Multiple Performance Heterogeneous Reliability Modeling via Copula Function and Wiener Processes

Huibing Hao, Chunping Li

***Abstract*—Modern products are increasingly complex and featuring numerous functions, which often results in multiple degradation performance characteristics (PCs). This research introduces a bivariate reliability model tailored for systems with two PCs, and different PCs are modeled by heterogeneous Wiener process. The interdependences between these PCs are captured by using the Frank copula. Utilizing AIC and BIC criteria, the best fitted marginal degradation process is obtained, and the unknown parameters are obtained utilizing a combination of MLE and MCMC methods. To validate the utility and effectiveness of the proposed model and method, a numerical example involving train wheel wear degradation data is provided.**

***Index Terms*—Copula function, Wiener process, MCMC method, MLE method**

I. Introduction

With advancements in science and technology, modern products now exhibit longer lifetimes and greater reliability. Traditional lifetime data theory (see Meeker and Escobar [1], Nelson [2]) is insufficient for accurately assessing their reliability. Compared to traditional lifetime data, degradation data offers a more appealing approach for reliability assessment.

Degradation is very common for most mechanical system, and it can be described by a continuous performance process in terms of time (see Zuo et al. [3]). Given the dynamic nature of failure mechanisms and operational environments, many studies employ stochastic processes to model degradation paths, utilizing methods including Gamma processes, Markov chain, and Wiener processes (the details see in Refs [4-8]).

Most previous studies on degradation analysis focus on a single performance characteristic (PC). However, modern products often have complex structures and multiple functions, leading to multiple degradation failure mechanisms. For instance, a train wheel system consists of a wheel on the left axle and a wheel on the right axle, each experiencing different wear patterns. This indicates that the train wheels system has two PCs (left wheel and right wheel), and these two PCs may be dependent each other because they are subjected to the same stress (e.g. weight). Consequently, accurately analyzing the reliability of such systems poses a significant challenge.

Several papers have been studied the reliability based on binary or multivariate degradation data, such as Crk [9] and Bagdonavicius et al. [10]. Some studies assume independence among multiple PCs or use a multivariate normal distribution (see in Refs [11-12]). Sari [13-14] developed a dependence system by using copula function. Base on Sari's work, Pan et al. [15-16] introduced a multiple PC degradation model by the Wiene process and copula function. Hao et al. [17] developed a multiple PC degradation model using a nonlinear diffusion process and copula. Wang et al. [18] devised bivariate gamma degradation model utilizing copula function. Peng et al. [19] developed a multiple PC degradation model via the copula and Inverse Gaussian process.

The aforementioned studies all assume that bivariate models share the same stochastic process governing both PCs. However, those papers overlook the possibility that each PC may exhibit different stochastic behaviors. As an example, consider a system equipped with two PCs, where the first PC follows one stochastic process, and the second PC may be governed by another stochastic process.

This paper introduces a novel bivariate reliability model by using the Frank copula function. Unknown parameters are obtained by using MLE and MCMC methods. An illustrative numerical example is included to demonstrate the proposed model and methodology.

II. Wiener process modeling

A. *Marginal model based on Wiener process*

Consider a system with two PCs, where degradation is characterized using a Wiener process. In the degradation test experiment, let n represent the unit number and m depict the measurement number. $X_{ik}(t_j)$ represents the ith unit of the kth PC at the corresponding time t_j, where $i = 1, \dots, n$, $j = 1, \dots, m$, and $k = 1, 2$. The degradation data can be expressed as

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$$X_{2n\times m}=\begin{pmatrix}X_1\\X_2\end{pmatrix}=\begin{pmatrix}X_{11}(t_1)&\cdots&X_{11}(t_m)\\\vdots&\ddots&\vdots\\X_{n1}(t_1)&\cdots&X_{n1}(t_m)\\X_{12}(t_1)&\cdots&X_{12}(t_m)\\\vdots&\ddots&\vdots\\X_{n2}(t_1)&\cdots&X_{n2}(t_m)\end{pmatrix}\tag{1}$$

In the next subsection, the heterogeneous Wiener processes are used to represent the above degradation data, including the random effect Wiener process and the Wiener process with measurement error.

B. *Different Wiener processes model*

If product degradation trajectory is conceptualized under a fixed effect Wiener process model as M_1, is

$$X(t)=\mu t+\sigma_B B(t)\tag{2}$$

Here, $B(t)$ denotes the standard Brownian motion that captures a time correlated structure, and μ and σ denote the drift and diffusion parameters, respectively.

If the degradation path is characterized by process M_1, and when the degradation path reaches the threshold value ξ , the product's lifetime T is defined as

$$T=\inf\{t\mid X(t)\geq\xi\}\tag{3}$$

The lifetime T follows an inverse Gaussian distribution, and its corresponding PDF and CDF can be given as

$$f_T(t\mid\mu,\sigma_B)=\frac{\xi}{\sqrt{2\pi\sigma_B^2t^3}}\exp\left(-\frac{(\xi-\mu t)^2}{2\sigma_B^2t}\right)\tag{4}$$

and

$$F_T(t\mid\mu,\sigma_B)=\Phi\left(\frac{\mu t-\xi}{\sqrt{\sigma_B^2t}}\right)+\exp\left(\frac{2\mu\xi}{\sigma_B^2}\right)\Phi\left(-\frac{\mu t+\xi}{\sqrt{\sigma_B^2t}}\right)\tag{5}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

In some degradation model applications, considering that individual items may experience various sources of variation throughout their operational life. To enhance the realism of a degradation model, it becomes essential to incorporate unit to unit variability into the modeling process. Random effect model is used to depict differences across units. For the sake of analytical tractability, assumed that the drift parameter μ is random effect follows $N(\mu_\beta,\sigma_\beta^2)$ and the diffusion parameter σ is defined as fixed effect. Then, the conventional random effect model can be derived as M_2

$$\begin{cases}X(t)=\mu t+\sigma_B B(t)\\\mu\sim N(\mu_\beta,\sigma_\beta^2)\end{cases}\tag{6}$$

Utilizing the degradation model (6), we can characterize the deterioration behavior and the unit to unit heterogeneity of the product. Additionally, we can also capture the temporal correlation structure inherent to the degradation process.

Under the degradation model (6), considering the threshold value ξ , the product's lifetime T_r can be delineated as

$$T_r=\inf\{t\mid X(t)\geq\xi\}\tag{7}$$

Considering the diffusion coefficient σ is a constant, but μ is a random variable, utilizing the total law of probability, the following equation is derived as follow

$$f_{T_r}(t\mid\sigma)=\int_{-\infty}^{+\infty}f_T(t\mid\mu,\sigma)\varphi\left(\frac{\mu-\mu_\beta}{\sigma_\beta}\right)d\mu$$

$$=\sqrt{\frac{\xi^2}{2\pi(\sigma^2+\sigma_\eta^2t)t^3}}\exp\left(-\frac{(\xi-\eta t)^2}{2(\sigma^2t+\sigma_\eta^2t^2)}\right)\tag{8}$$

Then, we can get

$$R_r(t)=\Phi\left(-\frac{\mu_\beta t-\xi}{\sqrt{\sigma_\beta^2t^2+\sigma^2t}}\right)$$

$$-\exp\left(\frac{2\mu_\beta\sigma^2\xi+2\sigma_\beta^2\xi^2}{\sigma^4}\right)\Phi\left(-\frac{2\sigma_\beta^2\xi t+\sigma^2(\mu_\beta t+\xi)}{\sigma^2\sqrt{\sigma_\beta^2t^2+\sigma^2t}}\right)\tag{9}$$

C. *Wiener process with measurement error*

Perfect measurement is usually costly or impossible. On the contrary, some measurement errors are inevitable during the observation. For example, degradation processes are often measured using imperfect instruments. Additionally, random environmental factors can also affect the measurements.

Let $Y(t)$ denote the Wiener process with measurement error, with ε representing the measurement error. Then, the observed degradation process $\{Y(t)\}$ is given as M_3

$$Y(t)=X(t)+\varepsilon=\mu t+\sigma_B B(t)+\varepsilon\tag{10}$$

Here, $X(t)$ represents the actual degradation, while ε denotes the measurement error and $\varepsilon\sim N(0,\sigma_\varepsilon^2)$.

Similarly, considering the threshold value ξ, the product's lifetime T_e is defined as

$$T_e=\inf\{t\mid Y(t)\geq\xi\}\tag{11}$$

From Ref. [20], we can get the PDF of the lifetime T_e as

$$f_{T_e}(t\mid\mu,\sigma_B,\sigma_\varepsilon)=\frac{\mu\sigma_\varepsilon^2+\xi\sigma_B^2}{\sqrt{2\pi\left(\sigma_\varepsilon^2+\sigma_B^2t\right)^3}}\exp\left(-\frac{(\xi-\mu t)^2}{2\left(\sigma_\varepsilon^2+\sigma_B^2t\right)}\right)\tag{12}$$

and the corresponding CDF is

$$F_{T_e}(t\mid\mu,\sigma_B,\sigma_\varepsilon)=1-\Phi\left(\frac{\xi-\mu t}{\sqrt{\sigma_\varepsilon^2+\sigma_B^2t}}\right)+\exp\left(\frac{2\mu\xi}{\sigma_B^2}+\frac{2\mu^2\sigma_\varepsilon^2}{\sigma_B^4}\right)$$

$$\times\Phi\left(-\frac{\xi+\mu t+2\mu\sigma_\varepsilon^2/\sigma_B^2}{\sqrt{\sigma_\varepsilon^2+\sigma_B^2t}}\right)\tag{13}$$

Similarly, the conventional random effect model with measurement error can be derived as M_4

$$\begin{cases}Y(t)=\mu t+\sigma_B B(t)+\varepsilon\\\mu\sim N(\mu_\beta,\sigma_\beta^2)\end{cases}\tag{14}$$

Then, considering the threshold value ξ, the product's lifetime T_{re} is defined as

$$T_{re}=\inf\{t\mid Y(t)\geq\xi\}\tag{15}$$

Lemma 1 If $Z\sim N(\mu,\sigma^2)$, and $A,C,D\in R$, then

$$E_Z[\exp(A-Z)\exp(-(B-Z)^2/2C)]$$
$$=\sqrt{\frac{C}{\sigma^2+C}}\cdot\left(A-\frac{\sigma^2B+\mu C}{\sigma^2+C}\right)\exp\left(-\frac{(B-\mu)^2}{2(\sigma^2+C)}\right)$$

The proof is detailed in Ref. [20].

Lemma 2 If $Z\sim N(\mu,\sigma^2)$, and $C,D\in R$, then

$$E_Z[\Phi(C+DZ)]=\Phi\left(\frac{C+D\mu}{\sqrt{1+D^2\sigma^2}}\right)$$

The proof is detailed in Ref. [20].

Lemma 3 If $Z\sim N(\mu,\sigma^2)$, and $C,D\in R$, then

$$E_Z[\exp(AZ+BZ^2)\Phi(C+DZ)]=\frac{1}{\sqrt{1-2B\sigma^2}}$$
$$\times\exp\left(\frac{2(A\mu+B\mu^2)+A^2\sigma^2}{2(1-2B\sigma^2)}\right)\Phi\left(\frac{C+D\mu+(AD-2BC)\sigma^2}{(1-2B\sigma^2)\sqrt{1+D^2\sigma^2}}\right)$$

Proof

$$E_Z[\exp(AZ+BZ^2)\Phi(C+DZ)]$$
$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}\exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right)\exp(Az+Bz^2)\Phi(C+DZ)dz$$
$$=\frac{1}{\sqrt{1-2B\sigma^2}}\exp\left(\frac{2(A\mu+B\mu^2)+A^2\sigma^2}{2(1-2B\sigma^2)}\right)$$
$$\times E_u\left[\Phi\left(C+\frac{D\mu+AD\sigma^2}{(1-2B\sigma^2)}+\frac{D\sigma}{\sqrt{1-2B\sigma^2}}u\right)\right]$$

By using Lemma 2, we can get

$$E_Z[\exp(AZ+BZ^2)\Phi(C+DZ)]$$
$$=\frac{1}{\sqrt{1-2B\sigma^2}}\exp\left(\frac{2(A\mu+B\mu^2)+A^2\sigma^2}{2(1-2B\sigma^2)}\right)\Phi\left(\frac{C+D\mu+(AD-2BC)\sigma^2}{(1-2B\sigma^2)\sqrt{1+D^2\sigma^2}}\right)$$

Theorem 1 Suppose that the degradation process $\{Y(t)\}$ is a continuously differentiable function of time t, then the PDF of the lifetime T_{re} can be obtained with an explicit form as

$$f_{T_e}(t\mid\mu,\sigma_B,\sigma_\varepsilon)=\frac{\mu\sigma_\varepsilon^2+\xi\sigma_B^2}{\sqrt{2\pi\left(\sigma_\varepsilon^2+\sigma_B^2t\right)^3}}\sqrt{\frac{\sigma_\varepsilon^2+\sigma_B^2t}{\sigma_\varepsilon^2+\sigma_B^2t+\sigma_\beta^2t^2}}$$
$$\times\left(\xi\sigma_B^2+\sigma_\varepsilon^2\frac{\sigma_\beta^2\xi t+\mu_\beta(\sigma_\varepsilon^2+\sigma_B^2t)}{\sigma_\varepsilon^2+\sigma_B^2t+\sigma_\beta^2t^2}\right)\exp\left(-\frac{(\xi-\mu_\beta t)^2}{2\left(\sigma_\varepsilon^2+\sigma_B^2t+\sigma_\beta^2t^2\right)}\right) \quad (16)$$

and the corresponding CDF is

$$F_{T_{re}}(t\mid\sigma_B,\sigma_\varepsilon)=1-\Phi\left(\frac{\xi-\mu_\beta t}{\sqrt{\sigma_\varepsilon^2+\sigma_B^2t+\sigma_\beta^2t^2}}\right)$$
$$+\frac{\sigma_B^2}{\sqrt{\sigma_B^4-4\sigma_\varepsilon^2\sigma_\beta^2}}\exp\left(\frac{2(\xi\mu_\beta\sigma_B^2+\mu_\beta^2\sigma_\varepsilon^2+\xi^2\sigma_\beta^2)}{\sigma_B^4-4\sigma_\varepsilon^2\sigma_\beta^2}\right)$$
$$\times\Phi\left(-\frac{\xi\sigma_B^4+(t\sigma_B^4+2\sigma_B^2\sigma_\varepsilon^2)\mu_\beta+2\xi t\sigma_B^2\sigma_\beta^2}{(\sigma_B^4-4\sigma_\varepsilon^2\sigma_\beta^2)\sqrt{\sigma_\varepsilon^2+(\sigma_B^2+\sigma_\beta^2)t+2\sigma_\varepsilon^2\sigma_\beta^2/\sigma_B^2}}\right) \quad (17)$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

Proof

From Equations (12) and (16), it is known that the findings in Equation (12) do not take into account the random effect of μ. Let's suppose the random effect

$$\mu\sim N(\mu_\beta,\sigma_\beta^2)$$

By applying the law of total probability, the PDF of the failure time T_{re} is obtained as

$$f_{T_e}(t\mid\sigma_B,\sigma_\varepsilon)=\int_{-\infty}^{+\infty}f_{T_e}(t\mid\mu,\sigma_B,\sigma_\varepsilon)p(\mu)d\mu$$
$$=E_\mu[f_{T_e}(t\mid\mu,\sigma_B,\sigma_\varepsilon)]$$

where $p(\mu)$ is the PDF of μ and $E_\mu[\cdot]$ represents the expectation with respect to μ. Equation (16) is derived using Equation (12) and Lemma 1. Similarly, Equation (17) is obtained using Equation (13) and Lemma 3.

It is note that if σ_β=0, the degradation model M_4 reduces to the M_3, and model M_2 reduces to the M_1. Likewise, if we let σ_ε=0, the model M_4 reduces to the M_2, and model M_3 reduces to the M_1.

D. *Model selection criteria* [21]

To compare the proposed different degradation models, some criteria should be given. For degradation models M_1, M_2, M_3, and M_4, some model selection criteria, sucn as log-LF criterion, AIC, and BIC, are adopted for model selection.

The AIC and BIC are calculated as follows

$$AIC=-2(\max l)+2p \quad (18)$$

and

$$BIC=-2(\max l)+p\ln(n) \quad (19)$$

Herein, l represents the value of the log-likelihood function, p depicts the number of parameters in the degradation model, and n denotes the number of samples.

When using the log-LF criterion, the upper log-LF value is preferred. Conversely, when using the AIC or BIC criterias, and lower AIC or BIC values are preferred.

F. *Copulas and their properties*

Copulas offer a highly convenient methodology for constructing and quantifying the interdependencies among multiple PCs. They describe the dependence structure that links the marginal distributions of each PC to their collective multivariate joint distribution.

Sklar's theorem (1959) offers a probabilistic approach to define the copula (detailed in Nelson [25]).

Theorem (Sklar, 1959) Let X and Y be random variables with continuous distribution $F(x)$ and $G(y)$, respectively, and $H(x, y)$ be the two dimensional cumulative distribution function. Hence, there exists a two dimensional copula $C(\cdot,\cdot)$ such that for all $x, y\in(-\infty,+\infty)$,

$$H(x,y)=C\left(F(x),G(y)\right) \quad (20)$$

In this investigation, the Frank copula is utilized to represent the dependence among multiple PCs as follows

$$C(u,v)=-\frac{1}{\alpha}\ln\left\{1+\frac{[\exp(-\alpha u)-1][\exp(-\alpha v)-1]}{\exp(-\alpha)-1}\right\} \quad (21)$$

Here, α is the Frank copula parameter and $\alpha\neq 0$.

III. Heterogeneous degradation model

Consider a product characterized by two PCs $\left(X_1(t),X_2(t)\right)$, and its corresponding failure threshold, denoted as $\xi=(\xi_1,\xi_2)$. Note that the degradation path is a decreasing function. Considering that the failure threshold ξ_k of the kth PC, the marginal reliability at time t can be presented as

$$R_k(t)=1-F_k(t)=\Pr\left(X_k(t)>\xi_k\right) \quad (22)$$

Then, the product reliability can be obtained as

$$R(t)=\Pr\left(X_1(t)>\xi_1,X_2(t)>\xi_2\right) \quad (23)$$

If the two PCs are independent, the Equation (23) can be obtained as

$$R(t)=\Pr\left(X_1(t)>\xi_1,X_2(t)>\xi_2\right)$$
$$=\Pr\left(X_1(t)>\xi_1\right)\times\Pr(X_2(t)>\xi_2)$$
$$=R_1(t)\times R_2(t) \quad (24)$$

However, when the two PCs exhibit interdependence, an accurate reliability assessment cannot be achieved. In this case, the copula approach is used to construct the dependent framework between those two PCs. Similarly to the approach of Sari et al. [13] and Pan et al. [15], the system reliability in Equation (23) can be obtained as

$$R(t)=C\left(R_1(t),R_2(t)\right) \quad (25)$$

where $C(\cdot,\cdot)$ is a Copula function which offer a highly convenient methodology for constructing and quantifying the dependence among multiple PCs. They describe the dependence structure that links the marginal distributions of each PC to their collective multivariate joint distribution.

Copula model method is widely used to model bivariate degradation data in many papers, see in Sari [13-14], pan et al. [15-16], Hao [17] and wang et al. [18].

If we describe the interdependence of the two PCs via bivariate Frank copula as

$$C(u,v)=-\frac{1}{\alpha}\ln\left\{1+\frac{[\exp(-\alpha u)-1][\exp(-\alpha v)-1]}{\exp(-\alpha)-1}\right\} \quad (26)$$

Then, we can get

$$C\left(R_1(t),R_2(t)\right)$$
$$=-\frac{1}{\alpha}\ln\left\{1+\frac{\left[\exp\left(-\alpha R_1(t)\right)-1\right]\left[\exp\left(-\alpha R_2(t)\right)-1\right]}{\exp(-\alpha)-1}\right\} \quad (27)$$

where $R_k(t)=1-F_k(t)$.

IV. Statistical inferential methods

A. Parameters estimation of marginal distribution

Let $t=(t_1, t_2, \cdots, t_m)^T$, $Y_{ki}=(Y_{ki}(t_1), Y_{ki}(t_2), \cdots, Y_{ki}(t_m))^T$, and $Y_k=(Y_{k1}^T, Y_{k2}^T, \cdots, Y_{kn}^T)^T$, where "$T$" denotes transposition. Assuming that the degradation progression of a product is defined by model M_4, it can be deduced that Y_{ki} follows a multivariate normal distribution with a specific mean and variance as

$$E(Y_{ki})=\mu_{k\beta}t$$

and

$$Cov(Y_{ki})=\Sigma_k=\sigma_{k\beta}^2tt^T+\Omega_k \quad (28)$$

where

$$\Omega_k=\sigma_{kB}^2Q+\sigma_{k\varepsilon}^2I_m,\quad Q=\begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_m \end{bmatrix}$$, and I_m is an identity matrix of order m.

Then, let $\eta_k=(\mu_{k\beta},\sigma_{k\beta},\sigma_{kB},\sigma_{k\varepsilon})$, the log-likelihood function can be obtained as

$$L(\eta_k\mid y_k)=-\frac{nm}{2}\ln(2\pi)-\frac{n}{2}\ln|\Sigma_k| -\frac{1}{2}\sum_{i=1}^{n}(y_{ki}-\mu_{k\beta})^T\Sigma_k^{-1}(y_{ki}-\mu_{k\beta}) \quad (29)$$

To simplify the terms in the log-likelihood, we use the results

$$|\Sigma_k|=|\Omega_k|\left(1+\sigma_{k\beta}^2t^T\Omega_kt\right), \quad (30)$$

and

$$\Sigma_k^{-1}=\Omega^{-1}-\frac{\sigma_{k\beta}^2}{1+\sigma_{k\beta}^2t^T\Omega^{-1}t}\Omega^{-1}tt^T\Omega^{-1} \quad (31)$$

Taking partial derivatives to the Equation (29), then we can get

$$\frac{\partial L(\eta_k\mid y_k)}{\partial\mu_{k\beta}}=\frac{\sum_{i=1}^{n}t^T\Omega^{-1}y_{ki}-\mu_{k\beta}nt^T\Omega^{-1}t}{1+\sigma_{k\eta}^2t^T\Omega^{-1}t}, \quad (32)$$

and

$$\frac{\partial L(\eta_k\mid y_k)}{\partial\sigma_{k\beta}}=-\frac{n\sigma_{k\beta}t^T\Omega^{-1}t}{1+\sigma_{k\eta}^2t^T\Omega^{-1}t} +\frac{\sigma_{k\beta}\sum_{i=1}^{n}(y_{ki}-\mu_{k\beta}t)^T\Omega^{-1}tt^T\Omega^{-1}(y_{ki}-\mu_{k\beta}t)}{(1+\sigma_{k\eta}^2t^T\Omega^{-1}t)^2} \quad (33)$$

Thus, for specified values of $(\sigma_{kB},\sigma_{k\varepsilon})$, we can get

$$\hat{\mu}_{k\beta}(\sigma_{kB},\sigma_{k\varepsilon})=\frac{1}{nt^T\Omega^{-1}t}\sum_{i=1}^{n}t^T\Omega^{-1}y_{ki}, \quad (34)$$

and

$$\hat{\sigma}_{k\beta}(\sigma_{kB},\sigma_{k\varepsilon})=\frac{1}{nt^T\Omega^{-1}t}\sum_{i=1}^{n}t^T\Omega^{-1}y_{ki}\left\{\frac{1}{n(t^T\Omega^{-1}t)^2}\times\sum_{i=1}^{n}(y_{ki}-\hat{\mu}_{k\beta}(\sigma_{kB},\sigma_{k\varepsilon})t)^T\Omega^{-1}tt^T\Omega^{-1}\times(y_{ki}-\hat{\mu}_{k\beta}(\sigma_{kB},\sigma_{k\varepsilon})t)-\frac{1}{t^T\Omega^{-1}t}\right\}^{1/2} \quad (35)$$

Substituting Equations (34) and (35) into Equation (29), the following equation is derived

$$L(\sigma_{kB},\sigma_{k\varepsilon}\mid y_k)=-\frac{nm}{2}\ln(2\pi)-\frac{n}{2}-\frac{n}{2}\ln|\Omega| -\frac{n}{2}\left\{\frac{\sum_{i=1}^{n}(t^T\Omega^{-1}y_{ki})^2}{nt^T\Omega^{-1}t}-\frac{(\sum_{i=1}^{n}t^T\Omega^{-1}y_{ki})^2}{n^2t^T\Omega^{-1}t}\right\} \quad (36)$$

Then, by using a two dimensional search in Equation (36), the MLE of $(\sigma_{kB},\sigma_{k\varepsilon})$ can be obtained as $(\hat{\sigma}_{kB},\hat{\sigma}_{k\varepsilon})$.

B. Parameters estimation of joint distribution

In this subsection, the parameter estimation methods in Equation (27) are discussed. The unknown parameters are

$$\theta=(\mu_{\beta1},\sigma_{\beta1},\sigma_{B1},\sigma_{\varepsilon1},\mu_{\beta2},\sigma_{\beta2},\sigma_{B2},\sigma_{\varepsilon2},\alpha)$$

Considering that the model has nine parameters, the Gibbs techniques to estimate model parameters, and WinBUGS[23] is utilized to implement the Gibbs sampling process.

The Gibbs sampling algorithm is

Step 1: Initialize $\theta^{(0)}=(\theta_1^{(0)},\theta_2^{(0)},\cdots,\theta_n^{(0)})$.

Step 2: Set $\theta=\theta^{(t-1)}$, and generate $\theta^{(t)}$.

Generate $\theta_1^{(t)}$ from $\pi_1^*(\theta_1\mid\theta_2^{(t-1)},\theta_3^{(t-1)},\cdots,\theta_n^{(t-1)},Y)$,

$\cdots$

Generate $\theta_n^{(t)}$ from $\pi_n^*(\theta_n\mid\theta_1^{(t-1)},\theta_2^{(t-1)},\cdots,\theta_{n-1}^{(t-1)},Y)$;

Step 3: Set $t=t+1$, and repeat step 2, $t=1,2,\cdots,N_1$.

Step 4: Estimate desired features based on the simulate sample $\theta^{(m)},\theta^{(m+1)},\cdots,\theta^{(N_1)}$, where m is the burn-in number.

V. Numerical example

The actual train wheel wear degradation data were acquired by assessing the work of Freitas *et al.* [24]. The wheel's diameter is a critical performance metric, with failure defined by exceeding a threshold. A new wheel has a diameter of 966 mm, and replacement occurs when it diminishes to 889 mm. Data from 14 samples were compiled, with mileage at 50,000 km. To represent the bivariate degradation model, 8 samples were selected, with half representing the first PC (PC1, left wheels) and the other half representing the second (PC2, right wheels).

The analysis utilized data up to 600,000 km or wheel failure, whichever occurred first. Figures 1-2 illustrate the cumulative degradation for both wheels, represented by the decrease from the initial diameter at time *t*. Wheel failure is designated at a degradation threshold of 77 mm.

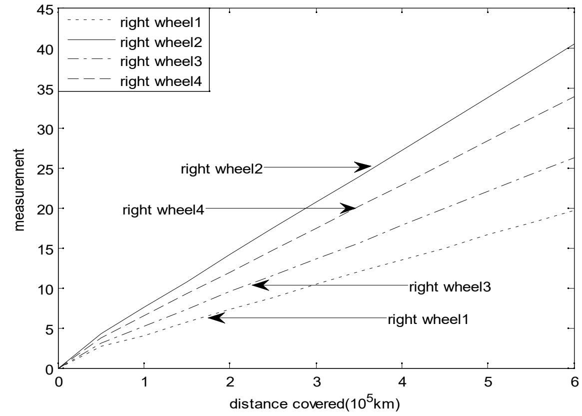

Fig. 1. The degradation paths of the right wheel

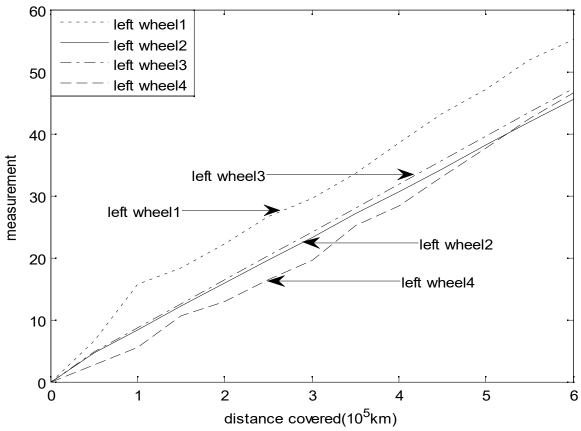

Fig. 2. The degradation paths of the left wheel

A. *Estimation of marginal distribution parameters*

To demonstrate the proposed model and method, some wheel wear degradation data sets are applied. First, different Wiener process models are used to fit the data for PC1 and PC2. Utilizing the approach outlined in Section 3, the estimated parameters for the left wheel are detailed in Table I.

For model comparison, the AIC and BIC criteria are adopted for model selection. The calculated criterion values for various degradation models are presented in Table II.

TABLE I
ESTIMATION OF THE UNKNOWN PARAMETERS OF RIGHT WHEEL

	unknown parameters			
M1	$\mu 1$= 8.097146		$\sigma 1B$ =1.481565	
M2	$\mu 1$= 8.096993	$\sigma 1$= 0.2051773	$\sigma 1B$ =1.474465	
M3	$\mu 1$=8.096877		$\sigma 1B$ =1.481512	$\sigma 1\varepsilon$ = 6.209586E-05
M4	$\mu 1$= 8.096993	$\sigma 1$=0.2051773	$\sigma 1B$ =1.474465	$\sigma 1\varepsilon$= 3.830997E-05

TABLE II
ESTIMATION OF THE UNKNOWN PARAMETERS OF RIGHT WHEEL

Model	M1	M2	M3	M4
Log-likelihood	-70.34228	-70.33086	-70.34228	-70.33086
AIC	144.6846	146.6617	146.6846	148.6617
BIC	143.4571	144.8206	144.8434	146.2026
Rang	1	2	3	4

TABLE III
ESTIMATION OF THE UNKNOWN PARAMETERS OF LEFT WHEEL

	unknown parameters			
M1	$\mu 2$= 5.008808		$\sigma 2B$ =1.0113956	
M2	$\mu 2$=5.008810	$\sigma 2$= 1.294029	$\sigma 2B$ =0.4308234	
M3	$\mu 2$=5.008707		$\sigma 2B$ =1.0113262	$\sigma 2\varepsilon$ = 4.049608E-05
M4	$\mu 2$=5.00810	$\sigma 2$= 1.2940029	$\sigma 2B$ =0.4308234	$\sigma 2\varepsilon$ =6.423012E-07

TABLE IV
ESTIMATION OF THE UNKNOWN PARAMETERS OF LEFT WHEEL

Model	M1	M2	M3	M4
Log-likelihood	-52.01741	-19.07401	-52.01741	-19.07401
AIC	108.0348	44.1480	110.0348	46.1480
BIC	106.8047	42.3069	108.1937	43.6932
Rang	3	1	4	2

TABLE V
PARAMETER ESTIMATION CONSIDERING THE DEPENDENCY

Parameter	Mean	Standard error	MC error	95% HPD Interval
$\mu 1$	8.105	0.411	0.0086	(7.267, 8.889)
$\sigma 1B$	1.669	0.215	0.0041	(1.319, 2.149)
$\mu 2$	4.983	1.225	0.01356	(2.64, 7.207)
$\sigma 2$	1.946	1.362	0.02805	(1.613, 5.162)
$\sigma 2B$	0.4682	0.056	0.00103	(0.4634, 0.5906)
α	-0.5937	3.391	0.1248	(-6.73, 4.935)

Based on the AIC and BIC values in Table II, model M_1, which has the lowest values, is identified as the most optimal fit among the considered Wiener process models.

Similarly, parameter estimation outcomes for the right wheel are provided in Table III and Table IV. Among the different Wiener process models, model M_2, which has the lowest value, demonstrates the best-fitting performance.

B. *Parameter estimation of joint distribution*

To account for the interdependence between the two PCs, the MCMC approach is used to generate 50,000 samples, and the initial 10,000 samples is applied to ensure convergence.

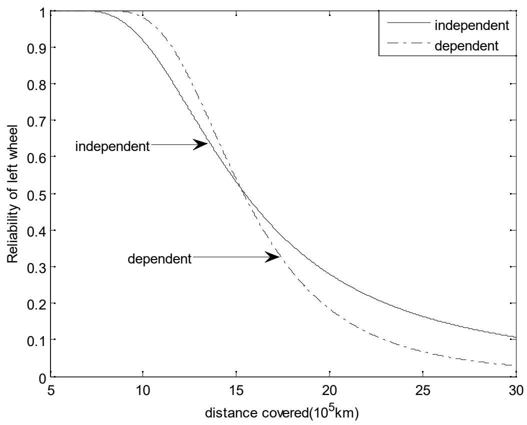

Fig. 3. The reliability curves of the right wheel under different cases

Subsequently, an additional 40,000 samples are drawn using the Gibbs sampling technique to estimate the parameters. Table V provides a comprehensive summary of the posterior distributions, including the posterior means, standard errors, and 95% HPD intervals for the parameters.

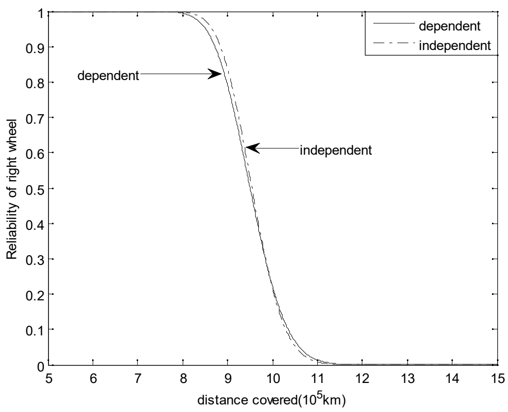

Fig. 4. The reliability curves of the left wheel under different cases

C. *Reliability assessment*

According to the estimated parameters in Table I and Table III, the marginal reliability curves for the right and left wheels are given in Figures 3 and 4 under the dependent case and independent case.

Figures 3-4 clearly demonstrate that both the left and right wheels exhibit different reliability under the independent and dependent cases. In other words, incorrect assumptions regarding dependence can lead to inaccuracies in reliability assessment.

Using the estimated parameters and Frank copula, the system reliability curves for both independent and dependent scenarios are depicted in Figure 5. Figure 5 reveals noticeable differences between the reliability predictions for the dependent and independent cases. Specifically, neglecting the interdependence between the left and right wheels can lead to divergent system reliability conclusions.

Consequently, it is imperative to consider the potential dependency of failure mechanisms and conduct the dependent reliability analysis to obtain more accurate and reliable results.

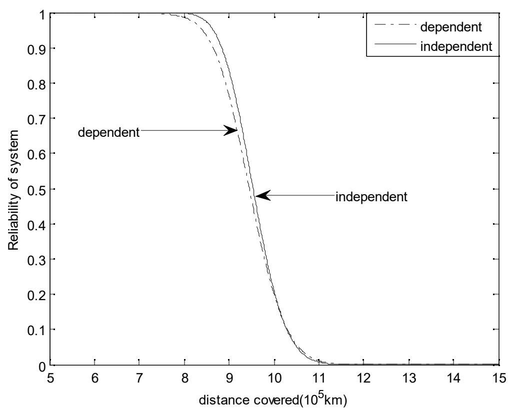

Fig. 5. System reliability curves

VI. CONCLUSION

This investigation attempted to devise a reliability model for a wheel system comprising two PCs. A random effect Wiener process represents one PC, whereas the other PC employs a fixed effect Wiener process to represent. The dependency between these PCs is characterized by a copula function. To obtain the unknown parameters, a hybrid approach combining MLE with MCMC methods is used.

The example in Section V elaborates the significance of the dependency. Neglecting it may yield divergent conclusions in reliability assessment.

The current research centers on scenarios where products have two marginal distributions, with each distribution governed by a heterogeneous Wiener process. In future, this framework could be extended to include other stochastic processes like the heterogeneous Gamma process or Geometric Brownian motion. Moreover, from a practical perspective, forthcoming studies should explore the effective implementation of the proposed estimation outcomes to make maintenance decisions for products with two PCs.

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