Some New Results on Laplace-Complex EE (Emad-Elaf) Integral Transform with Applications

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***Abstract*—Laplace transform has many applications in applied mathematics, science, engineering, and technology. Its further study will play an important role in developing new theories with applications. This paper deals with new results on the Laplace-Complex EE (Emad-Elaf) integral transform (LCEE) of two variables. Starting with definitions and standard results, we obtained various properties, including the shifting property and change of scale, etc. We developed new theorems on the Laplace-Complex EE integral transform of partial derivatives. Further, we applied our results to solve non-homogeneous telegraph equations. Finally, we illustrate our results with examples.**

***Index Terms*—Laplace Transform, Multiple Integral Transforms, Double Complex EE Integral Transform, Integral Transforms of Special Functions, Partial Derivatives.**

I. Introduction

INTEGRAL transforms, such as the Laplace transform [1], modified transform, and complex EE (Emad-Elaf) integral transforms [2], are useful tools for finding solutions to initial and boundary value problems [3], [4], [5], [6]. These transforms have extensive applications in mathematical sciences and engineering, including physics, and mechanics [7], [8], [9], [10], [11], [12]. They play an important role in solving integral equations and partial differential equations, which are useful in many physical phenomena [7], [8], [13], population, epidemic spread, pharmaceutics in biology and medicine [14], signal filtering in audio processing [8], control system design for robotics [15], [16], options pricing in finance [2], population growth modeling in ecology [17], and image restoration in medical imaging [18], [19], [20]. Moreover, the double integral transforms have more importance due to their numerous applications in science and engineering [21], [22], [23], [24], [25] and the double complex EE integral transform. Our approach is to develop a new theory of the Laplace-Complex EE (Emad-Elaf) transform and apply it to solve various problems. In this paper, we obtain the LCEE transform of first-order and second-order partial derivatives and use it to solve the partial differential equations with initial and boundary conditions.

II. Definitions and Notations

1. Laplace Transform:

The modified Laplace transform of a function $u(x)$ which is piecewise continuous and of exponential order is given by,

$$LE_{1,e}[u(x)] = \overline{u}(p) = \int_0^\infty e^{-px} u(x) dx,\ \mathcal{R}(p) > 0, \quad (1)$$

provided that the integral exists, and the corresponding inverse transform is

$$LE_{1,e}^{-1}[\overline{u}(p)] = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{px}\ u(p,e)\ dx,\ (r \geq 0),$$

[21], [22].

2. Complex EE (Emad-Elaf) Integral Transform:

The complex EE (Emad-Elaf) integral transform of a function $u(x)$ with $x > 0$ is given by [2],

$$LC_{1,c}[u(x)] = \int_0^\infty e^{-iq^n x} u(x)\ dx, \quad (2)$$

where q is a complex parameter, $n \in Z$, where Z is an integer, $i \in C$, set of complex number and every $\text{Im}(q^n) < 0$.
The corresponding inverse transform is

$$LC_{1,c}^{-1}[u(x)] = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{q^n x}\ u(q,c)\ dx,\ (r \geq 0).$$

3. Double Complex EE (Emad-Elaf) Integral Transform:

The double complex transform of a function $u(x,y)$ is given by [2],

$$LC_{2,c}[u(x,y)] = \int_0^\infty \int_0^\infty e^{-i(p^m x + q^n y)} u(x,y)\ dx\ dy, \quad (3)$$

where p and q are complex parameters, $m, n \in Z$, where Z is an integer, $i \in C$ and every $\text{Im}(p^m) < 0$ and $\text{Im}(q^n) < 0$.
Now we introduce the Laplace-Complex EE integral transform.

4. Heaviside Unit Step Function:

The $H(x,y)$ is Heaviside unit step function given by [20],

$$\begin{aligned} H(x-\alpha, y-\beta) &= 1,\ if\ x \geq \alpha,\ y \geq \beta, \\ &= 0,\ if\ x < \alpha,\ y < \beta. \end{aligned} \quad (4)$$

5. Gamma Function:

Gamma function is $\Gamma(\alpha)$, and is defined as [26],

$$\Gamma(\alpha) = \int_0^\infty e^{-s} x^{\alpha-1} ds. \quad (5)$$

and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, where α is constant.

6. Laplace-Complex EE (Emad-Elaf) Integral Transform:

The Laplace-Complex EE transform of a function $u(x,y)$ of two variable $x > 0$ and $y > 0$ is given by

$$LCEE[u(x,y)] = \int_0^\infty \int_0^\infty e^{-(px + iq^n y)}\ u(x,y)\ dx\ dy, \quad (6)$$

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where p and q are complex parameters, $m, n \epsilon Z$, $i \epsilon Z C$ and every $\text{Re}(p) > 0$ and $\text{Im}(q^n) < 0$.
The corresponding inverse transform is

$$L_{EC}^{-1}[\overline{\overline{u}}(p,q)] = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{px}\, dp$$
$$\frac{1}{2\pi i}\int_{l-i\infty}^{l+i\infty} e^{iq^n y}\, \overline{\overline{u}}(p,q)\, dq.$$

7. Function of Exponential Order:

[6], function u(x, y) is said to be of exponential order if there exists positive constant M, for all $x > X$, $y > Y$ such that

$$|u(x,y)| \le M e^{rx+ly},$$

and we write

$$u(x,y) < O\, e^{rx+ly} \qquad as \quad x \to \infty,\ y \to \infty.$$

Or,

$$\begin{aligned} &\lim_{x\to\infty,\ y\to\infty} e^{-(rx+ly)}|u(x,y)| \\ &\le M \lim_{x\to\infty,\ y\to\infty} e^{-(p-r)x} e^{-(iq^n-l)y} \\ &= 0,\ p > r,\ q^n > l. \end{aligned}$$

Now we use the following notations

$$\begin{gathered} (i)\overline{\overline{u}}(p,q) = LCEE[u(x,y)] \\ (ii)\ u_1(q) = LC_{1,c}[u(0,y)] \\ (iii)\ u_2(p) = LE_{1,e}[u(x,0)] \\ (iv)\ u_3(q) = LC_{1,c}[u_x(0,y)] \\ (v)\ u_4(p) = LE_{1,e}[u_y(x,0)] \\ (vi)\ LE_{1,e}[u_x(x,0)] = p\, L_{1,e}[u(x,0)] - u(0,0). \end{gathered}$$

III. Theorems of Laplace-Complex EE (Emad-Elaf) Integral Transform

In this paper, we assume that we $u(x,y)$ satisfied all conditions required for the existence of the Laplace-Complex EE integral transform.
It is obvious that LCEE is a linear transformation; that is

$$\begin{aligned} &LCEE[\alpha u_1(x,y) + \beta u_2(x,y)] \\ &= \alpha\, LCEE[u_1(x,y)] + \beta\, LCEE[u_2(x,y)] \end{aligned} \tag{7}$$

where α and β are constants.

Theorem 1. (Existence Theorem):
The Laplace-Complex EE (Emad-Elaf) transform of $u(x,y)$ exists for all p and q, where Re $(p) > r$ and Re $(q) > l$ and $u(x,y)$ is a piece-wise continuous and of exponential order defined in finite intervals $(0,X)$ and $(0,Y)$.

Theorem 2. (Shifting Property):
If $u(x,y)$ is of exponential order and piece-wise continuous, then

$$LCEE[e^{\alpha x+\beta y}\, u(x,y)] = LCEE[p-\alpha, iq^n - \beta]. \tag{8}$$

By definition 6, we have

$$\begin{aligned} &LCEE[e^{\alpha x+\beta y}\, u(x,y)] \\ &= \int_0^\infty\int_0^\infty e^{-(px+iq^n)}\, e^{\alpha x+\beta y}\, u(x,y)\, dx\, dy \\ &= \int_0^\infty\int_0^\infty e^{-[(p-\alpha)x+(iq^n-\beta)y]}\, u(x,y)\, dx\, dy \\ &= LCEE[p-\alpha, iq^n-\beta]. \end{aligned}$$

Theorem 3. (Change of Scale Property):

$$LCEE[u(\alpha x, \beta y)] = \frac{1}{\alpha\beta}\, LCEE\left[u\left(\frac{p}{\alpha}, \frac{iq^n}{\beta}\right)\right]. \tag{9}$$

By definition 6, we have

$$\begin{aligned} &LCEE[e^{\alpha x+\beta y}\, u(\alpha x,\beta y)] \\ &= \int_0^\infty\int_0^\infty e^{-(\alpha px+i\beta q^n y)}\, u(\alpha x,\beta y)\, dx\, dy \end{aligned}$$

$$\begin{gathered} \text{Put} \quad \alpha x = r, \quad \beta y = s \\ dx = \tfrac{1}{\alpha}dr, \quad dy = \tfrac{1}{\beta}ds \end{gathered}$$

We can also write it as

$$\begin{aligned} &LCEE[u(\alpha x,\beta y)] \\ &= \frac{1}{\alpha\beta}\int_0^\infty\int_0^\infty e^{-[(\frac{p}{\alpha})x+(\frac{iq^n}{\beta})y]}\, u\left(\frac{x}{\alpha},\frac{y}{\beta}\right) dx\, dy \\ &= \frac{1}{\alpha\beta}\, LCEE\left[u\left(\frac{p}{\alpha},\frac{iq^n}{\beta}\right)\right]. \end{aligned}$$

Theorem 4. (Unit Step Function):

$$LCEE[H(x-\alpha, y-\beta)] = -\frac{i}{p\, q^n}\, e^{-(p\alpha+iq^n\beta)}, \tag{10}$$

where $H(x,y)$ is defined by equation (5) and $p\, q^n > 0$.
By definition 6, we have

$$\begin{aligned} &LCEE[H(x-\alpha, y-\beta)] \\ &= \int_\alpha^\infty\int_\beta^\infty e^{-(px+iq^n y)} H(x-\alpha,y-\beta)\, dx\, dy. \end{aligned}$$

By definition 5, we have

$$\begin{aligned} &= \int_\alpha^\infty\int_\beta^\infty e^{-(px+iq^n y)}\ (1)\ dx\, dy \\ &= \left(\int_\alpha^\infty e^{-px}\, dx\right)\left(\int_\beta^\infty e^{-iq^n y}\, dy\right) \\ &= -\frac{i}{p\, q^n}\, e^{-(p\alpha+iq^n\beta)}. \end{aligned}$$

We will discuss Convolution theorem for two functions $u(x,y)$ and $v(x,y)$.
Convolution is defined as

$$(v{*}{*}u)(x,y) = \int_0^x\int_0^y v(x-\mu, y-\lambda)u(\mu,\lambda)\, d\lambda\, d\mu. \tag{11}$$

Theorem 5. (Convolution Theorem):
If $LCEE[v(x,y)] = \overline{\overline{v}}(p,q)$ and $LCEE[u(x,y)] = \overline{\overline{u}}(p,q)$, then

$$LCEE[(v**u)(x,y)] = \overline{\overline{v}}(p,q)\, \overline{\overline{u}}(p,q). \tag{12}$$

By definition 6, we have

$$\begin{aligned} &LCEE[(v**u)(x,y)] \\ &= \int_0^\infty\int_0^\infty e^{-(px+iq^n y)}(v**u)(x,y)\, dx\, dy \\ &= \int_0^\infty\int_0^\infty e^{-(px+iq^n y)}\Bigg[\int_0^x\int_0^y v(x-\mu,y-\lambda)u(\mu,\lambda) \\ &\qquad d\lambda\, d\mu\Bigg]\, dx\, dy \end{aligned}$$

Using Heaviside unit step function by definition 5 is,

$$= \int_0^\infty \int_0^\infty u(\mu,\lambda) e^{-(px+iq^n y)} \overline{\overline{v}}(p,q)\, d\lambda\, d\mu$$
$$= \overline{\overline{v}}(p,q) \int_0^\infty \int_0^\infty e^{-(px+iq^n y)} u(\mu,\lambda)\, d\lambda\, d\mu$$
$$= \overline{\overline{v}}(p,q)\, \overline{\overline{u}}(p,q).$$

IV. LCEE Integral Transform of Standard Functions

We consider some standard functions and obtain their Laplace-Complex EE (Emad-Elaf) transforms.

1.
$$LCEE[1] = -\frac{1}{p}\frac{i}{q^n}\ . \tag{13}$$

2. k is constant, then
$$LCEE[k] = -\frac{k}{p}\frac{i}{q^n}\ . \tag{14}$$

3.
$$LCEE[e^{ax+by}] = -\frac{1}{p-a}\frac{b+iq^n}{b^2+q^{2n}}\ . \tag{15}$$

4.
$$LCEE[x^r y^s] = \frac{r!}{p^{r+1}}\frac{s!}{(iq)^{(s+1)n}}\ . \tag{16}$$

5.
$$LCEE[e^{i(ax+by)}] = \frac{a-ip}{(p^2+a^2)(q^n-b)}\ . \tag{17}$$

Consequently,
$$LCEE[cos(ax+by)] = \frac{a}{(p^2+a^2)(q^n-b)}\ . \tag{18}$$

and
$$LCEE[sin(ax+by)] = -\frac{p}{(p^2+a^2)(q^n-b)}\ . \tag{19}$$

6.
$$LCEE[cosh(ax+by)] = -\frac{ab+ipq^n}{(p^2-a^2)(q^{2n}+b^2)}\ . \tag{20}$$

7.
$$LCEE[sinh(ax+by)] = -\frac{pb+iaq^n}{(p^2-a^2)(q^{2n}+b^2)}\ . \tag{21}$$

8.
$$LCEE[e^{2x+y} * * x^3 y^2] = -\frac{12(q^n+i)}{p^4 q^3 (p-2)(q^{2n}-1)}\ . \tag{22}$$

V. LCEE Integral Transforms of Partial Derivatives

We derive Laplace-Complex EE (Emad-Elaf) transform results for partial derivatives of $u(x,y)$, assuming piece-wise continuity and exponential order.

Theorem 6.
$$LCEE[u_x] = p\, \overline{\overline{u}}(p,q) - LC_{1,c}[u(0,y)]. \tag{23}$$

By definition 6, we have

$$LCEE[u_x]$$
$$= \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u_x(x,y)\, dx\, dy$$
$$= \int_0^\infty \left[e^{-(px+iq^n y)}\ u(x,y)\right]_0^\infty dy$$
$$-(-p) \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u(x,y)\, dx\, dy$$
$$= p\, \overline{\overline{u}}(p,q) - \overline{u}_1(q),$$

where $\overline{u}_1(q) = LC_{1,c}[u(0,y)]$.

Theorem 7.
$$LCEE[u_y] = iq^n\, \overline{\overline{u}}(p,q) - LE_{1,e}[u(x,0)]. \tag{24}$$

By definition 6, we have

$$LCEE[u_y]$$
$$= \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u_y(x,y)\, dx\, dy$$
$$= \int_0^\infty \left[e^{-(px+iq^n y)}\ u(x,y)\right]_0^\infty dx$$
$$-(-iq^n) \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u(x,y)\, dx\, dy$$
$$= iq^n\, \overline{\overline{u}}(p,q) - \overline{u}_2(p),$$

where $\overline{u}_2(p) = LE_{1,e}[u(x,0)]$.

Theorem 8.
$$LCEE[u_{xx}] = p^2\, \overline{\overline{u}}(p,q) - p\, LC_{1,c}[u(0,y)] \tag{25}$$
$$-LC_{1,c}[u_x(0,y)].$$

By definition 6, we have

$$LCEE[u_{xx}]$$
$$= \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u_{xx}(x,y)\, dx\, dy$$
$$= \int_0^\infty \left[e^{-(px+iq^n y)}\ u_x(x,y)\right]_0^\infty dy$$
$$-(-p) \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u_x(x,y)\, dx\, dy$$
$$= p\,[p\, \overline{\overline{u}}(p,q)] - p\, LC_{1,c}[u(0,y)] - LC_{1,c}[u_x(0,y)]$$
$$= p^2\, \overline{\overline{u}}(p,q) - p\, \overline{u}_1(q) - \overline{u}_3(q),$$

where $\overline{u}_3(q) = LC_{1,c}[u_x(0,y)]$.

Theorem 9.
$$LCEE[u_{yy}] = -q^{2n}\, \overline{\overline{u}}(p,q) - iq^n\, LE_{1,e}[u(x,0)] - LE_{1,e}[u_y(x,0)]. \tag{26}$$

By definition 6, we have

$$\begin{aligned}
&LCEE[u_{yy}] \\
&= \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u_{yy}(x,y)\ dx\ dy \\
&= \int_0^\infty \left[e^{-(px+iq^n y)}\ u_y(x,y) \right]_0^\infty dx \\
&- (-iq^n) \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u_y(x,y)\ dx\ dy \\
&= iq^n\ [iq^n\ \overline{\overline{u}}(p,q)] - iq^n\ LE_{1,e}[u(x,0)] - LE_{1,e}[u_y(x,0)] \\
&= -q^{2n}\ \overline{\overline{u}}(p,q) - iq^n\ \overline{u}_2(p) - \overline{u}_4(p).
\end{aligned}$$

where $\overline{u}_4(p) = LE_{1,e}[u_y(x,0)]$.

Theorem 10.

$$\begin{aligned}
LCEE[u_{xy}] = ipq^n\ \overline{\overline{u}}(p,q) - iq^n\ LE_{1,e}[u(0,y)] \\
- p\ LE_{1,e}[u(x,0)] + u(0,0). \qquad (27)
\end{aligned}$$

By definition 6, we have

$$\begin{aligned}
&LCEE[u_{xy}] \\
&= \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u_{xy}(x,y)\ dx\ dy \\
&= \int_0^\infty \left[e^{-(px+iq^n y)}\ u_x(x,y) \right]_0^\infty dx \\
&- (-iq^n) \int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u_x(x,y)\ dx\ dy \\
&= iq^n \left[\int_0^\infty \int_0^\infty e^{-(px+iq^n y)}\ u_x(x,y)\ dx\ dy \right] \\
&- LE_{1,e}[u_x(x,0)]. \\
&= ipq^n\ \overline{\overline{u}}(p,q) - iq^n\ \overline{u}_1(q) - p\ \overline{u}_2(p) + u(0,0).
\end{aligned}$$

Remark 1. Result of Loknath Debnath [1], equations (44) to (48) can be obtained from our results, if we put $n = 1$ and replace iq by q, in Theorems (4 to 8) as a special case.

Remark 2. Results of A. Issa, E. A. Kuffi [2], results (4.1 to 4.5) can be obtained from our results, if we replace p by ip^m, in Theorems (4 to 8) as a special case.

VI. Applications to Solve Partial Differential Equations

The Laplace transform is used to solve linear, homogeneous, and non-homogeneous partial differential equations of both first and second order. This includes equations such as the Laplace equation, wave equation, heat transfer equation, and telegraph equations. These mathematical tools are essential in various important applications across engineering, astronomy, physics, life sciences [3], and medical fields [14], [26].

Examples: (First and Second order PDE)
Now we will apply our results to the following problems.

Examples 1: Consider second order nonhomogeneous partial differential equation with constant coefficients,

$$a_0 u_{xx} + a_1 u_{yy} + a_2 u_x + a_3 u_y + a_4 u = g(x,y), \qquad (28)$$

Where a_1, a_1, a_2, a_3, a_4 are constants,
with initial and boundary conditions,

$$\begin{aligned}
u(x,0) = g_1(x), u_y(x,0) = g_2(x), \\
u(0,y) = g_3(x), u_x(0,y) = g_4(x). \qquad (29)
\end{aligned}$$

Apply the Laplace-Complex EE (Emad-Elaf) integral transform with linearity and derivative properties on (28),

$$\begin{aligned}
&a_0 LCEE[u_{xx}] + a_1 LCEE[u_{yy}] + a_2 LCEE[u_x] \\
&+ a_3 LCEE[u_y] + a_4 LCEE[u] \\
&= LCEE[g(x,y)]. \\
&a_0[p^2\ \overline{\overline{u}}(p,q) - p\ LC_{1,c}[u(0,y)] - LC_{1,c}[u_x(0,y)]] \\
&+ a_1[-q^{2n}\ \overline{\overline{u}}(p,q) - iq^n\ LE_{1,e}[u(x,0)] - LC_{1,e}[u_y(x,0)]] \\
&+ a_2[p\ \overline{\overline{u}}(p,q) - LC_{1,c}[u(0,y)]] + a_3[iq^n\ \overline{\overline{u}}(p,q) \\
&- LE_{1,e}[u(x,0)]] + a_4 \overline{\overline{u}}(p,q) \\
&= \overline{\overline{g}}(p,q), \\
&(a_0 p^2 - a_1 q^{2n} + a_2 p + i a_3 q^n + a_4)\ \overline{\overline{u}}(p,q) - a_0\ p\ \overline{g_3} \\
&- a_0\ \overline{g_4} - i a_1\ q^n\ \overline{g_1} - a_1\ \overline{g_2} - a_2\ \overline{g_3} - a_3\ \overline{g_1} = \overline{\overline{g}}(p,q), \\
&\overline{\overline{u}}(p,q) = \\
&\frac{\overline{\overline{g}}(p,q) + a_0(p\ \overline{g_3} + \overline{g_4}) + a_1(iq^n \overline{g_1} + \overline{g_2}) + a_2 \overline{g_3} + a_3 \overline{g_1}}{a_0 p^2 - a_1 q^{2n} + a_2 p + i a_3 q^n + a_4}.
\end{aligned} \qquad (30)$$

Applying the inverse transform to equation (30),
we get a solution to the partial differential equation (28).

Example 2 :
We consider first-order non-homogeneous partial differential equation given by

$$\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} = (1+x)e^y \qquad (31)$$

with given conditions,

$$u(x,0) = x, u(0,y) = 0. \qquad (32)$$

We apply Laplace-Complex EE (Emad-Elaf) integral transform on (31),

$$\begin{aligned}
&LCEE\left[\frac{\partial u(x,y)}{\partial x}\right] + LCEE\left[\frac{\partial u(x,y)}{\partial y}\right] \\
&= LCEE[(1+x)e^y][p\ \overline{\overline{u}}(p,q) - \overline{u}_1(q)] \\
&+ [iq^n\ \overline{\overline{u}}(p,q) - \overline{u}_2(p)], \\
&\frac{1}{p}\frac{1}{iq^n - 1} + \frac{1}{p^2}\frac{1}{iq^n - 1}(p + iq^n)\ \overline{\overline{u}}(p,q) \\
&- \frac{1}{p^2} = \frac{1}{p}\frac{1}{iq^n - 1} + \frac{1}{p^2}\frac{1}{iq^n - 1}, \\
&(p + iq^n)\ \overline{\overline{u}}(p,q) = \frac{1}{p}\frac{1}{iq^n - 1} + \frac{1}{p^2}\frac{1}{iq^n - 1} + \frac{1}{p^2}, \\
&(p + iq^n)\ \overline{\overline{u}}(p,q) = \frac{p + 1 + iq^n - 1}{p^2(iq^n - 1)}.
\end{aligned}$$

We have

$$\overline{\overline{u}}(p,q) = \frac{1}{p^2(iq^n - 1)}. \qquad (33)$$

Applying the inverse transform, we get a solution to the partial differential equation (33) as,

$$u(x,y) = x\ e^y \quad \text{(See Fig. 1)}. \qquad (34)$$

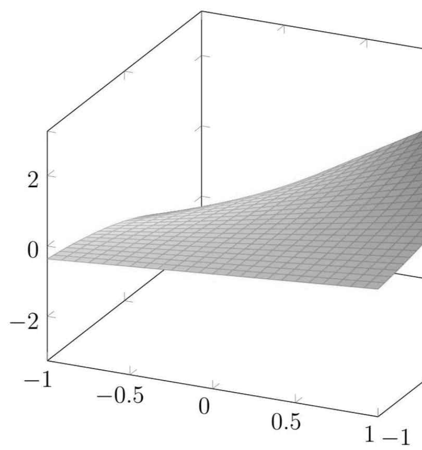

Fig. 1. Graph of $u(x,y) = x\ e^y, \quad \forall\ x\ \epsilon\ [-1,1],\ y\ \epsilon\ [-2,2]$.

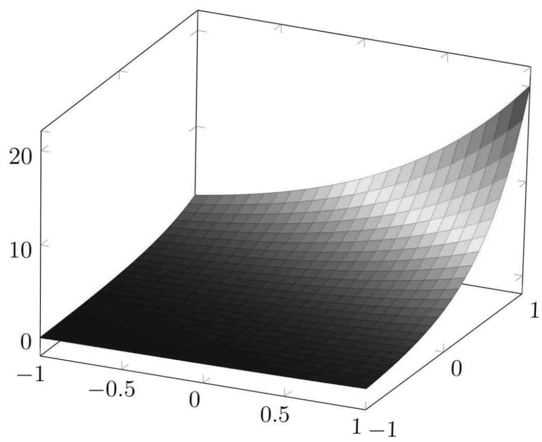

Fig. 2. Graph of $u(x,y) = e^{x+2y}, \forall\ x\ \epsilon\ [-1,1], \quad y\ \epsilon\ [0,20]$.

Example 3. (Telegraph Equation):

Consider the partial differential equation given by

$$\frac{\partial^2 u(x,y)}{\partial y^2} - \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial u(x,y)}{\partial y} + u(x,y) = 6\ e^{x+2y}, \tag{35}$$

with initial and boundary conditions,

$$u(x,0) = e^x, u_y(x,0) = 2e^x, \tag{36}$$
$$u(0,y) = e^{2y}, u_x(0,y) = e^{2y}.$$

Apply the Laplace-Complex EE (Emad-Elaf) integral transform on (35),

$$LCEE\left[\frac{\partial^2 u(x,y)}{\partial y^2}\right] - LCEE\left[\frac{\partial^2 u(x,y)}{\partial x^2}\right]$$
$$+ LCEE\left[\frac{\partial u(x,y)}{\partial y}\right] + LCEE[u(x,y)]$$
$$= 6\ LCEE\left[e^{x+2y}\right],$$
$$[-q^{2n}\ \overline{\overline{u}}(p,q) - iq^n\overline{u}_2(p) - \overline{u}_4(p)] - [p^2\ \overline{\overline{u}}(p,q)$$
$$-\ p\ \overline{u}_1(q) - \overline{u}_3(q)] + [iq^n\ \overline{\overline{u}}(p,q) - \overline{u}_2(p)] + \overline{\overline{u}}(p,q)$$
$$= \frac{6}{(p-1)(iq^n-2)},$$
$$(q^{2n} + p^2 - iq^n - 1)\ \overline{\overline{u}}(p,q) = p\left(\frac{1}{iq^n-2}\right) + \frac{1}{iq^n-2}$$
$$- iq^n\left(\frac{1}{p-1}\right) - \frac{2}{p-1} - \frac{1}{p-1} - 6\left(\frac{1}{p-1}\frac{1}{iq^n-2}\right),$$
$$(q^{2n} + p^2 - iq^n - 1)\ \overline{\overline{u}}(p,q) = \frac{(q^{2n} + p^2 - iq^n - 1)}{(p-1)(iq^n-2)}.$$

We have

$$\overline{\overline{u}}(p,q) = \frac{1}{(p-1)(iq^n-2)} \tag{37}$$

Applying the inverse transform, we get a solution to the partial differential equation (37) as,

$$u(x,y) = e^{x+2y} \quad \text{(See Fig. 2).} \tag{38}$$

VII. Conclusion

In this paper, we introduce a new integral transform called the Laplace-Complex EE (Emad-Elaf) integral transform (LCEE), and we obtained its fundamental properties. These results of the LCEE transform are useful to solve various partial differential equations, integral equations, and transforms. There is further scope for developing this theory and its corresponding applications.

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