On the Spectrum of the Harmonic Oscillator on $\mathbb{R}$ Perturbed by a Certain Scalar Potential

Mohamed El Hammaji, Arij Bouzelmate, Ilias Aarab and Mohamed Ali Tagmouti

***Abstract*—In this work, we examine the perturbation $\mathcal{P} = H + V$, where $H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$ is the harmonic oscillator in $\mathbb{R}$ and V is a particular scalar potential. The eigenvalues of $\mathcal{P}$ are given by $\lambda_k = k + \frac{1}{2} + \mu_k$. The main result of this paper is to give an asymptotic expansion of μ_k and to connect its coefficients to a specific transformation of V.**

***Index Terms*—Pseudo-differential operator, Harmonic oscillator, Perturbation theory, Spectral asymptotics, Averging method.**

I. Introduction

THE purpose of this paper is to study the harmonic oscillator H defined by:

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right) \tag{1}$$

It's a self-adjoint differential operator with a compact resolvent. Its spectrum is the sequence of simple eigenvalues $\left\{\lambda_k = k + \frac{1}{2}\right\}_{k\in\mathbb{N}}$. We are given an even scalar potential V that satisfies the following estimate for all $x \in \mathbb{R}$:

$$\left|V^{(k)}(x)\right| \leq c_k \left(1 + x^2\right)^{\frac{-s}{2}}, s \in]0, 1[\tag{2}$$

The operator $\mathcal{P} = H + V$ is self-adjoint with a compact resolvent [1]. Its spectrum consists of the sequence $\{\lambda_k + \mu_k\}_k$. Our goal is to analyze the asymptotic behavior of μ_k approaches infinity. Now, we present the main result demonstrated in this paper.

Theorem 1. *The asymptotic behavior of μ_k is:*

$$\mu_k = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} V(\sqrt{2\lambda_k}\cos t)dt + O\left(\lambda_k^{-(s-\eta)}\right), k \to +\infty \tag{3}$$

with $\eta \in \left]0, \frac{s}{2}\right[$

In [2] Gurarie studies the case of the harmonic oscillator on IR perturbed by a scalar potential B which admits the following asymptotic behavior:

$$B(x) \sim |x|^{\alpha} \sum_m a_m \cos \omega_m x \tag{4}$$

Where $\alpha > 0$, a_m and b_m are real numbers.
We can cite A. Pushnitski [13], who studied the same perturbation but with a potential q with compact support:

$$-\frac{d^2}{dx^2} + x^2 + q(x), \quad q \in C_0^{\infty}(\mathbb{R})$$

he proved that μ_k admits the following series development:

$$\mu_k = \sum_{j=1}^{+\infty} c_j \lambda_k^{-\frac{j}{2}}, \quad \lambda_k \longrightarrow +\infty$$

such that c_j are real coefficients, in particular we have:

$$c_1 = \frac{1}{\pi}\int_{-\infty}^{+\infty} q(x)dx, \quad c_2 = 0$$

Other authors have extended the study by treating the following superquadratic oscillator:

$$A = -\frac{d^2}{dx^2} + x^{2p}, \quad p \in \mathbb{N}^*$$

Among them, we can cite R. Imekraz [4], who studied the following superquadratic oscillator:

$$A = -\frac{d^2}{dx^2} + x^{2p} + \eta(x),$$

such that p is a natural integer ≥ 2 and η is a polynomial of degree < 2, verifying:

$$\inf\left(x^{2p} + \eta(x)\right) \geq 0$$

We can also cite A. Voros [6], who dealt with the particular case of the quartic oscillator:

$$A = -\frac{d^2}{dx^2} + x^4$$

We first recall that the latter is very useful in thermodynamics and chemistry to describe the interactions between molecules. Our study argument is the averaging method used by A. Weinstein [11]. It consists of replacing V in $\mathcal{P} = H + V$ by the average:

$$\overline{V} = \frac{1}{2\pi}\int_0^{2\pi} e^{-itH} V e^{itH} dt$$

It then turns out that the spectrum of $\overline{\mathcal{P}} = H + \overline{V}$ is very close to that of $\mathcal{P}$. More precisely $\overline{\mathcal{P}}$ and $\mathcal{P}$ are almost unitarily equivalent and $\left[H, \overline{V}\right] = 0$. We first study

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Mohamed El Hammaji is a PhD Student at the Department of Mathematics, Faculty of Science, Abdelmalek Essaadi University, B.P 2121, Tetuan, Morocco. (Corresponding author to provide e-mail: mohamedelhammaji@gmail.com).
Arij Bouzelmate is a Professor and Head of the Department of Mathematics, Faculty of Science, Abdelmalek Essaadi University, B.P 2121, Tetuan, Morocco. (E-mail: abouzelmate@uae.ac.ma).
Ilias Aarab is a PhD graduate in mathematics from the Faculty of Science, Abdelmalek Essaadi University, B.P 2121, Tetuan, Morocco. (E-mail: ilias.aarab1989@gmail.com).
Mohamed ali Tagmouti is a professor of mathematics at the Faculty of Science, Abdelmalek Essaadi University, B.P 2121, Tetuan, Morocco. (E-mail: tagmoutimohamedali@gmail.com).

the spectrum of $\overline{\mathcal{P}}$, then move on to that of $\mathcal{P}$. For an overview of this kind of problem, see [12]. The article is organized as follows. In the next section, we recall supplementary information regarding certain properties of pseudo-differential operators. In section III, we study the relation between the spectrum $\mathcal{P}$ and $\overline{\mathcal{P}}$. In the section IV, we study the asymptotic behavior of μ_k and in Section V, we present a concrete example to illustrate and clarify the application of Theorem 1.

Notes and remarks

N.1 $\Gamma_\rho^m(m \in \mathbb{R}, \rho \in [0,1])$ designates the symbol class associated with the weight tempered on $\mathbb{R}^2$: $(x,\xi) \longrightarrow \left(1+x^2+\xi^2\right)^{\frac{m}{2}}$ [14]. G_ρ^m is the corresponding class of pseudodifferential operators (ΨDO) .

R.1 The integral of (3) can be viewed as an $\widetilde{V}$ transform of V :

$$\widetilde{V}(x) = \frac{2}{\pi}\int_0^x \frac{V(u)}{\sqrt{x^2-u^2}}du$$

by a change of variable we can write:

$$\widetilde{V}(x) = \frac{1}{\pi}\int_0^{x^2} \frac{V(\sqrt{u})}{\sqrt{u}\sqrt{x^2-u}}du \tag{5}$$

$\widetilde{V}$ is none other than the Abel transforma applied to x^2 of the function $u \to (u)^{-\frac{1}{2}}V(\sqrt{u})$.

R.2 It is possible to extend Theorem 1 to the case of the operator

$$(-1)^p\frac{d^{2p}}{dx^{2p}} + x^{2q}, \quad p,q \in \mathbb{N}^*.$$

We plan to detail this result in future work.

II. Pseudo-differential operators

Definition 2. *(see [14]) Let $\rho \in [0,1]$, $m \in \mathbb{R}$ and $n \in \mathbb{N}^*$. A symbol is any function $a \in C^\infty\left(\mathbb{R}^2\right)$ verifying: $\forall \alpha,\beta \in \mathbb{N}, \exists\, c_{\alpha,\beta} > 0$ such that:*

$$|\partial_x^\alpha \partial_\xi^\beta a\left(x,\xi\right)| \leq c_{\alpha,\beta}(1+x^2+\xi^2)^{\frac{m-\rho(\alpha+\beta)}{2}}$$

We denote by $\mathcal{S}\left(\mathbb{R}\right)$ the space of rapidly decreasing functions on $\mathbb{R}$.

Definition 3. *(see [3]) We define a pseudo-differential operator A as follows: for $a \in \sum_\rho^m\left(\mathbb{R}^3\right)$ (where $\sum_\rho^m$ denotes an amplitude space), $m \in \mathbb{R}$, $\rho \in [0,1]$ and $u \in \mathcal{S}\left(\mathbb{R}\right)$*

$$Au(x) = \frac{1}{2\pi}\int\int e^{i(x-y)\xi}a(x,y,\xi)u(y)dyd\xi$$

We will employ the standard Weyl quantization of symbols. In particular, if $a \in \Gamma_\rho^m$, then for $u \in S\left(\mathbb{R}\right)$ the operator associated is defined by:

$$\begin{aligned} &op^w\left(a\right)u\left(x\right) \\ = \;& \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\xi}a\left(\frac{x+y}{2},\xi\right)u\left(y\right)dyd\xi \end{aligned} \tag{6}$$

Definition 4. *Let $a_j \in \Gamma_\rho^{p_j}$, $j \in \mathbb{N}$, we suppose that p_j is a decreasing sequence tending towards $-\infty$. We say that $a \in C^\infty\left(\mathbb{R}\times\mathbb{R}\right)$ has an asymptotic expansion and we write:*

$$a = \sum_{j=0}^{+\infty} a_j,$$

if

$$a - \sum_{j=0}^{r-1} a_j \in \Gamma_\rho^{p_r}, \qquad \forall r \geq 1.$$

Theorem 5. *(Calderon-Vaillancourt Theorem)*

If $a \in \Gamma_0^0$ then the operator $op^w\left(a\right)$ is bounded on $L^2\left(\mathbb{R}\right)$.

Theorem 6. *(Compactness) If $a \in \Gamma_\rho^p, p < 0$ and $\rho \in [0,1]$, then the operator $op^\omega(a)$ is compact on $L^2(\mathbb{R})$.*

III. Comparison of $\mathcal{P}$ and $\overline{\mathcal{P}}$

We use the averaging method. To do this, we first note that the Hamiltonian flow associated with the symbol

$$\sigma_H(x,\xi) = \frac{1}{2}(x^2+\xi^2)$$

of operator H is a one-parameter group whose elements are square matrices of order 2:

$$\chi_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{7}$$

Observe that this flow is periodic with a period of 2π. To initiate the averaging method, we introduce the following operators:

$$W(t) = e^{-itH}Ve^{itH}, \tag{8}$$

$$\overline{V} = \frac{1}{2\pi}\int_0^{2\pi} W(t)dt, \tag{9}$$

$$\overline{\overline{V}} = \frac{1}{4\pi i}\int_0^{2\pi}\int_0^t [W(t),W(r)]drdt. \tag{10}$$

Since H commute with $\overline{V}$, the spectrum of $\overline{\mathcal{P}}$ is $\{\lambda_k + \overline{\mu}_k\}$, where $\bar{\mu}_k$ is the k^{th} eigenvalue of $\overline{V}$. To compare μ_k and $\bar{\mu}_k$ we will need the following lemmas:

Lemma 7. $\left[H,\overline{V}\right] = 0$

Proof: After we derive $W(t)$, we obtain:

$$\frac{dW(t)}{dt} = \frac{1}{i}\left[H,W(t)\right] \tag{11}$$

Now, we have:

$$\left[H,\overline{V}\right] = \frac{i}{2\pi}\int_0^{2\pi}\frac{dW(t)}{dt}dt = \frac{i}{2\pi}\left(W(2\pi)-W(0)\right) \tag{12}$$

Since $e^{2\pi iH} = -id_{L^2(\mathbb{R})}$, we get $W(2\pi) = W(0)$
Finally, we have $\left[H,\overline{V}\right] = 0$. ■

Lemma 8.

$$i/\overline{V} \in G_0^{-s}, \quad ii/\overline{\overline{V}} \in G_0^{-2s+2\eta} \tag{13}$$

where $\eta \in \left]0, \frac{s}{2}\right[$

Proof: i/The Weyl symbol of the operator $W(t)$ is

$$\sigma_{W(t)} = V o \chi_t \tag{14}$$

where χ_t is the flow described in (7).
This result arises because, on the one hand, e^{itH} belongs to the Metaplectic group, and on the other hand, Weyl's quantization is invariant under this group ([8],[9]). The Weyl's symbol of $\overline{V}$ is obtained by integrating the symbol of $W(t)$ uniformly with respect to t.

$$\sigma_{\overline{V}}(x,\xi) = \frac{1}{2\pi}\int_0^{2\pi} V(x\cos t + \xi \sin t)dt. \tag{15}$$

By using (2), we get the following estimate, for $\alpha, \beta \in \mathbb{N}$ and $x, \xi \in \mathbb{R}$:

$$\begin{aligned} & \left|\partial_x^\alpha \partial_\xi^\beta \sigma_{\overline{V}}(x,\xi)\right| \\ \leq\ & C_{\alpha,\beta}\int_0^{2\pi}\left[1+(x\cos t+\xi\sin t)^2\right]^{\frac{-s}{2}} dt \\ \leq\ & C_{\alpha,\beta}\int_0^{2\pi}\left[1+(x^2+\xi^2)\cos^2 t\right]^{\frac{-s}{2}} dt \\ \leq\ & C_{\alpha,\beta}\left(\int_0^{\frac{\pi}{2}}(\cos t)^{-s}dt\right)\times\left(1+x^2+\xi^2\right)^{-\frac{s}{2}} \\ \leq\ & C_{\alpha,\beta}\left(1+x^2+\xi^2\right)^{-\frac{s}{2}} \end{aligned} \tag{16}$$

due to $\int_0^{\frac{\pi}{2}}(\cos t)^{-s}dt < +\infty$, if $s \in]0,1[$.
ii/ According to the previous calculations, the operator $B(t) = \int_0^t W(r)dr$ belongs to G_0^{-s}, its Weyl's symbole $\sigma_{B(t)}$ check:

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma_{B(t)}(x,\xi)\right| \leq C_{\alpha,\beta}(1+x^2+\xi^2)^{\frac{-s}{2}} \tag{17}$$

uniformly with respect to t.
Let us begin by clarifying the class of the operator $\int_0^{2\pi} W(t)B(t)dt$. At this point, we are focusing on the operator $W(t)B(t)$, its Weyl symbol c_t is given in [14] by:

$$\begin{aligned} c_t(x,\xi) \ =\ & \tfrac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}\sigma_{W(t)}(x+\omega,\xi+\rho) \\ & \times\sigma_{B(t)}(x+r,\xi+\tau)d\rho d\omega d\tau dr. \end{aligned} \tag{18}$$

We split the oscillator integral c_t into two parts $c_t^{(1)}$ and $c_t^{(2)}$, then we use the cutoff functions:

$$\omega_{1,\varepsilon}(x,\xi,\omega,\tau,r,\rho) = \chi\left[\frac{\omega^2+\rho^2+r^2+\tau^2}{\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}}\right] \text{ and } \omega_{2,\varepsilon} = 1-\omega_{1,\varepsilon}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ in $[-1,1]$, $\chi \equiv 0$ in $\mathbb{R}\backslash]-2,2[$, $R = \omega^2+\rho^2+r^2+\tau^2$, $\varepsilon > 0$ and $\eta \in \left]0,\frac{1}{2}\right[$. Let's consider

$$\begin{aligned} d_j(x,\xi,\omega,\tau,r,\rho) \ =\ & \omega_{j,\varepsilon}(x,\xi,\omega,\tau,r,\rho) \\ & \times\sigma_{W(t)}(x+\omega,\xi+\rho) \\ & \times\sigma_{B(t)}(x+r,\xi+\tau) \end{aligned} \tag{19}$$

$c_t^{(1)}$ (resp $c_t^{(2)}$) the integral obtained in (18) by replacing the amplitude by d_1 (resp d_2)

Study of $c_t^{(2)}$ **:**

On the support of d_2, we have $R \geq \varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}$. We make an integration by parts using the operator:

$$M = \tfrac{1}{2iR}(-\rho\partial_r - r\partial_\rho + \tau\partial_\omega + \omega\partial_\tau)$$

We have for all $k \in \mathbb{N}$

$$c_t^{(2)} = \frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}({}^tM)^k d_2\, d\rho\, d\omega\, d\tau\, dr$$

Then we get for all $k > 0$

$$\left|c_t^{(2)}\right| \leq C_k(1+x^2+\xi^2)^{\frac{-\eta k}{4}}$$

uniformly with respect to $t \in [0,2\pi]$

Study of $c_t^{(1)}$ **:**

On the support of d_1, we have

$$\begin{aligned} c_t^{(1)}(x,\xi) = \frac{1}{\pi^2}\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} & e^{-2i(r\rho-\omega\tau)} \\ & \times\sigma_{W(t)}(x+\omega,\xi+\rho) \\ & \times\sigma_{B(t)}(x+r,\xi+\tau)\omega_{1,\varepsilon}d\rho d\omega d\tau dr \end{aligned} \tag{20}$$

$$\begin{aligned} \int_0^{2\pi}\left|c_t^{(1)}\right|dt \leq c & \int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} d\rho d\omega d\tau dr \\ & \times\int_0^{2\pi}\left|\sigma_{W(t)}(x+\omega,\xi+\rho)\right|dt \\ & \times\int_0^{2\pi}\left|\sigma_{B(t)}(x+r,\xi+\tau)\right|dt. \end{aligned} \tag{21}$$

On the support of d_1, for ε small enough and since $\eta \in \left]0,\frac{1}{2}\right[$, there are positive constants c, c', C, C' such that:

$$\begin{cases} c(1+x^2+\xi^2)^{\frac{1}{2}} \leq (1+(x+\omega)^2+(\rho+\xi)^2)^{\frac{1}{2}} \\ (1+(x+\omega)^2+(\rho+\xi)^2)^{\frac{1}{2}} \leq C(1+x^2+\xi^2)^{\frac{1}{2}} \\ c'(1+x^2+\xi^2)^{\frac{1}{2}} \leq (1+(x+r)^2+(\tau+\xi)^2)^{\frac{1}{2}} \\ (1+(x+r)^2+(\tau+\xi)^2)^{\frac{1}{2}} \leq C'(1+x^2+\xi^2)^{\frac{1}{2}} \end{cases}$$

Therefore

$$\begin{aligned} \int_0^{2\pi} c_t^{(1)}dt \ \leq\ & C(1+x^2+\xi^2)^{-s} \\ & \times\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} d\rho d\omega d\tau dr \end{aligned} \tag{22}$$

Finally

$$\int_0^{2\pi} c_t^{(1)}dt \leq c(1+x^2+\xi^2)^{-s+\eta} \tag{23}$$

In the end, by denoting σ as the Weyl symbol of the operator $\int_0^{2\pi} W(t)B(t)dt$, we have:

$$\begin{aligned} |\sigma| &\leq \int_0^{2\pi} \left|c_t^{(1)}\right| dt + \int_0^{2\pi} \left|c_t^{(2)}\right| dt \\ &\leq C\left[(1+x^2+\xi^2)^{\frac{-\eta k}{4}} + (1+x^2+\xi^2)^{-s+\eta}\right] \\ &\leq C(1+x^2+\xi^2)^{\frac{-2s+2\eta}{2}} \end{aligned}$$

Finally, we deduce that $\overline{\overline{V}} \in G_0^{-2s+2\eta}$. ■

Lemma 9. *There exists a skew-symmetric operator $U \in G_0^{-s}$ such as the operator $(e^U \mathcal{P} e^{-U} - \overline{\mathcal{P}})H^{s-\eta}$ is bounded.*

The Proof of this Lemma is based on the following proposition that presents an extension of the symbolic calculus to boundary classes G_0^m.

Proposition 10. *(see [5])*
i) If $A \in G_1^{m_1}$ and $B \in G_0^{m_2}$ then the operator $AB \in G_0^{m_1+m_2}$. Its Weyl symbol admits an asymptotic development:

$$c = \sum_{j=0}^{+\infty} c_j, \quad c_j \in \Gamma_0^{m_1+m_2-j}$$

where

$$c_j = \frac{1}{2^j} \sum_{\alpha+\beta=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (\partial_\xi^\alpha \partial_x^\beta a)(\partial_x^\alpha \partial_\xi^\beta b)$$

ii) The commutator $[A, B] \in G_0^{m_1+m_2-1}$
iii) If $(B_i)_{i\in\{1,\cdots,n\}}$ is the family of operators such as $B_i \in G_0^{m_i}$. Then the operator

$$B_1 B_2 \cdots B_n H^{-\frac{m_1+\cdots+m_n}{2}}$$

is bounded.

In our work, we use the functional calculus of operators H, where the function f verifies the following estimates:
For $r \in \mathbb{R}, k \in \mathbb{N}$ and $\rho \in [\frac{1}{2}, 1]$

$$|f^{(k)}(x)| \leq C_k (1+|x|)^{r-\rho k}$$

Proposition 11. *$f(H)$ is a (ΨDO) included in $G_{1-2(1-\rho)}^{2r}$ and its weyl symbol admits the following development*

$$\sigma_{f(H)} = \sum_{j\geq 0} \sigma_{f(H),2j}$$

$$\sigma_{f(H),2j} = \sum_{k=2}^{3j} \frac{d_{jk}}{k!} f^{(k)}(\sigma_H), \quad \forall j \geq 1$$

where

$$d_{j,k} \in \Gamma_1^{2k-4j}, \quad \sigma_{f(H),2j} \in \Gamma_{1-2(1-\rho)}^{2r-j(6\rho-2)} \tag{24}$$

in particular

$$\sigma_{f(H),0} = f(\sigma_H)$$

Proof: For studying $f(H)$ we follow the same strategy in [7], using the Mellin transformation, the latter consists of the following steps:
(1) We prove by induction that $(H-\lambda)^{-1}$,$\lambda \in \ C$, is a (ΨDO) and its Weyl symbol admits the development $b_\lambda = \sum_{j=0}^{+\infty} b_{j,\lambda}$ where:

$$\begin{cases} b_{0,\lambda} = (\sigma_H - \lambda)^{-1}, \\ b_{2j+1,\lambda} = 0, \\ b_{2j,\lambda} = \sum_{k=2}^{3j} (-1)^k d_{j,k}.b_{0,\lambda}^{k+1}, \quad d_{j,k} \in \Gamma_1^{2k-4j}. \end{cases}$$

(2) We study the operator H^s using the Cauchy's integral formula

$$H^s = \frac{1}{2\pi i} \int_\Delta \lambda^s (H-\lambda)^{-1} d\lambda$$

Δ is the same domain defined in the article [7]
H^s is a (ΨDO) and its Weyl symbol is
$\sigma_s = \sum_{j=0}^{+\infty} \sigma_{s,2j}$ where $\sigma_{s,0} = \sigma_H^s$, and

$$\sigma_{s,2j} = \sum_{k=2}^{3j} d_{j,k}.\frac{s(s-1)\cdots(s-k+1)}{k!} \sigma_H^{s-k}$$

with

$$\sigma_{s,2j} \in \Gamma_1^{2s-4j}$$

(3) We study $f(H)$ using the representation formula:

$$f(H) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} M[f](s) H^{-s} ds$$

$\sigma \in [0, -r[$, $r < 0$ and $M[f]$ is the Mellin transformation of f. ■

Proof of Lemma 9

Proof: Consider the following antisymmetrical operator U:

$$U = U_1 + U_2 \tag{25}$$

where

$$U_1 = \frac{i}{2\pi} \int_0^{2\pi} (2\pi - t) W(t) dt$$

$$U_2 = \frac{-1}{4\pi} \int_0^{2\pi} (2\pi - t) \int_0^t [W(t), W(r)] dr dt$$

Using the same calculations as those in Lemma 8, we obtain: $U_1 \in G_0^{-s}$ and $U_2 \in G_0^{-2s+2\eta}$, finally $U \in G_0^{-s}$.
Before beginning the proof, we will need the following relations:

$$\begin{aligned} [U_1, H] &= \frac{i}{2\pi} \int_0^{2\pi} (2\pi - t) \frac{dW(t)}{dt} dt \\ &= \overline{V} - V \end{aligned} \tag{26}$$

and

$$\begin{aligned}[U_2,H] &= \frac{-1}{4\pi}\int_0^{2\pi}(2\pi-t)\int_0^t [[W(t),W(r)],H]drdt\\ &= \frac{i}{4\pi}\int_0^{2\pi}(2\pi-t)\\ &\times\int_0^t\left(\left[W(t),W^{'}(r)\right]+\left[W^{'}(t),W(r)\right]\right)drdt\end{aligned} \tag{27}$$

We set:

$$F(t)=\frac{1}{2\pi}\int_0^t W(r)dr$$

On the one hand :

$$\begin{aligned}&\frac{i}{4\pi}\int_0^{2\pi}(2\pi-t)\int_0^t\left[W(t),W^{'}(r)\right]drdt\\ &=\frac{i}{4\pi}\int_0^{2\pi}(2\pi-t)\left[W(t),\int_0^t W^{'}(r)dr\right]dt\\ &=\frac{-i}{4\pi}\int_0^{2\pi}(2\pi-t)\left[W(t),V\right]dt\\ &=\tfrac{-1}{2}\left[U_1,V\right].\end{aligned}$$

on the other hand :

$$\begin{aligned}&\frac{i}{4\pi}\int_0^{2\pi}(2\pi-t)\int_0^t\left[W^{'}(t),W(r)\right]drdt\\ &=\frac{i}{2}\int_0^{2\pi}(2\pi-t)\left[W^{'}(t),F(t)\right]dt\\ &=\frac{i}{2}\int_0^{2\pi}(2\pi-t)\frac{d}{dt}([W(t),F(t)])dt\\ &=\frac{i}{2}([(2\pi-t)\left[W(t),F(t)\right]]_0^{2\pi}+\int_0^{2\pi}[W(t),F(t)]dt)\\ &=-\overline{\overline{V}}\end{aligned}.$$

Finally, we have :

$$[U_2,H]=-\overline{\overline{V}}-\frac{1}{2}\left[U_1,V\right] \tag{28}$$

We notice $AdU.\mathcal{P}=[U,\mathcal{P}]$. The differential equation:

$$\begin{cases}\frac{dX}{dt}=[U,X]\\ X(0)=\mathcal{P}\end{cases} \tag{29}$$

has a unique solution

$$X(t)=e^{tADU}.\mathcal{P}=e^{tU}\mathcal{P}e^{-tU}$$

We deduce, taking into account (26) and (28) that :

$$\begin{aligned}e^U\mathcal{P}\,e^{-U}-\overline{\mathcal{P}} \;=\;& -\overline{\overline{V}}+\tfrac{1}{2}\left[U_2,V\right]\\ &+\tfrac{1}{2}\left[U,\overline{V}\right]+\tfrac{1}{4}\left[U,[U_1,V]\right]\\ &+\tfrac{1}{2}\left[U,[U_2,V]\right]-\tfrac{1}{2}\left[U,\overline{\overline{V}}\right]\\ &+\sum_{n\geq 0}\frac{(AdU)^n}{(n+3)!}\left[U,[U,[U,\mathcal{P}]]\right].\end{aligned} \tag{30}$$

We now apply Proposition 10, since $V\in G_0^0$, $\overline{V}\in G_0^{-s}$, $U_1,U\in G_0^{-s}$ and $U_2,\overline{\overline{V}}\in G_0^{-2s+2\eta}$, we get :

$$\begin{cases}\left\|\overline{\overline{V}}.H^{s-\eta}\right\| &\leq C\\ \left\|[U_2,V]\,H^{s-\eta}\right\| &\leq C\\ \left\|\left[U,\overline{V}\right]H^s\right\| &\leq C\\ \left\|[U,[U_1,V]]\,H^s\right\| &\leq C\\ \left\|[U,[U_2,V]]\,H^{\frac{3s}{2}-\eta}\right\| &\leq C\\ \left\|\left[U,\overline{\overline{V}}\right]H^{\frac{3s}{2}-\eta}\right\| &\leq C\\ \left\|\frac{(AdU)^n}{(n+3)!}\left[U,[U,[U,\mathcal{P}]]\right]H^{\frac{3s}{2}-\frac{1}{2}}\right\| &\leq C\left\|U\right\|^n\end{cases} \tag{31}$$

For the last inequality, we used the following identity

$$(AdU)^n.W=\sum_{p=0}^{n}(-1)^{n-p}C_n^pU^pWU^{n-p}$$

From (30) and (31) we deduce that : $(e^U\mathcal{P}e^{-U}-\overline{\mathcal{P}})H^{s-\eta}$ is bounded. ■

We can now compare μ_k and $\overline{\mu}_k$. From lemma 9, we deduce that there exists a constant $c>0$ such that:

$$-cH^{-(s-\eta)}\leq e^U\mathcal{P}e^{-U}-\overline{\mathcal{P}}\leq cH^{-(s-\eta)}$$

The min-max Theorem (see [10]) implies that:

$$\mu_k=\overline{\mu}_k+O(\lambda_k^{-(s-\eta)}), \tag{32}$$

where $\eta\in\left]0,\frac{s}{2}\right[$.

IV. The asymptotic behavior of μ_k

We begin by studying the asymptotic behavior of $\overline{\mu}_k$, as a result of using (32), we deduce that of μ_k. Let us first recall that $\overline{\mu}_k$ is the k^{th} eingenvalue of $\overline{V}$. In polar coordinates the identity (15) that presents the symbol of Weyl of $\overline{V}$ is written:

$$\sigma_{\overline{V}}(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}V(r\left(\cos(t-\theta)\right)dt$$

From the parity of V we get:

$$\sigma_{\overline{V}}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}V\left(r\cos(t)\right)dt=f(\sqrt{\sigma_H})$$

where

$$f(x)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}V\left(\sqrt{2x}\cos t\right)dt$$

,

A direct calculation shows that:

$$|f(x)|\leq c(1+|x|)^{-\frac{s}{2}}$$

and

$$\left|f^{(k)}(x)\right|\leq c_k(1+|x|)^{-\frac{s}{2}-\frac{k}{2}}$$

so f is in the class of Hörmander $S_{\frac{1}{2}}^{\frac{-s}{2}}$. By applying the Proposition 11, we have

$$f(H)\in G_0^{-s}$$

and

$$\overline{V} - f(H) \in G_0^{-s-1} \tag{33}$$

By combining the equation (33) and the Proposition 10-iii), we deduce that

$$(\overline{V} - f(H))H^{\frac{s+1}{2}}$$

is bounded.
Come back to the Proof of Theorem 1. Therefore, there exists a constant $c > 0$ such that

$$-cH^{-\frac{s+1}{2}} \leq \overline{V} - f(H) \leq cH^{-\frac{s+1}{2}}$$

According to the min-max Theorem [10], we get:

$$\overline{\mu}_k = f(\lambda_k) + O\left(\lambda_k^{-(\frac{s+1}{2})}\right) \tag{34}$$

By combining the equation (34) and (32) we deduce:

$$\mu_k = f(\lambda_k) + O(\lambda_k^{-(s-\eta)}),$$

Finaly, we have:

$$\mu_k = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} V\left(\sqrt{2\lambda_k}\cos t\right)dt + O(\lambda_k^{-(s-\eta)}),$$

where $\eta \in \left]0, \frac{s}{2}\right[$.

V. Example

In some cases of V, we can further improve the estimate given by (3) by determining $\sup\{p \in \ R_*^+/\lambda_k^p\mu_k \to 0; \quad \lambda_k \to +\infty\}$. To do this, it is sufficient to study the asymptotic behavior of the integral of (3). As an example, we consider the potential.

$$V(x) = \frac{1}{(1+x)^s}e^{-wx^2}$$

with $s \in]0, 1[$ and $w > 0$.
Since the function $x \to P(x)e^{-wx^2}$ tends to 0 as x tends to infinity for any polynomial P we have:

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} V\left(\sqrt{2\lambda_k}\cos t\right)dt \\ &= \int_0^{\varepsilon}\left(1+\sqrt{2\lambda_k}\sin t\right)^{-s} e^{-w(2\lambda_k)\sin^2 t}dt + O(\lambda_k^{-\infty})\end{aligned}. \tag{35}$$

where $\varepsilon > 0$ is near zero.
By Taylor's formula, we can write:

$$\begin{aligned}&\int_0^{\varepsilon}\left(1+\sqrt{2\lambda_k}\sin t\right)^{-s} e^{-w(2\lambda_k)\sin^2 t}dt \\ &= (2\lambda_k)^{-\frac{s}{2}}\int_0^{\varepsilon}(\sin t)^{-s}e^{-w(2\lambda_k)\sin^2 t}dt + O(\lambda_k^{-\frac{s}{2}-\frac{1}{2}})\end{aligned}. \tag{36}$$

Using Laplace's method [15], we have:

$$\begin{aligned}&\int_0^{\varepsilon}(\sin t)^{-s}e^{-w(2\lambda_k)\sin^2 t}dt \\ &= \frac{1}{2w^{\frac{1-s}{2}}(2\lambda_k)^{\frac{1-s}{2}}}\Gamma\left(\tfrac{1-s}{2}\right) + O\left(\lambda_k^{-\frac{s+1}{2}}\right)\end{aligned}. \tag{37}$$

Finally

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} V\left(\sqrt{2\lambda_k}\cos t\right)dt \\ &= \frac{1}{2w^{\frac{1-s}{2}}(2\lambda_k)^{\frac{1}{2}}}\Gamma\left(\tfrac{1-s}{2}\right) + O\left(\lambda_k^{-\frac{s+1}{2}}\right)\end{aligned} \tag{38}$$

and for $s \in]\frac{1}{2}, 1[$ we have:

$$\mu_k = \frac{1}{\pi w^{\frac{1}{2}}(2\lambda_k)^{\frac{1}{2}}}\Gamma\left(\frac{1-s}{2}\right) + O\left(\lambda_k^{-(s-\eta)}\right)$$

We can also study the case where the potential admits the asymptotic behavior given by (4), we can even suppose that the frequency w_m depends on x.

VI. Conclusion

The perturbed harmonic oscillator is a prominent problem in spectral theory, largely due to its wide-ranging applications in physics. While several techniques are available to handle such problems, we chose to apply the averaging method, given the periodic nature of the harmonic oscillator's flow. Our approach allowed us to describe the asymptotic behavior of its spectrum. Moving forward, we plan to explore the application of this method to anharmonic oscillators.

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