in terms of tripolar fuzzy interior ideals. In 2024, T. Promai et al. [13] studied tripolar fuzzy ideals in semigroups. In 2025, P. Khamrot et al. [14] find conditions of types of tripolar fuzzy ideals in semigroups.

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P. Khamrot is a lecturer at the Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University Technology Lanna Phitsanulok, Phitsanulok, Thailand. (e-mail: pk_g@rmutl.ac.th).

A. Iampan is a lecturer at the Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand. (e-mail: aiyared.ia@up.ac.th).

T. Gaketem is a lecturer at the Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand. (corresponding author to provide email: thiti.ga@up.ac.th).

In this paper, we give the definition of tripolar fuzzy bi-ideals in semigroups. We discuss the properties of tripolar fuzzy bi-ideals in semigroups together we will prove the relationship between bi-ideals and tripolar fuzzy bi-ideals. Moreover, we characterized regular semigroups in terms of tripolar fuzzy bi-ideals.

II. Preliminaries

In this section, we will recall some concepts and results, which help us study the next sections.

Definition 2.1. *A non-empty subset $\ddot{\mathcal{B}}$ of an SG (SG) $\ddot{\mathcal{S}}$ is called*

(1) *a subsemigroup (SSG) of $\ddot{\mathcal{S}}$ if $\ddot{\mathcal{B}}^2 \subseteq \ddot{\mathcal{B}}$,*
(2) *a left ideal (LI) of $\ddot{\mathcal{S}}$ if $\ddot{\mathcal{S}}\ddot{\mathcal{B}} \subseteq \ddot{\mathcal{B}}$*
(3) *a right ideal (RI) of $\ddot{\mathcal{S}}$ if $\ddot{\mathcal{B}}\ddot{\mathcal{S}} \subseteq \ddot{\mathcal{B}}$.*
(4) *an* ideal *(ID) $\ddot{\mathcal{B}}$ of $\ddot{\mathcal{S}}$ if it consistent with the* (2) *and* (3).
(5) *a* generalized bi-ideal *(GBI) of $\ddot{\mathcal{S}}$ if $\ddot{\mathcal{B}}\ddot{\mathcal{S}}\ddot{\mathcal{B}} \subseteq \ddot{\mathcal{B}}$.*
(6) *a* bi-ideal *(BI) of $\ddot{\mathcal{S}}$ if it consistent with SSG and GBI.*

For any $\ddot{h}_i \in [0,1]$, $i \in \ddot{\ddot{\mathcal{F}}}$, define

$$\underset{i\in\ddot{\mathcal{F}}}{\vee}\ddot{h}_i := \sup_{i\in\ddot{\mathcal{F}}}\{\ddot{h}_i\} \quad \text{and} \quad \underset{i\in\ddot{\mathcal{F}}}{\wedge}\ddot{h}_i := \inf_{i\in\ddot{\mathcal{F}}}\{\ddot{h}_i\}.$$

We see that for any $\ddot{h}, \ddot{r} \in [0,1]$, we have

$$\ddot{h} \vee \ddot{r} = \max\{\ddot{h}, \ddot{r}\} \quad \text{and} \quad \ddot{h} \wedge \ddot{r} = \min\{\ddot{h}, \ddot{r}\}.$$

A *fuzzy set (FS)* of a non-empty set $\ddot{\mathcal{E}}$ is a function $\ddot{\rho} : \ddot{\mathcal{E}} \to [0,1]$.

For any two FSs $\ddot{\rho}$ and $\ddot{\nu}$ of a non-empty set $\ddot{\mathcal{E}}$, define the symbol as follows:

(1) $\ddot{\rho} \leq \ddot{\nu} \Leftrightarrow \ddot{\rho}(\ddot{h}) \leq \ddot{\nu}(\ddot{h})$ for all $\ddot{h} \in \ddot{\mathcal{E}}$,
(2) $\ddot{\rho} = \ddot{\nu} \Leftrightarrow \ddot{\rho} \subseteq \ddot{\nu}$ and $\ddot{\nu} \subseteq \ddot{\rho}$,
(3) $(\ddot{\rho} \wedge \ddot{\nu})(\ddot{h}) = \ddot{\rho}(\ddot{h}) \wedge \ddot{\nu}(\ddot{h})$ and $(\ddot{\rho} \vee \ddot{\nu})(\ddot{h}) = \ddot{\rho}(\ddot{h}) \vee \ddot{\nu}(\ddot{h})$ for all $\ddot{h} \in \ddot{\mathcal{E}}$, For the symbol $\ddot{\rho} \leq \ddot{\nu}$, we mean $\ddot{\nu} \leq \ddot{\rho}$.

Let $\ddot{k}$ be an element of an SG $\ddot{\mathcal{S}}$. Then $\ddot{\mathcal{F}}_{\ddot{k}} := \{(\ddot{m}, \ddot{n}) \in \ddot{\mathcal{S}} \times \ddot{\mathcal{S}} \mid \ddot{k} = \ddot{m}\ddot{n}\}$.

For any two FSs $\ddot{\rho}$ and $\ddot{\nu}$ of a semigroup $\ddot{\mathcal{S}}$. The product of FSs $\ddot{\rho}$ and $\ddot{\nu}$ of $\ddot{\mathcal{S}}$ is defined as follows, for all $h \in \ddot{\mathcal{S}}$

$$(\ddot{\rho} \circ \ddot{\nu})(\ddot{h}) = \begin{cases} \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathcal{F}}_{\ddot{k}}} \{\ddot{\rho}(\ddot{m}) \wedge \ddot{\nu}(\ddot{n})\} & \text{if } \ddot{\mathcal{F}}_{\ddot{k}} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic function of a subset $\ddot{\mathcal{B}}$ of a non-empty set $\ddot{\mathcal{E}}$ is a fuzzy set of $\ddot{\mathcal{E}}$

$$\ddot{\lambda}_{\ddot{\mathcal{B}}}(\ddot{h}) = \begin{cases} 1 & \text{if } \ddot{h} \in \ddot{\mathcal{B}}, \\ 0 & \text{if } \ddot{h} \notin \ddot{\mathcal{B}}. \end{cases}$$

for all $\ddot{h} \in \ddot{\mathcal{S}}$.

Lemma 2.2. *[2] Let $\ddot{\mathcal{B}}$ and $\ddot{\mathcal{L}}$ be non-empty subsets of an SG $\ddot{\mathcal{S}}$. Then the following holds.*

(1) *If* $\ddot{\mathcal{B}} \subseteq \ddot{\mathcal{L}}$, *then* $\ddot{\lambda}_{\ddot{\mathcal{B}}} \subseteq \ddot{\lambda}_{\ddot{\mathcal{L}}}$
(2) $\ddot{\lambda}_{\ddot{\mathcal{B}}} \wedge \ddot{\lambda}_{\ddot{\mathcal{L}}} = \ddot{\lambda}_{\ddot{\mathcal{B}} \cap \mathcal{L}}$.
(3) $\ddot{\lambda}_{\ddot{\mathcal{B}}} \circ \ddot{\lambda}_{\ddot{\mathcal{L}}} = \ddot{\lambda}_{\ddot{\mathcal{B}}\mathcal{L}}$.

Definition 2.3. *[2] A FS $\ddot{\rho}$ of an SG $\ddot{\mathcal{S}}$ is said to be*
(1) *a fuzzy subsemigroup (FSSG) of $\ddot{\mathcal{S}}$ if $\ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{r}) \leq \ddot{\rho}(\ddot{h}\ddot{r})$, for all $\ddot{h}, \ddot{r} \in \ddot{\mathcal{S}}$.*
(2) *a fuzzy left ideal (FLI) of $\ddot{\mathcal{S}}$ if $\ddot{\rho}(\ddot{r}) \leq \ddot{\rho}(\ddot{h}\ddot{r})$, for all $\ddot{h}, \ddot{r} \in \ddot{\mathcal{S}}$.*
(3) *a fuzzy right ideal (FRI) of $\ddot{\mathcal{S}}$ if $\ddot{\rho}(\ddot{h}) \leq \ddot{\rho}(\ddot{h}\ddot{r})$), for all $\ddot{h}, \ddot{r} \in \ddot{\mathcal{S}}$.*
(4) *a fuzzy ideal of $\ddot{\mathcal{S}}$ if it is both a FLI and a FRI of $\ddot{\mathcal{S}}$.*

Definition 2.4. *[8] The* tripolar fuzzy set *(TFS) $\mathcal{TF}$ on a non-empty set $\ddot{\mathcal{S}}$ if*

$$\mathcal{TF} := \{(\ddot{h}, \ddot{\rho}(\ddot{h}), \ddot{\nu}(\ddot{h}), \ddot{\ddot{\delta}}(\ddot{h})) \mid \ddot{h} \in \ddot{\mathcal{S}}\},$$

where $\ddot{\rho}(\ddot{h}) : \ddot{\mathcal{S}} \to [0,1]$, $\ddot{\nu}(\ddot{h}) : \ddot{\mathcal{S}} \to [0,1]$ and $\ddot{\rho}(\ddot{h}) : \ddot{\mathcal{S}} \to [-1,0]$, such that $0 \leq \ddot{\rho}(\ddot{h}) + \ddot{\nu}(\ddot{h}) \leq 1$ for all $\ddot{h} \in \ddot{\mathcal{S}}$. The membership degree $\ddot{\rho}(\ddot{h})$ characterizes the extent that the element $\ddot{\mathcal{S}}$ satisfies the property corresponding to TFS $\mathcal{TF}$ $\ddot{\nu}(\ddot{h})$ characterizes the extent to the element $\ddot{\mathcal{E}}$ satisfies the not property (irrelevant) corresponding to tripolar fuzzy set $\ddot{\rho}$, and $\ddot{\delta}(h)$ characterizes the extent that the element $\ddot{\mathcal{E}}$ satisfies the implicit counter property corresponding to TFS $\mathcal{TF}$. For simplicity $\mathcal{TF} := (\ddot{\rho}, \ddot{\nu}, \ddot{\ddot{\delta}})$ has been used for $\mathcal{TF} := \{(\ddot{h}, \ddot{\rho}(\ddot{h}), \ddot{\nu}(\ddot{h}), \ddot{\delta}(\ddot{h})) \mid \ddot{h} \in \ddot{\mathcal{S}}\}$ such that $0 \leq \ddot{\rho}(\ddot{h}) + \ddot{\nu}(\ddot{h}) \leq 1$.

The *characteristic tripolar fuzzy set* (CTFS) $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ of a non-empty subset $\ddot{\mathcal{B}}$ of $\ddot{\mathcal{S}}$ is defined as follows:

$$\ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{k}) = \begin{cases} 1 & \text{if } \ddot{k} \in \ddot{\mathcal{B}}, \\ 0 & \text{if } \ddot{k} \notin \ddot{\mathcal{B}}, \end{cases}$$

$$\ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{k}) = \begin{cases} 0 & \text{if } \ddot{k} \in \ddot{\mathcal{B}}, \\ 1 & \text{if } \ddot{k} \notin \ddot{\mathcal{B}}, \end{cases}$$

$$\ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{k}) = \begin{cases} -1 & \text{if } \ddot{k} \in \ddot{\mathcal{B}}, \\ 0 & \text{if } \ddot{k} \notin \ddot{\mathcal{B}} \end{cases}$$

for all $\ddot{k} \in \ddot{\mathcal{S}}$. In this case of $\ddot{\mathcal{B}} = \ddot{\mathcal{S}}$ defined $\mathcal{TF}_{\ddot{\mathcal{B}}} = (1, 0, -1)$.

Definition 2.5. *[13] A TFS is called a $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called a tripolar fuzzy subsemigroup (TFSSG) of $\ddot{\mathcal{S}}$ if*
(1) $\ddot{\rho}(\ddot{h}\ddot{k}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{k})$
(2) $\ddot{\nu}(\ddot{h}\ddot{k}) \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{k})$
(3) $\ddot{\delta}(\ddot{h}\ddot{k}) \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{k})$
for all $\ddot{h}, \ddot{k} \in \ddot{\mathcal{S}}$.

Definition 2.6. *[13] A TFS is called a $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called a tripolar fuzzy left ideal (TFLI) of $\ddot{\mathcal{S}}$ if*
(1) $\ddot{\rho}(\ddot{h}\ddot{k}) \geq \ddot{\rho}(\ddot{k})$
(2) $\ddot{\nu}(\ddot{h}\ddot{k}) \leq \ddot{\nu}(\ddot{k})$
(3) $\ddot{\delta}(\ddot{h}\ddot{k}) \leq \ddot{\delta}(\ddot{k})$
for all $\ddot{h}, \ddot{k} \in \ddot{\mathcal{S}}$.

Definition 2.7. *[13] A TFS is called a $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called a tripolar fuzzy right ideal (TFRI) of $\ddot{\mathcal{S}}$ if*
(1) $\ddot{\rho}(\ddot{h}\ddot{k}) \geq \ddot{\rho}(\ddot{h})$
(2) $\ddot{\nu}(\ddot{h}\ddot{k}) \leq \ddot{\nu}(\ddot{h})$
(3) $\ddot{\delta}(\ddot{h}\ddot{k}) \leq \ddot{\delta}(\ddot{h})$
for all $\ddot{h}, \ddot{k} \in \ddot{\mathcal{S}}$.

Definition 2.8. *[13] A TFS is called a $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called a tripolar fuzzy ideal (TFID) of $\ddot{\mathcal{S}}$ if it is both TFLI and TFRI of $\ddot{\mathcal{S}}$.*

Example 2.9. *[13] Let $\ddot{\mathcal{S}} = \{\ddot{w}, \ddot{x}, \ddot{y}, \ddot{z}\}$ be an SG with the following Cayley table:*

$\cdot$	$\ddot{w}$	$\ddot{x}$	$\ddot{y}$	$\ddot{z}$
$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$
$\ddot{x}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$
$\ddot{y}$	$\ddot{w}$	$\ddot{w}$	$\ddot{x}$	$\ddot{w}$
$\ddot{z}$	$\ddot{w}$	$\ddot{w}$	$\ddot{x}$	$\ddot{x}$

Define $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\ddot{\delta}})$ by $\ddot{\rho}(\ddot{w}) = 0.4$, $\ddot{\rho}(\ddot{x}) = 0.7$, $\ddot{\rho}(\ddot{y}) = 0.8$, $\ddot{\rho}(\ddot{z}) = 0.3$; $\ddot{\nu}(\ddot{w}) = 0.5$, $\ddot{\rho}(\ddot{x}) = 0.2$, $\ddot{\rho}(\ddot{y}) = 0.1$, $\ddot{\rho}(\ddot{z}) = 0.4$ and $\ddot{\delta}(\ddot{w}) = -0.7$, $\ddot{\delta}(\ddot{x}) = -0.5$, $\ddot{\delta}(\ddot{y}) = -0.3$, $\ddot{\delta}(\ddot{z}) = -0.3$. Then $\mathcal{TF}$ is a TFLI of $\ddot{\mathcal{S}}$.

Theorem 2.10. *[13] Let $\ddot{\mathcal{B}}$ be a non-empty subset of an SG $\ddot{\mathcal{S}}$. Then*
(1) *$\ddot{\mathcal{B}}$ is a SSG of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ is a TFSSG of $\ddot{\mathcal{S}}$.*
(2) *$\ddot{\mathcal{B}}$ is a LI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ is a TFLI of $\ddot{\mathcal{S}}$.*
(3) *$\ddot{\mathcal{B}}$ is a RI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ is a TFRI of $\ddot{\mathcal{S}}$.*
(4) *$\ddot{\mathcal{B}}$ is a ID of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ is a TFI of $\ddot{\mathcal{S}}$.*

The *support* of $\mathcal{TF} := (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ tripolar fuzzy set instead of $\text{supp}(\mathcal{TF}) = \{\ddot{h} \in \ddot{\mathcal{E}} \mid \ddot{\rho}(\ddot{h}) \neq 0, \ddot{\nu}(\ddot{h}) \neq 1, \ddot{\ddot{\delta}}(\ddot{h}) \neq 0\}$.

Theorem 2.11. *[13] Let $\ddot{\rho}$, $\ddot{\nu}$ and $\ddot{\ddot{\delta}}$ be nonzero fuzzy sets of an SG $\ddot{\mathcal{S}}$. Then $\mathcal{TF} := (\ddot{\rho}, \ddot{\nu}, \ddot{\ddot{\delta}})$ is a TFSSG of $\ddot{\mathcal{S}}$ if and only if* $\text{supp}(\mathcal{TF})$ *is a SG of $\ddot{\mathcal{S}}$.*

For $\mathcal{TF}_1 = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ and $\mathcal{TF}_2 = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ be a TFSs. Defined the product $\mathcal{TF}_1 \circ \mathcal{TF}_2$ of an SG $\ddot{\mathcal{S}}$ as follows:

$$(\ddot{\rho} \circ \ddot{\lambda})(\ddot{k}) = \begin{cases} \bigvee_{(\ddot{m}, \ddot{n}) \in \ddot{\mathcal{F}}_{\ddot{k}}} \{\ddot{\rho}(\ddot{m}) \wedge \ddot{\lambda}(\ddot{n})\} & \text{if } \ddot{\mathcal{F}}_{\ddot{k}} \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\ddot{\nu} \circ \ddot{\mu})(\ddot{k}) = \begin{cases} \bigwedge_{(\ddot{m}, \ddot{n}) \in \ddot{\mathcal{F}}_{\ddot{k}}} \{\ddot{\nu}(\ddot{m}) \vee \ddot{\mu}(\ddot{n})\} & \text{if } \ddot{\mathcal{F}}_{\ddot{k}} \neq \emptyset, \\ 1 & \text{otherwise}, \end{cases}$$

$$(\ddot{\delta} \circ \ddot{\omega})(\ddot{k}) = \begin{cases} \bigwedge_{(\ddot{m}, \ddot{n}) \in \ddot{\mathcal{F}}_{\ddot{k}}} \{\ddot{\delta}(\ddot{m}) \vee \ddot{\omega}(\ddot{n})\} & \text{if } \ddot{\mathcal{F}}_{\ddot{k}} \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

for all $\ddot{k} \in \ddot{\mathcal{E}}$. It is easy to verify that the structure $(\mathcal{TF}_1, \circ)$ is an SG. In the set of all TFSs of $\ddot{\mathcal{S}}$ we define the order relation as follows: $\mathcal{TF}_1 \sqsubseteq \mathcal{TF}_2$ if and only if $\ddot{\rho}(\ddot{h}) \leq \ddot{\lambda}(\ddot{h})$, $\ddot{\nu}(\ddot{h}) \geq \ddot{\mu}(\ddot{h})$ and $\ddot{\delta}(\ddot{h}) \geq \ddot{\omega}(\ddot{h})$ for all $h \in \ddot{\mathcal{S}}$. Finally, we define a binary operation $\sqcap$ on $\mathcal{TF}$ as follows:

$$\mathcal{TF}_1 \sqcap \mathcal{TF}_2 := (\ddot{\rho} \wedge \ddot{\lambda}, \ddot{\nu} \vee \ddot{\mu}, \ddot{\delta} \vee \ddot{\omega}),$$

where $(\ddot{\rho} \wedge \ddot{\lambda})(\ddot{h}) := \ddot{\rho}(\ddot{h}) \wedge \ddot{\lambda}(\ddot{h})$, $(\ddot{\nu} \vee \ddot{\mu})(\ddot{h}) := \ddot{\nu}(\ddot{h}) \vee \ddot{\mu}(\ddot{h})$ and $(\ddot{\delta} \vee \ddot{\omega})(\ddot{h}) := \ddot{\delta}(\ddot{h}) \vee \ddot{\omega}(\ddot{h})$ for all $\ddot{h} \in \ddot{\mathcal{S}}$.

Theorem 2.12. *[13] Let $\{\mathcal{TF}_{\ddot{i}|\ddot{i}\in\ddot{\mathcal{I}}}\}$ be a family of TFS of an SG $\ddot{\mathcal{S}}$. If $\{\mathcal{TF}_{\ddot{i}|\ddot{i}\in\ddot{\mathcal{I}}}\}$ be a family of TFSSG of $\ddot{\mathcal{S}}$, then the TFS $\bigcap_{\ddot{i}\in\ddot{\mathcal{I}}}\mathcal{TF}_i := (\bigcap_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\rho}, \bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\nu}, \bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\delta})$ of $\ddot{\mathcal{S}}$ is a TFSSG of $\ddot{\mathcal{S}}$.*

III. Tripolar fuzzy Bi-ideals in semigroups

In this section, we mentioned the definition of tripolar fuzzy bi-ideals in semigroups, and some of their properties are investigated.

Definition 3.1. *A TFS is called a $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called a tripolar fuzzy generalized bi-ideal (TFGIB) of $\ddot{\mathcal{S}}$ if*

(1) $\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \geq \ddot{\rho}(\ddot{k})$
(2) $\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\nu}(\ddot{k})$
(3) $\ddot{\delta}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\delta}(\ddot{k})$

for all $\ddot{h}, \ddot{k}, \ddot{r} \in \ddot{\mathcal{S}}$.

Definition 3.2. *A TFS is called a $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called a tripolar fuzzy bi-ideal (TFBI) of $\ddot{\mathcal{S}}$ if it is a TFSSG and a TFGBI.*

It is clearly that every TFBI is a TFGBI in semigroups.
The following example is a TFGBI of an SG.

Example 3.3. *[13] Let $\ddot{\mathcal{S}} = \{\ddot{w}, \ddot{x}, \ddot{y}, \ddot{z}\}$ be an SG with the following Cayley table:*

$\cdot$	$\ddot{w}$	$\ddot{x}$	$\ddot{y}$	$\ddot{z}$
$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$
$\ddot{x}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$
$\ddot{y}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{x}$
$\ddot{z}$	$\ddot{w}$	$\ddot{w}$	$\ddot{x}$	$\ddot{y}$

Define $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ by $\ddot{\rho}(\ddot{w}) = 0.6$, $\ddot{\rho}(\ddot{x}) = 0.3$, $\ddot{\rho}(\ddot{y}) = 0.4$, $\ddot{\rho}(\ddot{z}) = 0.1$; $\ddot{\nu}(\ddot{w}) = 0.1$, $\ddot{\rho}(\ddot{x}) = 0.3$, $\ddot{\rho}(\ddot{y}) = 0.2$, $\ddot{\rho}(\ddot{z}) = 0.4$ and $\ddot{\delta}(\ddot{w}) = -0.1$, $\ddot{\delta}(\ddot{x}) = -0.4$, $\ddot{\delta}(\ddot{y}) = -0.3$, $\ddot{\delta}(\ddot{z}) = -0.5$. Then $\mathcal{TF}$ is a TFLI of $\ddot{\mathcal{S}}$.

Definition 3.4. *[15] An SG $\ddot{\mathcal{S}}$ is called an* regular *if for each $\ddot{h} \in \ddot{\mathcal{S}}$, there exist $\ddot{v} \in \ddot{\mathcal{S}}$ such that $\ddot{h} = \ddot{h}\ddot{v}\ddot{h}$.*

Definition 3.5. *[15] An SG $\ddot{\mathcal{S}}$ is called an* intra-regular *if for each $\ddot{h} \in \ddot{\mathcal{S}}$, there exist $\ddot{k}, \ddot{v} \in \ddot{\mathcal{S}}$ such that $\ddot{h} = \ddot{v}\ddot{h}^2\ddot{k}$.*

Theorem 3.6. *In regular and intra-regular of $\ddot{\mathcal{X}}$, the TFBIs and TFGBIs coincide.*

Proof: Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be a BCF GBI of a regular of $\ddot{\mathcal{S}}$ and let $\ddot{h}, \ddot{k} \in \ddot{\mathcal{S}}$. Since $\ddot{\mathcal{S}}$ is regular, we have there exists $\ddot{m} \in \ddot{\mathcal{S}}$ such that $\ddot{k} = \ddot{k}\ddot{m}\ddot{k}$. Thus,
$\ddot{\rho}(\ddot{h}\ddot{k}) = \ddot{\rho}(\ddot{h}\ddot{k}\ddot{m}\ddot{k}) = \ddot{\rho}(\ddot{h}(\ddot{k}\ddot{m})\ddot{k}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{k})$,
$\ddot{\nu}(\ddot{h}\ddot{k}) = \ddot{\nu}(\ddot{h}\ddot{k}\ddot{m}\ddot{k}) = \ddot{\nu}(\ddot{h}(\ddot{k}\ddot{m})\ddot{k}) \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{k})$ and $\ddot{\delta}(\ddot{h}\ddot{k}) = \ddot{\delta}(\ddot{h}\ddot{k}\ddot{m}\ddot{k}) = \ddot{\delta}(\ddot{h}(\ddot{k}\ddot{m})\ddot{k}) \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{k})$.

Hence, $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFSSG of $\ddot{\mathcal{S}}$. By Definition 3.2, $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFBI of $\ddot{\mathcal{S}}$.

Similarly, we can prove the other cases also. ∎

Theorem 3.7. *Every TFID of an SG $\ddot{\mathcal{S}}$ is a TFBI of $\ddot{\mathcal{S}}$.*

Proof: Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be a TFID of $\ddot{\mathcal{S}}$ and let $\ddot{h}, \ddot{k} \in \ddot{\mathcal{S}}$. Then $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFLI and TFRI of $\ddot{\mathcal{S}}$. Thus,
$\ddot{\rho}(\ddot{h}\ddot{k}) \geq \ddot{\rho}(\ddot{k})$ $\ddot{\nu}(\ddot{h}\ddot{k}) \leq \ddot{\nu}(\ddot{k})$ and $\ddot{\delta}(\ddot{h}\ddot{k}) \leq \ddot{\delta}(\ddot{k})$. Hence, $\ddot{\rho}(\ddot{h}\ddot{k}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{k})$ $\ddot{\nu}(\ddot{h}\ddot{k}) \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{k})$ and $\ddot{\delta}(\ddot{h}\ddot{k}) \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{k})$. This show that $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFSSG of $\ddot{\mathcal{S}}$.

Let $\ddot{h}, \ddot{k}, \ddot{r} \in \ddot{\mathcal{S}}$. Then, $\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \geq \ddot{\rho}(\ddot{r})$ $\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\nu}(\ddot{r})$ and $\ddot{\delta}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\delta}(\ddot{r})$. Thus, $\ddot{\rho}(\ddot{h}\ddot{k}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{r})$ $\ddot{\nu}(\ddot{h}\ddot{k}) \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{r})$ and $\ddot{\delta}(\ddot{h}\ddot{k}) \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r})$. Therefore, $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFBI of $\ddot{\mathcal{S}}$. ∎

Corollary 3.8. *Every TFID of $\ddot{\mathcal{S}}$ is a TFGID of $\ddot{\mathcal{X}}$.*

The following theorem shows that the BCF BIs and BCF IDs coincide for some types of semigroups.

Theorem 3.9. *In regular of an SG $\ddot{\mathcal{S}}$, the TFBIs and TFIDs coincide.*

Proof: Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be a BCF BI of a regular of $\ddot{\mathcal{S}}$ and let $\ddot{h}, \ddot{k} \in \ddot{\mathcal{S}}$. Since $\ddot{\mathcal{S}}$ is regular, we have $\ddot{h}\ddot{k} \in (\ddot{h}\ddot{\mathcal{S}}\ddot{h})\ddot{\mathcal{X}} \subseteq \ddot{h}\ddot{\mathcal{S}}\ddot{h}$ which that $\ddot{h}\ddot{k} = \ddot{h}\ddot{s}\ddot{k}$ for some $\ddot{s} \in \ddot{\mathcal{S}}$. Thus, $\ddot{\rho}(\ddot{h}\ddot{k}) = \ddot{\rho}(\ddot{h}\ddot{s}\ddot{k}) \geq \ddot{\rho}(\ddot{h})$ $\ddot{\nu}(\ddot{h}\ddot{k}) = \ddot{\nu}(\ddot{h}\ddot{s}\ddot{k}) \leq \ddot{\nu}(\ddot{h})$ and $\ddot{\delta}(\ddot{h}\ddot{k}) = \ddot{\delta}(\ddot{h}\ddot{s}\ddot{k}) \leq \ddot{\delta}(\ddot{h})$.

Hence, $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFRI of $\ddot{\mathcal{S}}$. Similarly, we can prove that $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFLI of $\ddot{\mathcal{S}}$. Thus, $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFID of $\ddot{\mathcal{S}}$. ∎

Corollary 3.10. *In regular of $\ddot{\mathcal{S}}$, the TFGBIs and TFIDs coincide.*

Theorem 3.11. *Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be a TFS in an SG $\ddot{\mathcal{S}}$. Then the following statements hold.*

(1) *$\mathcal{TF} = (\ddot{\rho}, \overline{\ddot{\rho}}, \ddot{\delta})$ is a TFSSG of $\ddot{\mathcal{S}}$.*
(2) *$\mathcal{TF} = (\ddot{\rho}, \overline{\ddot{\rho}}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{S}}$.*
(3) *$\mathcal{TF} = (\ddot{\rho}, \overline{\ddot{\rho}}, \ddot{\delta})$ is a TFBI of $\ddot{\mathcal{S}}$.*

Proof: Let $\overline{\ddot{\rho}} = 1 - \ddot{\rho}$ and $\ddot{h}, \ddot{k} \in \ddot{\mathcal{S}}$. Then

(1) $\overline{\ddot{\rho}}(\ddot{h}\ddot{k}\ddot{r}) = 1 - \ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \leq 1 - (\ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{k})) = 1 - \ddot{\rho}(\ddot{h}) \vee 1 - \ddot{\rho}(\ddot{k}) = \overline{\ddot{\rho}}(\ddot{h}) \vee \overline{\ddot{\rho}}(\ddot{k})$. Thus, $\mathcal{TF} = (\ddot{\rho}, \overline{\ddot{\rho}}, \ddot{\delta})$ is a TFSSG of $\ddot{\mathcal{S}}$.
(2) $\overline{\ddot{\rho}}(\ddot{h}\ddot{k}\ddot{r}) = 1 - \ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \leq 1 - \ddot{\rho}(\ddot{k}) = \overline{\ddot{\rho}}(\ddot{k})$. Thus, $\mathcal{TF} = (\ddot{\rho}, \overline{\ddot{\rho}}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{S}}$.
(3) By (1) and (2), we have (3) is true.

∎

The following theorems show the connection between GBIs and TFBIs in semigroups.

Theorem 3.12. *Let $\ddot{\mathcal{B}}$ be a non-empty subset of an SG $\ddot{\mathcal{S}}$. Then the following statement holds;*

(1) *$\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ is a TFGBI of $\ddot{\mathcal{S}}$.*
(2) *$\ddot{\mathcal{B}}$ is a BI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ is a TFBI of $\ddot{\mathcal{S}}$.*

Proof:

(1) Suppose that $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$ and let $\ddot{h}, \ddot{k}, \ddot{r} \in \ddot{\mathcal{S}}$. Then $\mathcal{B}\ddot{\mathcal{S}}\mathcal{B} \subseteq \ddot{\mathcal{B}}$.
If $\ddot{h}, \ddot{r}, \in \ddot{\mathcal{B}}$, then $\ddot{h}\ddot{k}\ddot{r} \in \ddot{\mathcal{B}}$. Thus, $\ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}) = \ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{r}) = \ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) = 1$, $\ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}) = \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{r}) = \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) = 0$ and $\ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}) = \ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{r}) = \ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) = -1$. Hence, $\ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) \geq \ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}) \wedge \ddot{\rho}_{\ddot{\mathcal{B}}}(r)$, $\ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}) \vee \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{r})$ and $\ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}) \vee \ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{r})$.
If $\ddot{h} \notin \ddot{\mathcal{B}}$ or $\ddot{r} \notin \ddot{\mathcal{B}}$, then $\ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) \geq \ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}) \wedge \ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{r})$, $\ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}) \vee \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{r})$ and $\ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}) \vee \ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{r})$.
Therefore, $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ is a TFGBI of $\ddot{\mathcal{S}}$.
For the converse, assume that $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ is a TFSSG of $\ddot{\mathcal{S}}$, let $\ddot{h}, \ddot{k}, \ddot{r} \in \ddot{\mathcal{S}}$ with $\ddot{h}, \ddot{r} \in \ddot{\mathcal{B}}$. If $\ddot{h}\ddot{k}\ddot{r} \notin$

$\ddot{\mathcal{B}}$. Then $\ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}) = \ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{r}) = 1$, $\ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}) = \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{r}) = 0$, $\ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}) = \ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{r}) = -1$, and $\ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) = 0$, $\ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) = 1$, $\ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) = 0$. Thus, $0 = \ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) \geq \ddot{\rho}_{\ddot{\mathcal{B}}}(\ddot{h}) \wedge \ddot{\rho}_{\ddot{\mathcal{B}}}(r) = 1$, $1 = \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{h}) \vee \ddot{\nu}_{\ddot{\mathcal{B}}}(\ddot{r}) = 0$ and $0 = \ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\delta}_{\ddot{\mathcal{B}}}(\ddot{h}) \vee \ddot{\delta}_{\ddot{\mathcal{B}}}(r) = -1$. It is a contradiction so, $\ddot{h}\ddot{k}\ddot{r} \in \ddot{\mathcal{B}}$. Therefore $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$.

(2) It follows from Theorem 2.10 and (1).

■

Theorem 3.13. *Let $\ddot{\rho}$, $\ddot{\nu}$ and $\ddot{\delta}$ be nonzero fuzzy sets of an SG $\ddot{\mathcal{S}}$. Then the following statement holds;*

(1) *$\mathcal{TF} := (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{S}}$ if and only if* $\text{supp}(\mathcal{TF})$ *is a GBI of $\ddot{\mathcal{S}}$.*
(2) *$\mathcal{TF} := (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFBI of $\ddot{\mathcal{S}}$ if and only if* $\text{supp}(\mathcal{TF})$ *is a BI of $\ddot{\mathcal{S}}$.*

Proof:

(1) Suppose that $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{S}}$ and $\ddot{h}, \ddot{k}, \ddot{r} \in \ddot{\mathcal{S}}$ with $\ddot{h}, \ddot{r} \in \text{supp}(\mathcal{TF})$. Then $\ddot{\rho}(\ddot{h}) \neq 0, \ddot{\rho}(\ddot{r}) \neq 0$, $\ddot{\nu}(\ddot{h}) \neq 1, \ddot{\nu}(\ddot{r}) \neq 1$ and $\ddot{\delta}(\ddot{h}) \neq 0$, $\ddot{\delta}(\ddot{r}) \neq 0$. By assumption, $\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{r})$, $\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{r})$ and $\ddot{\delta}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r})$. Thus, $\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \neq 0$, $\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) \neq 1$ and $\ddot{\delta}(\ddot{h}\ddot{k}\ddot{r}) \neq 0$. So $\ddot{h}\ddot{k}\ddot{r} \in \text{supp}(\mathcal{TF})$. Hence, $\text{supp}(\mathcal{TF})$ is a GBI of $\ddot{\mathcal{S}}$.
For the converse, suppose that $\text{supp}(\mathcal{TF})$ is a SG of $\ddot{\mathcal{S}}$ and let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is not a TFGBI of $\ddot{\mathcal{S}}$. Then there exist $\ddot{h}, \ddot{k}, \ddot{r} \in \ddot{\mathcal{S}}$ such that $\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) < \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{r})$, $\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) > \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{r})$ and $\delta(\ddot{h}\ddot{k}\ddot{r}) > \delta(\ddot{h}) \vee \ddot{\delta}(\ddot{r})$. Since $\text{supp}(\mathcal{TF})$ is a GBI of $\ddot{\mathcal{S}}$ we have $\ddot{h}\ddot{k}\ddot{r} \in \text{supp}(\mathcal{TF})$. Thus, $\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \neq 0$, $\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) \neq 1$ and $\delta(\ddot{h}\ddot{k}\ddot{r}) \neq 0$.
If $\ddot{h}, \ddot{r} \in \text{supp}(\mathcal{TF})$, then $\ddot{\rho}(\ddot{h}) \neq 0$, $\ddot{\rho}(\ddot{r}) \neq 0$, $\ddot{\nu}(\ddot{h}) \neq 1$, $\ddot{\nu}(\ddot{k}) \neq 1$ and $\ddot{\delta}(\ddot{h}) \neq 0$, $\ddot{\delta}(\ddot{r}) \neq 0$. Thus, $\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{r})$, $\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{r})$, and $\ddot{\delta}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r})$. It is a contradiction.
If $\ddot{h} \notin \text{supp}(\mathcal{TF})$ or $\ddot{r} \notin \text{supp}(\mathcal{TF})$, then $\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{r})$, $\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{r})$ and $\ddot{\delta}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r})$. It is a contradiction.
Therefore, $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{S}}$.
(2) It follows from Theorem 2.11 and (1).

■

Next, we give the definition of a $\ddot{\rho}$-level $\ddot{\beta}$-cut, $\ddot{\nu}$-level $\ddot{\beta}$-cut and $\ddot{\delta}$-level $\ddot{\beta}$-cut. And we prove the set $\ddot{\rho}$-level $\ddot{\beta}$-cut, $\ddot{\nu}$-level $\ddot{\beta}$-cut and $\ddot{\delta}$-level $\ddot{\beta}$-cut are ideals of SGs.

Definition 3.14. *[13] Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be a TFS of an SG $\ddot{\mathcal{S}}$ and $\ddot{\beta} \in [0, 1]$. Then the set $\ddot{\rho}_{\ddot{\beta}} = \{h \in \ddot{\mathcal{S}} : \ddot{\rho}(h) \geq \ddot{\beta}\}$, $\ddot{\nu}_{\ddot{\beta}} = \{h \in \ddot{\mathcal{S}} : \ddot{\nu}(h) \leq \ddot{\beta}$, and $\ddot{\delta}_{\ddot{\beta}} = \{h \in \ddot{\mathcal{S}} : \ddot{\delta}(h) \leq -\ddot{\beta}$ are called a $\ddot{\rho}$-level $\ddot{\beta}$-cut, $\ddot{\nu}$-level $\ddot{\beta}$-cut and $\ddot{\delta}$-level $\ddot{\beta}$-cut of $\ddot{\mathcal{S}}$ respectively.*

Theorem 3.15. *[13] Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be a TFS of an SG $\ddot{\mathcal{S}}$. If $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFSSG of $\ddot{\mathcal{S}}$, then the $\ddot{\rho}$-level $\ddot{\beta}$-cut, $\ddot{\nu}$-level $\ddot{\beta}$-cut and $\ddot{\delta}$-level $\ddot{\beta}$-cut SGs of $\ddot{\mathcal{S}}$, for every $\ddot{\beta} \in Im(\ddot{\rho}) \cap Im(\ddot{\nu}) \subseteq [0, 1]$ and $-\ddot{\beta} \in Im(\ddot{\delta})$.*

Theorem 3.16. *Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be a TFS of an SG $\ddot{\mathcal{S}}$. Then the following statement holds;*

(1) *If $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{S}}$, then the $\ddot{\rho}$-level $\ddot{\beta}$-cut, $\ddot{\nu}$-level $\ddot{\beta}$-cut and $\ddot{\delta}$-level $\ddot{\beta}$-cut GBIs of $\ddot{\mathcal{S}}$, for every $\ddot{\beta} \in Im(\ddot{\rho}) \cap Im(\ddot{\nu}) \subseteq [0, 1]$ and $-\ddot{\beta} \in Im(\ddot{\delta})$.*
(2) *If $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFBI of $\ddot{\mathcal{S}}$, then the $\ddot{\rho}$-level $\ddot{\beta}$-cut, $\ddot{\nu}$-level $\ddot{\beta}$-cut and $\ddot{\delta}$-level $\ddot{\beta}$-cut BIs of $\ddot{\mathcal{S}}$, for every $\ddot{\beta} \in Im(\ddot{\rho}) \cap Im(\ddot{\nu}) \subseteq [0, 1]$ and $-\ddot{\beta} \in Im(\ddot{\delta})$.*

Proof: Let $\ddot{\beta} \in \text{Im}(\ddot{\rho}) \cap \text{Im}(\ddot{\nu}) \subseteq [0, 1]$ and $-\ddot{\beta} \in \text{Im}(\ddot{\delta})$ with $h, k \in \mathcal{TF} := (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$

(1) If $\ddot{h}, \ddot{r} \in \ddot{\rho}_{\ddot{\beta}}$, then $\ddot{\rho}(\ddot{h}) \geq \ddot{\beta}$, $\ddot{\rho}(\ddot{r}) \geq \ddot{\beta}$. Thus, $\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{r}) \geq \ddot{\beta}$. Hence, $\ddot{h}\ddot{k}\ddot{r} \in \ddot{\rho}_{\ddot{\beta}}$.
If $\ddot{h}, \ddot{r} \in \ddot{\nu}_{\ddot{\beta}}$, then $\ddot{\nu}(\ddot{h}) \leq \ddot{\beta}$, $\ddot{\nu}(\ddot{r}) \leq \ddot{\beta}$. Thus, $\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{r}) \leq \ddot{\beta}$. Hence, $\ddot{h}\ddot{k}\ddot{r} \in \ddot{\nu}_{\ddot{\beta}}$.
If $\ddot{h}, \ddot{k} \in \ddot{\delta}_{\ddot{\beta}}$, then $\ddot{\delta}(\ddot{h}) \leq -\ddot{\beta}$, $\ddot{\delta}(\ddot{r}) \leq -\ddot{\beta}$. Thus, $\ddot{\delta}(\ddot{h}\ddot{k}\ddot{r}) \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r}) \leq -\ddot{\beta}$. Hence, $\ddot{h}\ddot{k}\ddot{r} \in \ddot{\delta}_{\ddot{\beta}}$.
Therefore, $\ddot{\rho}$-level $\ddot{\beta}$-cut, $\ddot{\nu}$-level $\ddot{\beta}$-cut and $\ddot{\delta}$-level $\ddot{\beta}$-cut GBIs of $\ddot{\mathcal{S}}$.
(2) It follows Theorem 3.15 and (1).

■

Theorem 3.17. *Let $\{\mathcal{TF}_i \mid i \in I\}$ be a family of TFS of an SG $\ddot{\mathcal{S}}$. Then the following statement holds;*

(1) *If $\{\mathcal{TF}_i \mid i \in I\}$ be a family of TFGBI of $\ddot{\mathcal{S}}$, then the TFS $\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \mathcal{TF}_i := (\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\rho}, \bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\nu}, \bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\delta})$ of $\ddot{\mathcal{S}}$ is a TFGBI of $\ddot{\mathcal{S}}$.*
(2) *If $\{\mathcal{TF}_i \mid i \in I\}$ be a family of TFBI of $\ddot{\mathcal{S}}$, then the TFS $\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \mathcal{TF}_i := (\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\rho}, \bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\nu}, \bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\delta})$ of $\ddot{\mathcal{S}}$ is a TFBI of $\ddot{\mathcal{S}}$.*

Proof: Note that we defined
$\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \mathcal{TF}_i := (\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\rho}, \bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\nu}, \bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\delta})$ as follows:

$$\left(\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\rho}\right)(\ddot{h}) := \bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\rho}_i(\ddot{h}) : \inf\{\ddot{\rho}_i(\ddot{h}) \in I\},$$

$$\left(\bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\nu}\right)(\ddot{h}) := \bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\nu}_i(\ddot{h}) : \sup\{\ddot{\nu}_i(\ddot{h}) \in I\}$$

and

$$\left(\bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\delta}\right)(\ddot{h}) := \bigcup_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\delta}_i(\ddot{h}) : \sup\{\ddot{\delta}_i(\ddot{h}) \in I\},$$

for all $\ddot{h} \in \ddot{\mathcal{S}}$.

(1) Let $\ddot{h}, \ddot{k}, \ddot{r} \in \ddot{\mathcal{S}}$. Then

$$\begin{aligned}
\left(\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\rho}\right)(\ddot{h}\ddot{k}\ddot{r}) &= \bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\rho}_i(\ddot{h}\ddot{k}\ddot{r}) \\
&= \inf\{\ddot{\rho}_i(\ddot{h}) \mid i \in I\} \\
&\geq \inf\{\ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}_i(\ddot{r}) \mid i \in I\} \\
&= \bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} (\ddot{\rho}_i(\ddot{h}) \wedge \ddot{\rho}_i(\ddot{r})) \\
&= \left(\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\rho}\right)(\ddot{h}) \wedge \left(\bigcap_{\ddot{i} \in \ddot{\mathcal{I}}} \ddot{\rho}\right)(\ddot{r})
\end{aligned}$$

$$\left(\bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\nu}\right)(\ddot{h}\ddot{k}\ddot{r}) = \bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\nu}_i(\ddot{h}\ddot{k}\ddot{r})$$
$$= \sup\{\ddot{\nu}_i(\ddot{h}\ddot{k}\ddot{r}) \mid i \in I\}$$
$$\geq \sup\{\ddot{\nu}_i(\ddot{h}) \vee \ddot{\nu}_i(\ddot{r}) \mid i \in I\}$$
$$= \bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}(\ddot{\nu}_i(\ddot{h}) \vee \ddot{\nu}_i(\ddot{r}))$$
$$= \left(\bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\nu}\right)(\ddot{h}) \vee \left(\bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\nu}\right)(\ddot{r})$$

and

$$\left(\bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\delta}\right)(\ddot{h}\ddot{k}\ddot{r}) = \bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\delta}_i(\ddot{h}\ddot{k}\ddot{r})$$
$$= \sup\{\ddot{\delta}_i(\ddot{h}\ddot{k}\ddot{r}) \mid i \in I\}$$
$$\geq \sup\{\ddot{\delta}_i(\ddot{h}) \vee \ddot{\delta}_i(\ddot{r}) \mid i \in I\}$$
$$= \bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}(\ddot{\delta}_i(\ddot{h}) \vee \ddot{\delta}_i(\ddot{r}))$$
$$= \left(\bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\delta}\right)(\ddot{h}) \vee \left(\bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\delta}\right)(\ddot{r})$$

Thus, $\bigcap_{\ddot{i}\in\ddot{\mathcal{I}}}\mathcal{TF}_i := (\bigcap_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\rho}, \bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\nu}, \bigcup_{\ddot{i}\in\ddot{\mathcal{I}}}\ddot{\delta})$ of $\ddot{\mathcal{S}}$ is a TFGBI of $\ddot{\mathcal{S}}$.

(2) It follows Theorem 2.12 and (1).

■

Theorem 3.18. *[13] Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be TFS of an SG $\ddot{\mathcal{S}}$. Then $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFSSG of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF} \circ \mathcal{TF} \sqsubseteq \mathcal{TF}$.*

Theorem 3.19. *Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be TFS of an SG $\ddot{\mathcal{S}}$. Then the following statement holds;*

(1) *$\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF} \circ \mathcal{TF}_{\ddot{\mathcal{S}}} \circ \mathcal{TF} \sqsubseteq \mathcal{TF}$.*

(2) *$\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFBI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF} \circ \mathcal{TF} \sqsubseteq \mathcal{TF}$ and $\mathcal{TF} \circ \mathcal{TF}_{\ddot{\mathcal{S}}} \circ \mathcal{TF} \sqsubseteq \mathcal{TF}$, where $\mathcal{TF}_{\ddot{\mathcal{S}}} = (\ddot{\rho}_{\ddot{\mathcal{S}}}, \ddot{\nu}_{\ddot{\mathcal{S}}}, \ddot{\delta}_{\ddot{\mathcal{S}}})$.*

Proof:

(1) Assume that $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{S}}$ and let $\ddot{h} \in \ddot{\mathcal{S}}$.
If $\mathcal{F}_{\ddot{h}} = \emptyset$. Then $(\ddot{\rho} \circ \ddot{\rho}_{\ddot{\mathcal{S}}} \circ \ddot{\rho})(\ddot{h}) = 0 \leq \ddot{\rho}(\ddot{h})$, $(\ddot{\nu} \circ \ddot{\nu}_{\ddot{\mathcal{S}}} \circ \ddot{\nu})(\ddot{h}) = 1 \geq \ddot{\nu}(\ddot{h})$ and $(\ddot{\delta} \circ \ddot{\delta}_{\ddot{\mathcal{S}}} \circ \ddot{\delta})(\ddot{h}) = 0 \geq \ddot{\delta}(\ddot{h})$.
If $\mathcal{F}_{\ddot{h}} \neq \emptyset$. Then

$$((\ddot{\rho} \circ \ddot{\rho}_{\mathcal{S}}) \circ \ddot{\rho})(\ddot{h}) = \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{(\ddot{\rho} \circ \ddot{\rho}_{\mathcal{S}})(\ddot{m}) \wedge \ddot{\rho})(\ddot{n})\}$$
$$= \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{\bigvee_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\rho}(\ddot{p}) \wedge \ddot{\rho}_{\mathcal{S}}(\ddot{q})\} \wedge \ddot{\rho}(\ddot{n})\}$$
$$= \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{\bigvee_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\rho}(\ddot{p}) \wedge 1\} \wedge \ddot{\rho}(\ddot{n})\}$$
$$= \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{\bigvee_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\rho}(\ddot{p})\} \wedge \ddot{\rho}(\ddot{n})\}$$
$$= \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\bigvee_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\rho}(\ddot{p}) \wedge \ddot{\rho}(\ddot{n})\} \leq \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\ddot{\rho}(\ddot{p}\ddot{m}\ddot{n})$$
$$= \ddot{\rho}(\ddot{h}),$$
$$((\ddot{\nu} \circ \ddot{\nu}_{\mathcal{S}}) \circ \ddot{\nu})(\ddot{h}) = \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{(\ddot{\nu} \circ \ddot{\nu}_{\mathcal{S}})(\ddot{m}) \vee \ddot{\nu})(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\nu}(\ddot{p}) \vee \ddot{\nu}_{\mathcal{S}}(\ddot{q})\} \vee \ddot{\rho}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\nu}(\ddot{p}) \vee 0\} \vee \ddot{\nu}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\nu}(\ddot{p})\} \vee \ddot{\nu}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\nu}(\ddot{p}) \vee \ddot{\nu}(\ddot{n})\} \geq \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\ddot{\nu}(\ddot{p}\ddot{m}\ddot{n})$$
$$= \ddot{\nu}(\ddot{h}),$$
$$((\ddot{\delta} \circ \ddot{\delta}_{\mathcal{S}}) \circ \ddot{\delta})(\ddot{h}) = \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{(\ddot{\delta} \circ \ddot{\nu}_{\mathcal{S}})(\ddot{m}) \vee \ddot{\delta})(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\delta}(\ddot{p}) \vee \ddot{\delta}_{\mathcal{S}}(\ddot{q})\} \vee \ddot{\delta}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\delta}(\ddot{p}) \vee -1\} \vee \ddot{\delta}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\delta}(\ddot{p})\} \vee \ddot{\delta}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\delta}(\ddot{p}) \vee \ddot{\delta}(\ddot{n})\} \geq \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}}}\ddot{\delta}(\ddot{p}\ddot{m}\ddot{n})$$
$$= \ddot{\delta}(\ddot{h}),$$

Hence, $\mathcal{TF} \circ \mathcal{TF}_{\ddot{\mathcal{S}}} \circ \mathcal{TF} \sqsubseteq \mathcal{TF}$.
Conversely, assume that $\mathcal{TF} \circ \mathcal{TF}_{\ddot{\mathcal{S}}} \circ \mathcal{TF} \sqsubseteq \mathcal{TF}$ and $\ddot{h}, \ddot{k}, \ddot{r} \in \ddot{\mathcal{S}}$. Then

$$\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r}) \geq ((\ddot{\rho} \circ \ddot{\rho}_{\mathcal{S}}) \circ \ddot{\rho})(\ddot{h}\ddot{k}\ddot{r})$$
$$= \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\{(\ddot{\rho} \circ \ddot{\rho}_{\mathcal{S}})(\ddot{m}) \wedge \ddot{\rho})(\ddot{n})\}$$
$$= \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\{\bigvee_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\rho}(\ddot{p}) \wedge \ddot{\rho}_{\mathcal{S}}(\ddot{q})\} \wedge \ddot{\rho}(\ddot{n})\}$$
$$= \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\{\bigvee_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\rho}(\ddot{p}) \wedge 1\} \wedge \ddot{\rho}(\ddot{n})\}$$
$$= \bigvee_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\bigvee_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\rho}(\ddot{p}) \wedge \ddot{\rho}(\ddot{n})\} \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{r}),$$
$$\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r}) \leq ((\ddot{\nu} \circ \ddot{\nu}_{\mathcal{S}}) \circ \ddot{\nu})(\ddot{h}\ddot{k}\ddot{r})$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\{(\ddot{\nu} \circ \ddot{\nu}_{\mathcal{S}})(\ddot{m}) \vee \ddot{\nu})(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\nu}(\ddot{p}) \vee \ddot{\nu}_{\mathcal{S}}(\ddot{q})\} \vee \ddot{\nu}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\nu}(\ddot{p}) \vee 0\} \vee \ddot{\nu}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\nu}(\ddot{p}) \vee \ddot{\nu}(\ddot{n})\} \leq \ddot{\nu}(\ddot{h}) \wedge \ddot{\nu}(\ddot{r}),$$
$$\ddot{\delta}(\ddot{h}\ddot{k}\ddot{r}) \leq ((\ddot{\delta} \circ \ddot{\delta}_{\mathcal{S}}) \circ \ddot{\delta})(\ddot{h}\ddot{k}\ddot{r})$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\{(\ddot{\delta} \circ \ddot{\nu}_{\mathcal{S}})(\ddot{m}) \vee \ddot{\delta})(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\delta}(\ddot{p}) \vee \ddot{\delta}_{\mathcal{S}}(\ddot{q})\} \vee \ddot{\delta}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\{\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\delta}(\ddot{p}) \vee 0\} \vee \ddot{\delta}(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n})\in\ddot{\mathfrak{F}}_{\ddot{h}\ddot{k}\ddot{r}}}\bigwedge_{(\ddot{p},\ddot{q})\in\ddot{\mathfrak{F}}_{\ddot{m}}}\{\ddot{\delta}(\ddot{p}) \vee \ddot{\delta}(\ddot{n})\} \leq \ddot{\delta}(\ddot{h}) \wedge \ddot{\delta}(\ddot{r}).$$

Thus, $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{S}}$.

(2) It follows Theorem 3.18 and (1).

■

Definition 3.20. *[13] Let $\varphi : \ddot{\mathcal{X}} \to \ddot{\mathcal{Y}}$ be a map and let $\mathcal{TF}_1 = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ and $\mathcal{TF}_2 = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ be a TFSs in $\ddot{\mathcal{X}}$ and $\ddot{\mathcal{Y}}$, respectively. The pre-image of $\mathcal{TF}_2$ under φ, denoted by $\varphi^{-1}(\mathcal{TF}_2)$ is a TFS in $\ddot{\mathcal{X}}$ defined by:*

$$\varphi^{-1}(\mathcal{TF}_2) := (\varphi^{-1}(\ddot{\lambda}), \varphi^{-1}(\ddot{\mu}), \varphi^{-1}(\ddot{\omega})),$$

whrer $\varphi^{-1}(\ddot{\lambda}) = \ddot{\lambda}(\varphi)$, $\varphi^{-1}(\ddot{\mu}) = \ddot{\mu}(\varphi)$, $\varphi^{-1}(\ddot{\omega}) = \ddot{\omega}(\varphi)$.

Theorem 3.21. *[13] Let $\varphi : \ddot{\mathcal{X}} \to \ddot{\mathcal{Y}}$ be a homomorphism of semigroups. If $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFSSG of $\mathcal{Y}$, then $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\ddot{\rho}), \varphi^{-1}(\ddot{\nu}), \varphi^{-1}(\ddot{\delta}))$ is a TFSSG of $\ddot{\mathcal{X}}$.*

Theorem 3.22. *Let $\varphi : \ddot{\mathcal{X}} \to \ddot{\mathcal{Y}}$ be a homomorphism of semigroups. Then the following statement holds;*

(1) *If $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{Y}}$, then $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\ddot{\rho}), \varphi^{-1}(\ddot{\nu}), \varphi^{-1}(\ddot{\delta}))$ is a TFGBI of $\ddot{\mathcal{X}}$.*
(2) *If $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFBI of $\ddot{\mathcal{Y}}$, then $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\ddot{\rho}), \varphi^{-1}(\ddot{\nu}), \varphi^{-1}(\ddot{\delta}))$ is a TFBI of $\ddot{\mathcal{X}}$.*

Proof:

(1) Assume that $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ is a TFGBI of $\ddot{\mathcal{Y}}$ and $\ddot{h}, \ddot{k}, \ddot{r} \in \ddot{\mathcal{X}}$. Then

$$\begin{aligned}\varphi^{-1}(\ddot{\rho}(\ddot{h}\ddot{k}\ddot{r})) &= \ddot{\rho}(\varphi(\ddot{h}\ddot{k}\ddot{r})) = \ddot{\rho}(\varphi(\ddot{h})\varphi(\ddot{k}))\varphi(\ddot{r}) \\ &\geq \ddot{\rho}(\varphi(\ddot{h})) \wedge \ddot{\rho}(\varphi(\ddot{r})) \\ &= \varphi^{-1}(\ddot{\rho}\varphi(\ddot{h})) \wedge \varphi^{-1}(\ddot{\rho}\varphi(\ddot{r})),\end{aligned}$$

$$\begin{aligned}\varphi^{-1}(\ddot{\nu}(\ddot{h}\ddot{k}\ddot{r})) &= \ddot{\nu}(\varphi(\ddot{h}\ddot{k}\ddot{r})) = \ddot{\nu}(\varphi(\ddot{h})\varphi(\ddot{k})\varphi(\ddot{r})) \\ &\leq \ddot{\nu}(\varphi(\ddot{h})) \vee \ddot{\nu}(\varphi(\ddot{k})\varphi(\ddot{r})) \\ &= \varphi(\ddot{\nu}(\varphi(\ddot{h}))) \vee \varphi^{-1}(\ddot{\nu}(\varphi(\ddot{r}))),\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(\ddot{\delta}(\ddot{h}\ddot{k}\ddot{r})) &= \ddot{\delta}(\varphi(\ddot{h}\ddot{k}\ddot{r})) = \ddot{\delta}(\varphi(\ddot{h})\varphi(\ddot{r})) \\ &\geq \ddot{\delta}(\varphi(\ddot{h})) \vee \ddot{\delta}(\varphi(\ddot{r})) \\ &= \varphi^{-1}(\ddot{\delta}(\varphi(\ddot{h}))) \vee \varphi^{-1}(\ddot{\delta}(\varphi(\ddot{r}))).\end{aligned}$$

Thus, $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\ddot{\rho}), \varphi^{-1}(\ddot{\nu}), \varphi^{-1}(\ddot{\delta}))$ is a TFGBI of $\ddot{\mathcal{X}}$.
(2) By Theorem 3.21 and (1).

■

Lemma 3.23. *Let $\ddot{\mathcal{K}}$ and $\ddot{\mathcal{L}}$ be non-empty subsets of an semigroup $\ddot{\mathcal{S}}$. Then the following statements are true*

(1) $\mathcal{TF}_{\ddot{\mathcal{K}}} \sqcap \mathcal{TF}_{\ddot{\mathcal{L}}} = \mathcal{TF}_{\ddot{\mathcal{K}} \cap \ddot{\mathcal{L}}}$.
(2) $\mathcal{TF}_{\ddot{\mathcal{K}}} \circ \mathcal{TF}_{\ddot{\mathcal{L}}} = \mathcal{TF}_{\ddot{\mathcal{K}}\ddot{\mathcal{L}}}$

where $\mathcal{TF}_{\ddot{\mathcal{K}}} = (\ddot{\rho}_{\ddot{\mathcal{K}}}, \ddot{\nu}_{\ddot{\mathcal{K}}}, \ddot{\delta}_{\ddot{\mathcal{K}}})$ and $\mathcal{TF}_{\ddot{\mathcal{L}}} = (\ddot{\rho}_{\ddot{\mathcal{L}}}, \ddot{\nu}_{\ddot{\mathcal{L}}}, \ddot{\delta}_{\ddot{\mathcal{L}}})$.

The following lemma will be used to prove in Theorem 3.25.

Lemma 3.24. *[15] Let $\ddot{\mathcal{S}}$ be an SG. Then the following are equivalent:*

(1) *$\ddot{\mathcal{S}}$ is a regular,*
(2) *$\ddot{\mathcal{A}} \cap \ddot{\mathcal{L}} \subseteq \ddot{\mathcal{A}}\ddot{\mathcal{L}}$, for every GBI $\ddot{\mathcal{A}}$ and every LI $\ddot{\mathcal{L}}$.*

Theorem 3.25. *Let $\ddot{\mathcal{S}}$ be an SG. Then the following are equivalent:*

(1) *$\ddot{\mathcal{S}}$ is regular,*
(2) *$\mathcal{TF}_1 \sqcap \mathcal{TF}_2 \sqsubseteq \mathcal{TF}_1 \circ \mathcal{TF}_2$, for every TFGBI $\mathcal{TF}_1 = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ and every TFLI $\mathcal{TF}_2 = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ of $\ddot{\mathcal{S}}$.*

Proof: (1) ⇒ (2) Let $\mathcal{TF}_1 = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ and $\mathcal{TF}_2 = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ be a TFGBI and a TFLI of $\ddot{\mathcal{S}}$ respectively and let $\ddot{h} \in \ddot{\mathcal{S}}$. Since $\ddot{\mathcal{S}}$ is regular, there exists $\ddot{k} \in \ddot{\mathcal{S}}$ such that $\ddot{h} = \ddot{h}\ddot{k}\ddot{h}$. Thus,

$$(\ddot{\rho} \circ \ddot{\lambda})(\ddot{h}) = \bigvee_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{\ddot{h}}} \{(\ddot{\rho}(\ddot{m}) \wedge \ddot{\lambda})(\ddot{n})\}$$
$$= \bigvee_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{\ddot{h}}} \{(\ddot{\rho}(\ddot{m}) \wedge \ddot{\lambda})(\ddot{n})\} \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\lambda}(\ddot{k}\ddot{h}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\lambda}(\ddot{h})$$
$$= (\ddot{\rho} \wedge \ddot{\lambda})(\ddot{h})$$

$$(\ddot{\nu} \circ \ddot{\mu})(\ddot{h}) = \bigwedge_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{\ddot{h}}} \{(\ddot{\nu}(\ddot{m}) \vee \ddot{\mu})(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{\ddot{h}}} \{(\ddot{\nu}(\ddot{m}) \vee \ddot{\mu})(\ddot{n})\} \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\mu}(\ddot{k}\ddot{h})$$
$$\leq \ddot{\nu}(\ddot{h}) \vee \ddot{\mu}(\ddot{h}) = (\ddot{\nu} \vee \ddot{\mu})(\ddot{h})$$

$$(\ddot{\delta} \circ \ddot{\omega}_{\mathcal{L}})(\ddot{h}) = \bigwedge_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{\ddot{h}}} \{(\ddot{\delta}(\ddot{m}) \vee \ddot{\omega})(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{\ddot{h}}} \{(\ddot{\delta}(\ddot{m}) \vee \ddot{\omega})(\ddot{n})\} \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\omega}(\ddot{k}\ddot{h})$$
$$\leq \ddot{\delta}(\ddot{h}) \vee \ddot{\omega}(\ddot{h}) = (\ddot{\delta} \vee \ddot{\omega})(\ddot{h}).$$

So, $(\ddot{\rho} \circ \ddot{\lambda})(\ddot{h}) \geq (\ddot{\rho} \wedge \ddot{\lambda})(\ddot{h})$,
$(\ddot{\nu} \circ \ddot{\mu})(\ddot{h}) \leq (\ddot{\nu} \vee \ddot{\mu})(\ddot{h})$ and $(\ddot{\delta} \circ \ddot{\omega})(\ddot{h}) \leq (\ddot{\delta} \vee \ddot{\omega})(\ddot{h})$.
Hence, $\mathcal{TF}_1 \sqcap \mathcal{TF}_2 \sqsubseteq \mathcal{TF}_1 \circ \mathcal{TF}_2$.

(2) ⇒ (1) Let $\ddot{\mathcal{A}}$ and $\ddot{\mathfrak{L}}$ be a GBI and a LI of $\ddot{\mathcal{S}}$ respectively. Then by Theorem 3.12 and 2.10, $\mathcal{TF}_{\ddot{\mathcal{A}}} = (\ddot{\rho}_{\ddot{\mathcal{A}}}, \ddot{\nu}_{\ddot{\mathcal{A}}}, \ddot{\delta}_{\ddot{\mathcal{A}}})$ and $\mathcal{TF}_{\ddot{\mathcal{L}}} = (\ddot{\rho}_{\ddot{\mathcal{L}}}, \ddot{\nu}_{\ddot{\mathcal{L}}}, \ddot{\delta}_{\ddot{\mathcal{L}}})$ is a TFGBI and TFLI of $\ddot{\mathcal{S}}$ respectively. By supposition and Lemma 3.23, we have

$$\begin{aligned}\mathcal{TF}_{\ddot{\mathcal{A}} \cap \ddot{\mathcal{L}}} &= (\mathcal{TF}_{\ddot{\mathcal{A}}}) \sqcap (\mathcal{TF}_{\ddot{\mathcal{L}}}) \\ &\sqsubseteq (\mathcal{TF}_{\ddot{\mathcal{A}}}) \circ (\mathcal{TF}_{\ddot{\mathcal{L}}}) \\ &= \mathcal{TF}_{\ddot{\mathcal{A}}\ddot{\mathcal{L}}}.\end{aligned}$$

Thus, $\ddot{h} \in \ddot{\mathcal{A}}\ddot{\mathcal{L}}$ and so, $\ddot{\mathcal{A}} \cap \ddot{\mathcal{L}} \subseteq \ddot{\mathcal{A}}\ddot{\mathcal{L}}$. It follows that by Lemma 3.24, $\ddot{\mathcal{S}}$ is regular. ■

The following lemma will be used to prove in Theorem 3.27.

Lemma 3.26. *[15] Let $\ddot{\mathcal{S}}$ be an SG. Then the following are equivalent:*

(1) *$\ddot{\mathcal{S}}$ is regular,*
(2) *$\ddot{\mathcal{B}} = \ddot{\mathcal{B}}\ddot{\mathcal{S}}\ddot{\mathcal{B}}$, for each BI $\ddot{\mathcal{B}}$ of $\ddot{\mathcal{S}}$,*
(3) *$\ddot{\mathcal{A}} = \ddot{\mathcal{A}}\ddot{\mathcal{S}}\ddot{\mathcal{A}}$, for each GBI $\ddot{\mathcal{A}}$ of $\ddot{\mathcal{S}}$.*

Theorem 3.27. *Let $\ddot{\mathcal{S}}$ be an SG. Then the following are equivalent:*

(1) *$\ddot{\mathcal{S}}$ is regular,*
(2) *$\mathcal{TF} \circ \mathcal{TF}_{\ddot{\mathcal{S}}} \circ \mathcal{TF} = \mathcal{TF}$, for every TFBI $\mathcal{TF}_1 = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of $\ddot{\mathcal{S}}$.*
(3) *$\mathcal{TF} \circ \mathcal{TF}_{\ddot{\mathcal{S}}} \circ \mathcal{TF} = \mathcal{TF}$, for every TFFGBI $\mathcal{TF}_1 = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of $\ddot{\mathcal{S}}$.*

Proof: (1) ⇒ (3) Let $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ be a TFBI of $\ddot{\mathcal{S}}$ and let $\ddot{h} \in \ddot{\mathcal{S}}$. Then there exists $\ddot{r} \in \ddot{\mathcal{S}}$ such that $\ddot{h} = \ddot{h}\ddot{x}\ddot{h} = (\ddot{h}\ddot{x}\ddot{h})\ddot{x}\ddot{h}$. Thus,

$$((\ddot{\rho} \circ \ddot{\rho}_{\mathcal{S}}) \circ \ddot{\rho})(\ddot{h}) = \bigvee_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{\ddot{h}}} \{(\ddot{\rho} \circ \ddot{\rho}_{\mathcal{S}})(\ddot{m}) \wedge \ddot{\rho})(\ddot{n})\}$$
$$= \bigvee_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{(\ddot{h}\ddot{x}\ddot{h})\ddot{x}\ddot{h}}} \{(\ddot{\rho} \circ \ddot{\rho}_{\mathcal{S}})(\ddot{m}) \wedge \ddot{\rho})(\ddot{n})\}$$
$$\geq (\ddot{\rho} \circ \ddot{\rho}_{\mathcal{S}})((\ddot{h}\ddot{x}\ddot{h})\ddot{x}) \wedge \ddot{\rho})(\ddot{h})$$
$$= \bigvee_{(\ddot{u},\ddot{d}) \in \ddot{\mathfrak{F}}_{(\ddot{h}\ddot{x}\ddot{h})\ddot{x}}} \{\ddot{\rho}(\ddot{u}) \wedge \ddot{\rho}_{\mathcal{S}}(\ddot{d})\} \wedge \ddot{\rho}(\ddot{n})$$
$$\geq (\ddot{\rho}(\ddot{h}\ddot{x}\ddot{h}) \wedge \ddot{\rho}_{\mathcal{S}}(\ddot{x})) \wedge \ddot{\rho}(\ddot{h}) = (\ddot{\rho}(\ddot{h}\ddot{x}\ddot{h}) \wedge 1) \wedge \ddot{\rho}(\ddot{h})$$
$$= \ddot{\rho}(\ddot{h}\ddot{x}\ddot{h}) \wedge \ddot{\rho}(\ddot{h}) \geq \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{h}) \wedge \ddot{\rho}(\ddot{h}) = \ddot{\rho}(\ddot{h}),$$

$$((\ddot{\nu} \circ \ddot{\nu}_{\mathcal{S}}) \circ \ddot{\nu})(\ddot{h}) = \bigwedge_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{\ddot{h}}} \{(\ddot{\nu} \circ \ddot{\nu}_{\mathcal{S}})(\ddot{m}) \vee \ddot{\nu})(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{(\ddot{h}\ddot{x}\ddot{h})\ddot{x}\ddot{h}}} \{(\ddot{\nu} \circ \ddot{\nu}_{\mathcal{S}})(\ddot{m}) \vee \ddot{\nu})(\ddot{n})\}$$
$$\leq (\ddot{\nu} \circ \ddot{\nu}_{\mathcal{S}})((\ddot{h}\ddot{x}\ddot{h})\ddot{x}) \vee \ddot{\nu})(\ddot{h})$$
$$= \bigwedge_{(\ddot{u},\ddot{d}) \in \ddot{\mathfrak{F}}_{(\ddot{h}\ddot{x}\ddot{h})\ddot{x}}} \{\ddot{\nu}(\ddot{u}) \vee \ddot{\nu}_{\mathcal{S}}(\ddot{d})\} \vee \ddot{\nu}(\ddot{n})$$

$\leq (\ddot{\nu}(\ddot{h}\ddot{x}\ddot{h}) \vee \ddot{\nu}_{\mathcal{S}}(\ddot{x})) \vee \ddot{\nu}(\ddot{h}) = (\ddot{\nu}(\ddot{h}\ddot{x}\ddot{h}) \vee 0) \vee \ddot{\nu}(\ddot{h})$
$= \ddot{\nu}(\ddot{h}\ddot{x}\ddot{h}) \vee \ddot{\nu}(\ddot{h}) \leq \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{h}) \vee \ddot{\nu}(\ddot{h}) = \ddot{\nu}(\ddot{h})$

and

$$((\ddot{\delta} \circ \ddot{\delta}_{\mathcal{S}}) \circ \ddot{\delta})(\ddot{h}) = \bigwedge_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{\ddot{h}}} \{(\ddot{\delta} \circ \ddot{\delta}_{\mathcal{S}})(\ddot{m}) \vee \ddot{\delta})(\ddot{n})\}$$
$$= \bigwedge_{(\ddot{m},\ddot{n}) \in \ddot{\mathfrak{F}}_{(\ddot{h}\ddot{x}\ddot{h})\ddot{x}\ddot{h}}} \{(\ddot{\delta} \circ \ddot{\delta}_{\mathcal{S}})(\ddot{m}) \vee \ddot{\delta})(\ddot{n})\}$$
$$\leq (\ddot{\delta} \circ \ddot{\delta}_{\mathcal{S}})((\ddot{h}\ddot{x}\ddot{h})\ddot{x}) \vee \ddot{\delta})(\ddot{h})$$
$$= \bigwedge_{(\ddot{u},\ddot{d}) \in \ddot{\mathfrak{F}}_{(\ddot{h}\ddot{x}\ddot{h})\ddot{x}}} \{\ddot{\delta}(\ddot{u}) \vee \ddot{\delta}_{\mathcal{S}}(\ddot{d})\} \vee \ddot{\delta}(\ddot{n})$$
$$\leq (\ddot{\delta}(\ddot{h}\ddot{x}\ddot{h}) \vee \ddot{\delta}_{\mathcal{S}}(\ddot{x})) \vee \ddot{\delta}(\ddot{h}) = (\ddot{\delta}(\ddot{h}\ddot{x}\ddot{h}) \vee -1) \vee \ddot{\delta}(\ddot{h})$$
$$= \ddot{\delta}(\ddot{h}\ddot{x}\ddot{h}) \vee \ddot{\delta}(\ddot{h}) \leq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{h}) = \ddot{\delta}(\ddot{h}).$$

Hence, $((\ddot{\rho} \circ \ddot{\rho}_{\mathcal{S}}) \circ \ddot{\rho})(\ddot{h}) \geq \ddot{\rho}(\ddot{h})$, $((\ddot{\nu} \circ \ddot{\nu}_{\mathcal{S}}) \circ \ddot{\nu})(\ddot{h}) \leq \ddot{\nu}(\ddot{h})$ and $(((\ddot{\delta} \circ \ddot{\delta}_{\mathcal{S}}) \circ \ddot{\delta})(\ddot{h}) \leq \ddot{\delta}(\ddot{h})$. It implies that, $\mathcal{TF} \sqsubseteq \mathcal{TF} \circ \mathcal{TF}_{\ddot{\mathcal{S}}} \circ \mathcal{TF}$. By Theorem 3.19 we have, $\mathcal{TF} \circ \mathcal{TF}_{\ddot{\mathcal{S}}} \circ \mathcal{TF} \sqsubseteq \mathcal{TF}$. Thus, $\mathcal{TF} \circ \mathcal{TF}_{\ddot{\mathcal{S}}} \circ \mathcal{TF} = \mathcal{TF}$.

(3) $\Rightarrow$ (2) This is obvious because every TFBI of $\ddot{\mathcal{S}}$ is a TFGBI of $\ddot{\mathcal{S}}$.

(2) $\Rightarrow$ (1). Let $\ddot{\mathcal{B}}$ be a BI of $\ddot{\mathcal{S}}$. Then by Theorem 3.12, $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\rho}_{\ddot{\mathcal{B}}}, \ddot{\nu}_{\ddot{\mathcal{B}}}, \ddot{\delta}_{\ddot{\mathcal{B}}})$ is a TFBI of $\ddot{\mathcal{S}}$. By supposition and Lemma 3.23, we have

$$\mathcal{TF}_{\ddot{\mathcal{B}} \circ \ddot{\mathcal{S}} \circ \ddot{\mathcal{B}}} = (\mathcal{TF}_{\ddot{\mathcal{B}}}) \circ (\mathcal{TF}_{\ddot{\mathcal{S}}}) \circ (\mathcal{TF}_{\ddot{\mathcal{B}}}) = \mathcal{TF}_{\ddot{\mathcal{B}}}.$$

Thus, $\ddot{h} \in \ddot{\mathcal{B}}$ and so, $\ddot{\mathcal{B}} = \ddot{\mathcal{B}}\ddot{\mathcal{S}}\ddot{\mathcal{B}}$. It follows that by Lemma 3.24, $\ddot{\mathcal{S}}$ is regular. ■

IV. Minimal and Maximal Tripolar Fuzzy Generalized Bi-Ideal

Definition 4.1. *An GBI $\ddot{\mathcal{B}}$ of an SG $\ddot{\mathcal{S}}$ is called*

(1) *a* minimal *if for every GBI of $\ddot{\mathcal{J}}$ of $\ddot{\mathcal{S}}$ such that $\ddot{\mathcal{J}} \subseteq \ddot{\mathcal{B}}$, we have $\ddot{\mathcal{J}} = \ddot{\mathcal{B}}$,*

(2) *a* maximal *if for every GBI of $\ddot{\mathcal{J}}$ of $\ddot{\mathcal{S}}$ such that $\ddot{\mathcal{B}} \subseteq \ddot{\mathcal{J}}$, we have $\ddot{\mathcal{B}} = \ddot{\mathcal{J}}$.*

Definition 4.2. *A TFGBI $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called*

(1) *a* minimal *if for every TFGBI of $\mathcal{TF}_1 = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ of $\ddot{\mathcal{S}}$ such that $\mathcal{TF}_1 \subseteq \mathcal{TF}$, we have $\mathrm{supp}(\mathcal{TF}_1) = \mathrm{supp}(\mathcal{TF})$,*

(2) *a* maximal *if for every TFGBI of $\mathcal{TF}_1 = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ of $\ddot{\mathcal{S}}$ such that $\mathcal{TF} \subseteq \mathcal{TF}_1$, we have $\mathrm{supp}(\mathcal{TF}_1) = \mathrm{supp}(\mathcal{TF})$.*

Theorem 4.3. *Let $\ddot{\mathcal{B}}$ be a non-empty subset of an SG $\ddot{\mathcal{S}}$. Then the following statement holds.*

(1) *$\ddot{\mathcal{B}}$ is a minimal GBI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}} \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a minimal TFGBI of $\ddot{\mathcal{S}}$,*

(2) *$\ddot{\mathcal{B}}$ is a maximal GBI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}} \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a maximal TFGBI of $\ddot{\mathcal{S}}$.*

Proof:

(1) Suppose that $\ddot{\mathcal{B}}$ is a minimal GBI of $\ddot{\mathcal{S}}$. Then $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$. Thus, by Theorem 3.12, $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}} \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a TRGBI of $\ddot{\mathcal{S}}$. Let $\mathcal{TF} = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ be a TRGBI of $\ddot{\mathcal{S}}$ such that $\mathcal{TF} \subseteq \mathcal{TF}_{\ddot{\mathcal{B}}}$. Then $\mathrm{supp}(\mathcal{TF}) \subseteq \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}})$. Thus, $\mathrm{supp}(\mathcal{TF}) \subseteq \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}}) = \ddot{\mathcal{B}}$. Hence, $\mathrm{supp}(\mathcal{TF}) \subseteq \ddot{\mathcal{B}}$. Since $\mathcal{TF} = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ is a TRGBI of $\ddot{\mathcal{S}}$ we have $\mathrm{supp}(\mathcal{TF})$ is a GBI of $\ddot{\mathcal{S}}$. By assumption, $\mathrm{supp}(\mathcal{TF}) \subseteq \ddot{\mathcal{B}} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}})$. So, $\mathrm{supp}(\mathcal{TF}) = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}})$. Hence, $\mathcal{TF}_{\ddot{\mathcal{B}}}$ is a minimal TRGBI of $\ddot{\mathcal{S}}$.
Conversely, $\mathcal{TF}_{\ddot{\mathcal{B}}}$ is a minimal TRGBI of $\ddot{\mathcal{S}}$. Then $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}} \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a TRGBI of $\ddot{\mathcal{S}}$. By Theorem 3.12, $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$. Let $\ddot{\mathcal{J}}$ be a GBI of $\ddot{\mathcal{S}}$ such that $\ddot{\mathcal{J}} \subseteq \ddot{\mathcal{B}}$. Then by Theorem 3.12, $\mathcal{TF}_{\mathfrak{J}} = (\ddot{\lambda}_{\ddot{\mathcal{J}}}, \ddot{\mu}_{\ddot{\mathcal{J}}} \ddot{\omega}_{\ddot{\mathcal{J}}})$ is a TRFI of $\mathcal{S}$ such that $\mathcal{TF}_{\ddot{\mathcal{J}}} \subseteq \mathcal{TF}_{\ddot{\mathcal{B}}}$. Hence, $\ddot{\mathcal{J}} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{J}}}) \subseteq \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}}) = \ddot{\mathcal{B}}$. By assumption, $\ddot{\mathcal{B}} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}}) = \ddot{\mathcal{J}} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{J}}}) = \ddot{\mathcal{J}}$. So, $\ddot{\mathcal{B}} = \ddot{\mathcal{J}}$. Hence, $\ddot{\mathcal{B}}$ is a minimal GBI of $\ddot{\mathcal{S}}$.

(2) Suppose that $\ddot{\mathcal{B}}$ is a maximal GBI of $\ddot{\mathcal{S}}$. Then $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$. Thus, by Theorem 3.12, $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}} \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a TRGBI of $\ddot{\mathcal{S}}$. Let $\mathcal{TF} = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ be a TRGBI of $\ddot{\mathcal{S}}$ such that $\mathcal{TF}_{\ddot{\mathcal{B}}} \subseteq \mathcal{TF}$. Then $\mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}}) \subseteq \mathrm{supp}(\mathcal{TF})$. Thus, $\ddot{\mathcal{B}} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}}) \subseteq \mathrm{supp}(\mathcal{TF})$. Hence, $\ddot{\mathcal{B}} \subseteq \mathrm{supp}(\mathcal{TF})$. Since $\mathcal{TF} = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ is a TRGBI of $\ddot{\mathcal{S}}$ we have $\mathrm{supp}(\mathcal{TF})$ is a GBI of $\ddot{\mathcal{S}}$. By assumption, $\mathrm{supp}(\mathcal{TF}) \subseteq \ddot{\mathcal{B}} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}})$. So, $\mathrm{supp}(\mathcal{TF}) = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}})$. Hence, $\mathcal{TF}_{\ddot{\mathcal{B}}}$ is a maximal TRGBI of $\ddot{\mathcal{S}}$.
Conversely, $\mathcal{TF}_{\ddot{\mathcal{B}}}$ is a maximal TRGBI of $\ddot{\mathcal{S}}$. Then $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}} \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a TRGBI of $\ddot{\mathcal{S}}$. By Theorem 3.12, $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$. Let $\ddot{\mathcal{J}}$ be a GBI of $\ddot{\mathcal{S}}$ such that $\ddot{\mathcal{B}} \subseteq \ddot{\mathcal{J}}$. Then by Theorem 3.12, $\mathcal{TF}_{\mathfrak{J}} = (\ddot{\lambda}_{\ddot{\mathcal{J}}}, \ddot{\mu}_{\ddot{\mathcal{J}}} \ddot{\omega}_{\ddot{\mathcal{J}}})$ is a TRFI of $\mathcal{S}$ such that $\mathcal{TF}_{\ddot{\mathcal{B}}} \subseteq \mathcal{TF}_{\ddot{\mathcal{J}}}$. Hence, $\mathcal{B} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}}) \subseteq \ddot{\mathcal{J}} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{J}}})$ By assumption, $\ddot{\mathcal{B}} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{B}}}) = \ddot{\mathcal{J}} = \mathrm{supp}(\mathcal{TF}_{\ddot{\mathcal{J}}}) = \ddot{\mathcal{J}}$. So, $\ddot{\mathcal{B}} = \ddot{\mathcal{J}}$. Hence, $\ddot{\mathcal{B}}$ is a maximal GBI of $\ddot{\mathcal{S}}$.

■

Definition 4.4. *An BI $\ddot{\mathcal{B}}$ of an SG $\ddot{\mathcal{S}}$ is called*

(1) *a* minimal *if for every BI of $\ddot{\mathcal{J}}$ of $\ddot{\mathcal{S}}$ such that $\ddot{\mathcal{J}} \subseteq \ddot{\mathcal{B}}$, we have $\ddot{\mathcal{J}} = \ddot{\mathcal{B}}$,*

(2) *a* maximal *if for every BI of $\ddot{\mathcal{J}}$ of $\ddot{\mathcal{S}}$ such that $\ddot{\mathcal{B}} \subseteq \ddot{\mathcal{J}}$, we have $\ddot{\mathcal{B}} = \ddot{\mathcal{J}}$.*

Definition 4.5. *A TFBI $\mathcal{TF} = (\ddot{\rho}, \ddot{\nu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called*

(1) *a* minimal *if for every TFBI of $\mathcal{TF}_1 = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ of $\ddot{\mathcal{S}}$ such that $\mathcal{TF}_1 \subseteq \mathcal{TF}$, we have $\mathrm{supp}(\mathcal{TF}_1) = \mathrm{supp}(\mathcal{TF})$,*

(2) *a* maximal *if for every TFBI of $\mathcal{TF}_1 = (\ddot{\lambda}, \ddot{\mu}, \ddot{\omega})$ of $\ddot{\mathcal{S}}$ such that $\mathcal{TF} \subseteq \mathcal{TF}_1$, we have $\mathrm{supp}(\mathcal{TF}_1) = \mathrm{supp}(\mathcal{TF})$.*

Theorem 4.6. *Let $\ddot{\mathcal{B}}$ be a non-empty subset of an SG $\ddot{\mathcal{S}}$. Then the following statement holds.*

(1) *$\ddot{\mathcal{B}}$ is a minimal BI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}} \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a minimal TFBI of $\ddot{\mathcal{S}}$,*

(2) *$\ddot{\mathcal{B}}$ is a maximal BI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}} \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a maximal TFBI of $\ddot{\mathcal{S}}$.*

V. Prime and Semiprime Tripolar Fuzzy Generalized Bi-Ideal

Definition 5.1. *An GBI $\ddot{\mathcal{B}}$ of an SG $\ddot{\mathcal{S}}$ is called*

(1) *a* prime *if $\ddot{h}\ddot{r} \in \ddot{\mathcal{B}}$, then $\ddot{h} \in \ddot{\mathcal{B}}$ or $\ddot{r} \in \ddot{\mathcal{B}}$ for all $\ddot{h}, \ddot{r} \in \ddot{\mathcal{S}}$,*

(2) *a* semiprime *if $\ddot{h}^2 \in \ddot{\mathcal{B}}$, then $\ddot{h} \in \ddot{\mathcal{B}}$ for all $\ddot{h} \in \ddot{\mathcal{S}}$*

Definition 5.2. *A TFGBI $\mathcal{TF} = (\ddot{\rho}, \ddot{\mu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called*

(1) *a* prime *if* $\ddot{\rho}(\ddot{h}\ddot{r}) \leq \ddot{\rho}(\ddot{h}) \vee \ddot{\rho}(\ddot{r})$, $\ddot{\mu}(\ddot{h}\ddot{r}) \geq \ddot{\mu}(\ddot{h}) \wedge \ddot{\mu}(\ddot{r})$ *and* $\ddot{\delta}(\ddot{h}\ddot{r}) \geq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r})$ *for all* $\ddot{h}, \ddot{r} \in \ddot{\mathcal{S}}$,

(2) *a* semiprime *if* $\ddot{\rho}(\ddot{h}\ddot{h}) \leq \ddot{\rho}(\ddot{h})$, $\ddot{\mu}(\ddot{h}\ddot{h}) \geq \ddot{\mu}(\ddot{h})$ *and* $\ddot{\delta}(\ddot{h}\ddot{h}) \geq \ddot{\delta}(\ddot{h})$ *for all* $\ddot{h} \in \ddot{\mathcal{S}}$.

Theorem 5.3. *Let $\ddot{\mathcal{B}}$ be a non-empty subset of an SG $\ddot{\mathcal{S}}$. Then the following statement holds.*

(1) *$\ddot{\mathcal{B}}$ is a prime GBI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}}\, \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a prime TFGBI of $\ddot{\mathcal{S}}$,*

(2) *$\ddot{\mathcal{B}}$ is a semiprime GBI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}}\, \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a semiprime TFGBI of $\ddot{\mathcal{S}}$.*

Proof:

(1) Suppose that $\ddot{\mathcal{B}}$ is a prime GBI of $\ddot{\mathcal{S}}$. Then $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$. Thus, by Theorem 3.12, $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}}\, \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a TRGBI of $\ddot{\mathcal{S}}$. Let $\ddot{h}, \ddot{r} \in \ddot{\mathcal{S}}$. If $\ddot{h}\ddot{r} \in \ddot{\mathcal{B}}$, then $\ddot{h} \in \ddot{\mathcal{B}}$ or $\ddot{r} \in \ddot{\mathcal{B}}$. Thus, $\ddot{\rho}(\ddot{h}\ddot{r}) = 1 = \ddot{\rho}(\ddot{h}) = \ddot{\rho}(\ddot{r})$, $\ddot{\mu}(\ddot{h}\ddot{r}) = 0 = \ddot{\mu}(\ddot{h}) = \ddot{\mu}(\ddot{r})$ and $\ddot{\delta}(\ddot{h}\ddot{r}) = -1 = \ddot{\delta}(\ddot{h}) = \ddot{\delta}(\ddot{r})$. So $\ddot{\rho}(\ddot{h}\ddot{r}) = 1 \leq \ddot{\rho}(\ddot{h}) \vee \ddot{\rho}(\ddot{r})$, $\ddot{\mu}(\ddot{h}\ddot{r}) = 0 \geq \ddot{\mu}(\ddot{h}) \wedge \ddot{\mu}(\ddot{r})$ and $\ddot{\delta}(\ddot{h}\ddot{r}) = -1 \geq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r})$.
If $\ddot{h}\ddot{r} \notin \ddot{\mathcal{B}}$, then $\ddot{\rho}(\ddot{h}\ddot{r}) = 0 \leq \ddot{\rho}(\ddot{h}) \vee \ddot{\rho}(\ddot{r})$, $\ddot{\nu}(\ddot{h}\ddot{r}) = 1 \geq \ddot{\mu}(\ddot{h}) \wedge \ddot{mu}(\ddot{r})$ and $\ddot{\delta}(\ddot{h}\ddot{r}) = 0 \geq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r})$. Hence, $\mathcal{TF}_{\ddot{\mathcal{B}}}$ is a prime TRGBI of $\ddot{\mathcal{S}}$.
Conversely, $\mathcal{TF}_{\ddot{\mathcal{B}}}$ is a prime TRGBI of $\ddot{\mathcal{S}}$. Then $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}}\, \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a TRGBI of $\ddot{\mathcal{S}}$. By Theorem 3.12, $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$. Let $\ddot{h}, \ddot{r} \in \ddot{\mathcal{S}}$ with $\ddot{h}\ddot{r} \in \ddot{\mathcal{B}}$. Then $\ddot{\rho}(\ddot{h}\ddot{r}) = 1$, $\ddot{\mu}(\ddot{h}\ddot{r}) = 0$ and $\ddot{\delta}(\ddot{h}\ddot{r}) = -1$. If $\ddot{h} \notin \ddot{\mathcal{B}}$ and $\ddot{r} \notin \ddot{\mathcal{B}}$, then $\ddot{\rho}(\ddot{h}) = 0 = \ddot{\rho}(\ddot{r})$, $\ddot{\mu}(\ddot{h}) = 1 = \ddot{\mu}(\ddot{r})$ and $\ddot{\delta}(\ddot{h}) = 0 = \ddot{\delta}(\ddot{r})$. Thus, $1 = \ddot{\rho}(\ddot{h}\ddot{r}) \leq \ddot{\rho}(\ddot{h}) \vee \ddot{\rho}(\ddot{r}) = 0$, $0 = \ddot{\mu}(\ddot{h}\ddot{r}) \geq \ddot{\mu}(\ddot{h}) \wedge \ddot{\mu}(\ddot{r}) = 1$ and $-1 = \ddot{\delta}(\ddot{h}\ddot{r}) \geq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r}) = 0$. It is a contradiction so, $\ddot{h} \in \ddot{\mathcal{B}}$ or $\ddot{r} \in \ddot{\mathcal{B}}$. Hence, $\ddot{\mathcal{B}}$ is a prime GBI of $\ddot{\mathcal{S}}$.

(2) Suppose that $\ddot{\mathcal{B}}$ is a semiprime GBI of $\ddot{\mathcal{S}}$. Then $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$. Thus, by Theorem 3.12, $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}}\, \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a TRGBI of $\ddot{\mathcal{S}}$. Let $\ddot{h}, \in \ddot{\mathcal{S}}$. If $\ddot{h}\ddot{h} \in \ddot{\mathcal{B}}$, then $\ddot{h} \in \ddot{\mathcal{B}}$. Thus, $\ddot{\rho}(\ddot{h}\ddot{h}) = 1 = \ddot{\rho}(\ddot{h})$, $\ddot{\mu}(\ddot{h}\ddot{h}) = 0 = \ddot{\mu}(\ddot{h})$ and $\ddot{\delta}(\ddot{h}\ddot{r}) = -1 = \ddot{\delta}(\ddot{h})$. So $\ddot{\rho}(\ddot{h}\ddot{h}) = 1 \leq \ddot{\rho}(\ddot{h})$, $\ddot{\mu}(\ddot{h}\ddot{h}) = 0 \geq \ddot{\mu}(\ddot{h})$ and $\ddot{\delta}(\ddot{h}\ddot{h}) = -1 \geq \ddot{\delta}(\ddot{h})$.
If $\ddot{h}\ddot{h} \notin \ddot{\mathcal{B}}$, then $\ddot{\rho}(\ddot{h}\ddot{h}) = 0 \leq \ddot{\rho}(\ddot{h})$, $\ddot{\nu}(\ddot{h}\ddot{h}) = 1 \geq \ddot{\mu}(\ddot{h})$ and $\ddot{\delta}(\ddot{h}\ddot{h}) = 0 \geq \ddot{\delta}(\ddot{h})$. Hence, $\mathcal{TF}_{\ddot{\mathcal{B}}}$ is a semiprime TRGBI of $\ddot{\mathcal{S}}$.
Conversely, $\mathcal{TF}_{\ddot{\mathcal{B}}}$ is a semiprime TRGBI of $\ddot{\mathcal{S}}$. Then $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}}\, \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a TRGBI of $\ddot{\mathcal{S}}$. By Theorem 3.12, $\ddot{\mathcal{B}}$ is a GBI of $\ddot{\mathcal{S}}$. Let $\ddot{h}, \in \ddot{\mathcal{S}}$ with $\ddot{h}\ddot{h} \in \ddot{\mathcal{B}}$. Then $\ddot{\rho}(\ddot{h}\ddot{h}) = 1$, $\ddot{\mu}(\ddot{h}\ddot{h}) = 0$ and $\ddot{\delta}(\ddot{h}\ddot{h}) = -1$. If $\ddot{h} \notin \ddot{\mathcal{B}}$, then $\ddot{\rho}(\ddot{h}) = 0$, $\ddot{\mu}(\ddot{h}) = 1$ and $\ddot{\delta}(\ddot{h}) = 0$. Thus, $1 = \ddot{\rho}(\ddot{h}\ddot{h}) \leq \ddot{\rho}(\ddot{h}) = 0$, $0 = \ddot{\mu}(\ddot{h}\ddot{h}) \geq \ddot{\mu}(\ddot{h}) = 1$ and $-1 = \ddot{\delta}(\ddot{h}\ddot{h}) \geq \ddot{\delta}(\ddot{h}) = 0$. It is a contradiction so, $\ddot{h} \in \ddot{\mathcal{B}}$. Hence, $\ddot{\mathcal{B}}$ is a semiprime GBI of $\ddot{\mathcal{S}}$.

■

Definition 5.4. *An BI $\ddot{\mathcal{B}}$ of an SG $\ddot{\mathcal{S}}$ is called*

(1) *a* prime *if $\ddot{h}\ddot{r} \in \ddot{\mathcal{B}}$, then $\ddot{h} \in \ddot{\mathcal{B}}$ or $\ddot{r} \in \ddot{\mathcal{B}}$ for all $\ddot{h}, \ddot{r} \in \ddot{\mathcal{S}}$,*

(2) *a* semiprime *if $\ddot{h}^2 \in \ddot{\mathcal{B}}$, then $\ddot{h} \in \ddot{\mathcal{B}}$ for all $\ddot{h} \in \ddot{\mathcal{S}}$*

Definition 5.5. *A TFBI $\mathcal{TF} = (\ddot{\rho}, \ddot{\mu}, \ddot{\delta})$ of an SG $\ddot{\mathcal{S}}$ is called*

(1) *a* prime *if* $\ddot{\rho}(\ddot{h}\ddot{r}) \leq \ddot{\rho}(\ddot{h}) \vee \ddot{\rho}(\ddot{r})$, $\ddot{\mu}(\ddot{h}\ddot{r}) \geq \ddot{\mu}(\ddot{h}) \wedge \ddot{\mu}(\ddot{r})$ *and* $\ddot{\delta}(\ddot{h}\ddot{r}) \geq \ddot{\delta}(\ddot{h}) \vee \ddot{\delta}(\ddot{r})$ *for all* $\ddot{h}, \ddot{r} \in \ddot{\mathcal{S}}$,

(2) *a* semiprime *if* $\ddot{\rho}(\ddot{h}\ddot{h}) \leq \ddot{\rho}(\ddot{h})$, $\ddot{\mu}(\ddot{h}\ddot{h}) \geq \ddot{\mu}(\ddot{h})$ *and* $\ddot{\delta}(\ddot{h}\ddot{h}) \geq \ddot{\delta}(\ddot{h})$ *for all* $\ddot{h} \in \ddot{\mathcal{S}}$.

Theorem 5.6. *Let $\ddot{\mathcal{B}}$ be a non-empty subset of an SG $\ddot{\mathcal{S}}$. Then the following statement holds.*

(1) *$\ddot{\mathcal{B}}$ is a prime BI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}}\, \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a prime TFBI of $\ddot{\mathcal{S}}$,*

(2) *$\ddot{\mathcal{B}}$ is a semiprime BI of $\ddot{\mathcal{S}}$ if and only if $\mathcal{TF}_{\ddot{\mathcal{B}}} = (\ddot{\lambda}_{\ddot{\mathcal{B}}}, \ddot{\mu}_{\ddot{\mathcal{B}}}\, \ddot{\omega}_{\ddot{\mathcal{B}}})$ is a semiprime TFGBI of $\ddot{\mathcal{S}}$.*

VI. Conclusion

In paper, we study concpet tripolar fuzzy ideals in semigroup and connection between generalized bi-ideals and tripolar fuzzy generalized bi-ideals in semigroups. In the important results, regular and intra-regular semigroups are characterized in terms of tripolar fuzzy generalized bi-ideals are provided. In the future work, we can study tripolar fuzzy interior ideals in semigroups and their fuzzifications in other algebraic structures.

References

[1] L.A. Zadeh “Fuzzy sets,” Information and Control, vol. 8, pp.338-353, 1965.

[2] N. Kuroki, “Fuzzy bi-ideals in semigroup,” Commentarii Mathematici Universitatis Sancti Pauli, vol. 5, pp.128-132, 1979.

[3] K. T. Atanassov, “Intuitionistic fuzzy sets,” Fuzzy sets and Systems, vol. 20, pp.87-96, 1986;

[4] W. Zhang “Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis,” In Proceedings of IEEE conference, pp.305-309, 1994.

[5] K. Lee, “Bipolar-valued fuzzy sets and their operations,” In proceeding International Conference on Intelligent Technologies Bangkok, Thailand. pp.307-312, 2000.

[6] T. Gaketem, and P. Khamrot, “On some semigroups characterized in terms of bipolar fuzzy weakly interior ideals,” IAENG International Journal of Computer Science, vol. 48, no. 2 pp.250-256, 2021.

[7] T. Gaketem, N. Deetae and P. Khamrot “Some semigroups characterized in terms of cubic bipolar fuzzy ideals”, Engineering Letters, vol. 30, no. 4, pp.1260-1268, 2022.

[8] M. M. K. Rao, “Tripolar fuzzy interior ideals of Γ-semigroup,” Annals of Fuzzy Mathematics Informatics, vol. 15, pp.199–206, 2018.

[9] M. M. K. Rao, “ Tripolar fuzzy ideals of Γ-semirings,” Asia Pacific Journal of Mathematics. vol. 15, pp.192-207, 2018

[10] M. M. K. Rao, “Tripolar fuzzy interior ideals and Tripolar fuzzy soft interior ideals over Γ-semirings,” Facta Universitatis, Series: Mathematics and Informatics, vol. 35, pp.29-42, 2020

[11] N. Wattanasiripong, J. Mekwian, H. Sanpan, S. Lekkoksung, “On Tripolar fuzzy pure ideals in ordered semigroups,” International Journal of Analysis and Applications. vol. 20, no. 49, pp.1-19, 2022.

[12] N. Wattanasiripong, N. Lekkoksung, S. Lekkoksung, “On Tripolar fuzzy interior ideals in ordered semigroups,” International Journal of Innovative Computing, Information and Control, vol. 28, no. 4, pp.1291-304, 2022.

[13] T. Promai, A. Iampun, T. Gaketem, “Tripolar fuzzy ideals in semigroups,” IAENG International Journal of Applied Mathematics, vol. 54, no. 12, pp.2775-2782, 2024.

[14] P. Khamrot, A. Iampun, T. Gaketem, “Tripolar fuzzy ideals which coincide in semigroups,” IAENG International Journal of Applied Mathematics, vol. 55, no. 2, pp.361-368, 2025.

[15] J. N. Mordeson, D. S. Malik, and N. Kuroki, “Fuzzy semigroup,” Springer Science and Business Media, 2003.