An Unconditionally Stable Implicit Finite Difference Method on a Mathematical Model for Evaluating the Risk of Airborne Infection Among Bus Passengers

Jenjira Sooknum, and Nopparat Pochai

***Abstract*—Carbon dioxide from human breath contributes significantly to airborne diseases. Breathing can expose us to usually dangerous airborne infections, which rapidly spread. By using a bus, there is a chance of contracting an infection. This study takes into account a mathematical model of airborne infection caused by human breath. The purpose of this research is to propose an unconditionally stable implicit finite difference method for evaluating the probability that passengers in a bus with ventilation systems may well get an airborne infection. The model can be divided into five submodels, such as an exhaled air concentration measurement model for a bus with a variable number of passengers, the volume fraction of exhaled air model, the concentration of airborne infectious particles model, the number of airborne infectious particles model, and the risk of airborne infection model. The model's solution might be used to determine the probability that susceptible people will get an airborne infection. An unconditionally stable Crank-Nicolson implicit finite difference method and an explicit forward-time centered-space finite difference method are used to approximate the solution. The Crank-Nicolson finite difference technique is not restricted by the stability condition; it is able to use several flexible grid spacings and time increments. The proposed technique produces better simulations that are able to be applied in several required scenarios. In order to reduce the risk of airborne infection and improve ventilation, the provided mathematical models were used to assess the risk of airborne infection among bus passengers using ventilation systems. Better air quality control that balances the number of passengers allowed to travel on a bus will be among the ventilation's main advantages. The simulation has more passengers in the back zone of the bus than in all zones, so the carbon dioxide concentration is higher, and the risk of airborne infection is lower than in other zones. The estimated carbon dioxide concentration in the air on the bus increases or decreases with the number of passengers, and long bus rides increase the risk of airborne infection among passengers. We estimate using the Crank-Nicolson technique, which is unconditionally stable in all scenarios in this study. This makes it easier to modify than the previously recommended methods due to allowing the user to simply modify the spatial resolution and improve the simulation. The proposed technique enables more accurate simulation of scenarios, lowering the risk of airborne infection and improving ventilation. One of the primary benefits of the ventilation system for reducing the risk of airborne infection among bus passengers would be improved air quality control, balanced with the number of passengers permitted to travel on the bus.**

***Index Terms*—Airborne infection, bus passenger, implicit, risk, unconditionally stable**

I. Introduction

They used a portable carbon dioxide meter to monitor unstable carbon dioxide levels in the classroom. They estimated the tuberculosis transmission criteria by using the carbon dioxide risk equation. The recommendation to attain a 1000 ppm carbon dioxide level by natural ventilation was necessary, considering the elevated smear-positive rate among teenagers in South Africa's high schools. In high-prevalence areas, it is also an inexpensive method of helping to manage the TB outbreak [1].

They focused on creating and presenting flexible mathematical models that can predict the likelihood of infectious diseases spreading via the environment in both non-steady and steady-state conditions. By using the thought of the exhaled air accumulation rate and the air exhaled by an infected person in a limited area to determine the mean volume percentage of air directly breathed in each region and demonstrated that the probability of tuberculosis transmission and the generation rate of airborne infectious particles are correlated mathematically, ventilation rate, mean expiratory volume ratio, tuberculosis prevalence, and duration of contact with an infected person [2].

Healthcare facilities evaluate the appropriateness of ventilation to measure carbon dioxide levels at multiple points in acute care hospitals (as done in [3]). The amount of air that participants had previously exhaled indoors was verified. An unstable Wells-Riley formulation has developed and proven beneficial in poorly ventilated environments. They demonstrate how inhaling indoor air can contain significant fractions of air preventing the flu from occurring and is less than the airborne transmission of common respiratory diseases [4].

The potential use of natural ventilation in controlling infection inwards was demonstrated by field measurements

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of a naturally ventilated ward in Hong Kong. It represents that high ventilation rates are possible with natural ventilation, especially when opening the patient room doors and windows. For this reason, they advise against using it in appropriate medical wards to control infection. The research also suggests that mechanical exhaust fans might installed to temporarily create an isolation chamber in an existing ward currently being ventilated naturally [5].

They used the fourth-order Runge-Kutta (RK4) model to estimate the model solution simulating air quality control by modifying the inlet and exhaust ventilation rates under the condition of the surrounding number of people with a personal categorization factor. Numerical models can be used to describe the dynamic spread of airborne infectious diseases in outpatient settings. They will also be able to control airborne illnesses in more complex structures. The probability of infection varies as the population grows, the ventilation rate, and the efficacy of every vaccination [6].

They developed a model for severe infectious disease epidemic occurrences based on the severe acute respiratory syndrome (SARS) outbreak and offered two methods for calculating the model's parameters. The first technique depends on parameter estimates collected from the literature. The second technique uses parameters calculated from a common susceptible infection eradication (SIR) model. Both models predict similar outcomes. Their models predict significantly more severe outbreaks [7].

In research on the airborne transmission of respiratory diseases on subway platforms, they examined airflow patterns and infection hazards on island platforms under standard ventilation modes using computational fluid dynamics (CFD) models. They provide practical recommendations for reducing respiratory infections on subway platforms [8]. They conducted research and used a mathematical model to assess the dangers to the public's health associated with breathing in airborne contaminants when riding in public transportation [9]. The impact of environmental factors was estimated using the Wells-Riley model. There will be variations in the length of interaction and the quantity of passengers. All infected individuals will wear a surgical mask unless they choose not to. Transmission danger decreased. According to their findings, increasing ventilation in public trains seems to be a practical and successful strategy to stop influenza infections.

If there is an outbreak of airborne infectious diseases, traveling by bus may cause infection due to crowding and infected people traveling by bus. We will consider the airborne infection risk at that time and place using a mathematical model to estimate the airborne infectious disease risk among passengers on a bus equipped with a ventilation system using an explicit forward-time centered-space finite difference method, which helps us to determine the balance between the number of passengers allowed on the bus and the air quality management [28], [29]. In this research, we will use an unconditionally stable Crank-Nicolson implicit finite difference method and an explicit forward-time centered-space finite difference method to approximate the solution to develop a more flexible model.

II. Governing Equation

According to [11], [14], and [15], the raised concentration of indoor carbon dioxide is generally influenced by the rate of ventilation and exhalation per person. An infected person's exhaled breath contains infectious airborne particles [1], [13], [14], [15], and [16]. Carbon dioxide levels can be used instead of exhaled air. The average atmospheric concentration of carbon dioxide is 400 parts per million [1], [4], and [15].

A. A One-Dimensional Model for Measuring the Concentration of Exhaled Air: A Bus Carrying a Set Quantity of Passengers

An internal bus volume of V is what we assume. Airborne infectious particle concentration in exhaled air may tend to rise in the presence of infectors, depending on the ventilation velocity Q and bus passenger count.

The 48-seater air-conditioned bus has two exhaust fans. The structure of the simulated bus is approximately 12 meters long, 2.5 meters wide, and 2.5 meters high, as shown in Figure 1.

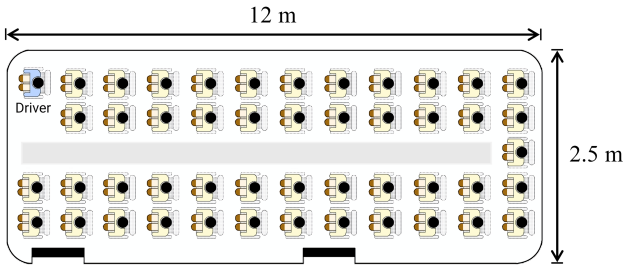

Fig. 1. Seating plan.

The advection-diffusion equation can model several environmental issues, including [17], [18], [19], [20], [21], [22], [23], and [24]. We can only assume that bus passengers produce a large amount of carbon dioxide as a sign of the air they breathe. The fundamental equation for the accumulation rate of exhaled air concentration in a bus with carbon dioxide is the exhaled air rate generated by passengers combined with the diffusion rate of carbon dioxide mixed with airflow velocity and minus the ventilation rate that eliminates exhaled air:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} + u\frac{\partial C}{\partial x} + npC_a - QC \tag{1}$$

for all $(x,t) \in \Omega$, where $\Omega = [0,L]\times[0,T]$, C is the air concentration within the bus's exhaled (ppm), D is The diffusion coefficient of carbon dioxide (m^2/s), u is airflow velocity (m/s), p is the breathing rate (L/s) for each passenger in the bus, C_a is the carbon dioxide fraction of exhaled air and Q is the ventilation rate, t is the amount of time, T is the stationary simulation time, and L is the length of a considered bus. The initial condition is $C(x,0) = C_0$, where C_0 is the latent carbon dioxide concentration. The boundary conditions are given by

$\frac{\partial C}{\partial x} = C_F$, where $x = 0$ and C_F are given constant and by $\frac{\partial C}{\partial x} = C_B$, where $x = L$ and C_B are given constant.

B. An Exhaled Air Concentration Measurement Model in One-Dimensional: A Bus with a Changeable Passenger Count

In the basic case, the number of passengers depends on a given time $n(x,t)$. If the number of passengers is uncertain, we use (1) as follows in this study:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} + u\frac{\partial C}{\partial x} + n(x,t)pC_a - QC \tag{2}$$

for all $(x,t) \in \Omega$.

C. The Exhaled Air's Volume Fraction

We determine the concentration, $C(x,t)$, of the exhaled air sample in the specified region. The volume fraction of exhaled air, $f(x,t)$, is obtained by dividing the sample carbon dioxide concentration by the carbon dioxide in the air produced by exhalation. We get

$$f(x,t) = \frac{C(x,t)}{C_a} \tag{3}$$

for all $(x,t) \in \Omega$.

D. The Airborne Infectious Particle Concentration

The likelihood of infection for susceptible individuals is somewhat high if infectious particles created by infectors reach the host's target infection site at a specific threshold [12]. On the other hand, some infectious particles can lodge in the upper respiratory tract and disperse to different parts of the body. Let μ be the infection particle mortality rate (particles/s) caused by the infector that does not reach the alveoli, and let β be the total infectious particle creation rate (particles/s) released into the air by the infector.

The volume fraction of air exhaled by an infected person ($If(x,t)/n(x,t)$), multiplied by the concentration of infectious particles in the air released by the infected person in the area that reaches the target area of infection of the respiratory system ($(\beta-\mu)/p$), equals the concentration of airborne infectious particles, is N, that cause infection as the following shown equation:

$$N(x,t) = \frac{If(x,t)(\beta-\mu)}{n(x,t)p}, \text{ and } \beta - \mu \geq 1, \tag{4}$$

where I is the possible infectors rate in the bus and for all $(x,t) \in \Omega$.

E. The Number of Infectious Particles in the Air

As all infected particles can reach and deposit in the alveoli, let θ represent the respiratory deposition fraction of airborne infectious particles that settle in the host's targeted infection site through respiratory deposition. Consequently, the product of the susceptible volume of breathed air (pt) and the concentration of airborne infectious particles released by infectors, N, yield the number of infectious particles in the air, $\lambda(x,t)$, that cause infection as follows:

$$\lambda(x,t) = pt\theta N(x,t), \quad 0 < \theta < 1, \tag{5}$$

where t is the duration of stay around infection and for all $(x,t) \in \Omega$.

F. The Risk of Airborne Infection

The Poisson distribution controls tuberculosis transmission [31] and [32]. We express the probability of airborne infection risk for susceptible individuals as follows:

$$P(x,t) = 1 - e^{-\lambda(x,t)}, \tag{6}$$

where $(x,t) \in \Omega$.

III. Numerical Techniques

We cannot find a continuous approximation to solution $C(x,t)$, but we can construct approximations C at values called mesh points in the interval $[0,T]$.

After interpolation, the estimated solution at further interval points can be determined. Initially, we stipulate that the mesh points are spread uniformly over the interval $[0,T]$. This condition is ensured by choosing a positive integer M and selecting the mesh point $t_n = n\Delta t$ for each $n = 0, 1, 2, ..., M$, where $\Delta t = T/M$ is called the time step. This condition is ensured by choosing a positive integer N and $x_m = m\Delta x$ for each $m = 0, 1, 2, ..., N$. The general distance between points $\Delta x = L/N$ is called the step size.

A. Initial Condition Setting

$$C(x,0) = f(x) \tag{7}$$

for all $(x,t) \in \Omega$ and $f(x)$ is a given function of the concentration of residual exhaled air in an empty bus.

B. Boundary Conditions Setting

Assuming there is no absorbance mechanism on the front and back of the bus considered. The left boundary condition (LBC):

$$\frac{\partial C}{\partial x} = C_F \tag{8}$$

for all $t > 0$ and $x = 0$.

The right boundary condition (RBC):

$$\frac{\partial C}{\partial x} = C_B \tag{9}$$

for all $t > 0$ and $x = L$.

C. Total Number of Passengers on the Bus Throughout the 60-Minute of the Bus Journey

Multiple linear regression where y is a linear function of x_1 and x_2, as shown in

$$y = a_0 + a_1 x_1 + a_2 x_2 + e. \tag{10}$$

The best values of the coefficients are defined by formulating the sum of the squares of the residuals in (11) and differentiating each of the unknown coefficients in (12) to (14).

$$S_r = \sum_{i=1}^{k} (\mathrm{y}_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i})^2, \tag{11}$$

$$\frac{\partial S_r}{\partial a_0} = -2\sum_{i=1}^{k} (\mathrm{y}_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i}), \tag{12}$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum_{i=1}^{k} x_{1,i} (y_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i}), \tag{13}$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum_{i=1}^{k} x_{2,i} (y_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i}). \tag{14}$$

By setting the partial derivatives to zero and putting the result in matrix form as,

$$\begin{bmatrix} k & \sum x_{1,i} & \sum x_{2,i} \\ \sum x_{1,i} & \sum x_{1,i}^2 & \sum x_{1,i} x_{2,i} \\ \sum x_{2,i} & \sum x_{1,i} x_{2,i} & \sum x_{2,i}^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_{1,i} y_i \\ \sum x_{2,i} y_i \end{Bmatrix}. \tag{15}$$

Next, we define the function $n(x,t)$ as the function of the total number of passengers on the bus throughout the 60 minutes of the bus journey by using multiple linear regression. Multiple linear regression is the case where $n(x,t)$ is a linear function of two independent variables, x and t, as shown in the following equation:

$$n(x,t) = a_0 + a_1 x + a_2 t. \tag{16}$$

D. A Forward-Time Centered-Space Finite Difference Method for a One-Dimensional Exhaled Air Concentration Measurement Model: A Bus with a Variable Number of Passengers

The FTCS finite difference equation is obtained as follows.

$$C_m^{n+1} = \left(\alpha - \frac{r}{2}\right) C_{m-1}^n - \left(2\alpha + Q\Delta t - 1\right) C_m^n + \left(\alpha + \frac{r}{2}\right) C_{m+1}^n + n(x,t)\, pC_a \Delta t, \tag{17}$$

where $\alpha = \dfrac{D\Delta t}{(\Delta x)^2}$ is the diffusion number, $r = \dfrac{u\Delta t}{\Delta x}$ is the convection number, and $p = 0.12$ (L/s) is the breathing rate.

The stability condition of (17) is [10] $0 < \alpha \le \dfrac{1}{2}$.

E. An Unconditionally Stable Implicit Crank-Nicolson Method for a One-Dimensional Exhaled Air Concentration Measurement Model: A Bus with a Variable Number of Passengers

Retaining in $\dfrac{\partial C}{\partial t}$, $\dfrac{\partial C}{\partial x}$ and $\dfrac{\partial^2 C}{\partial x^2}$ in (2). The Crank-Nicolson implicit method for approximating the solution of (2):

$$\frac{C_m^{n+1} - C_m^n}{\Delta t} = \frac{D}{2}\left(\frac{C_{m-1}^n - 2C_m^n + C_{m+1}^n + C_{m-1}^{n+1} - 2C_m^{n+1} + C_{m+1}^{n+1}}{(\Delta x)^2} \right)$$
$$+ \frac{u}{2}\left(\frac{C_{m+1}^n - C_{m-1}^n + C_{m+1}^{n+1} - C_{m-1}^{n+1}}{2\Delta x} \right) + n(x,t)\, pC_a - \frac{Q}{2}\left(C_m^n + C_m^{n+1}\right),$$

$$C_m^{n+1} - C_m^n = \frac{\alpha}{2}\left(C_{m-1}^n - 2C_m^n + C_{m+1}^n + C_{m-1}^{n+1} - 2C_m^{n+1} + C_{m+1}^{n+1}\right)$$
$$+ \frac{r}{4}\left(C_{m+1}^n - C_{m-1}^n + C_{m+1}^{n+1} - C_{m-1}^{n+1}\right) + n(x,t)\, pC_a \Delta t - \frac{Q\Delta t}{2}\left(C_m^n + C_m^{n+1}\right).$$

We get the Crank-Nicolson implicit equation becomes

$$\left(\frac{r}{4} - \frac{\alpha}{2}\right) C_{m-1}^{n+1} + \left(1 + \alpha + \frac{Q\Delta t}{2}\right) C_m^{n+1} - \left(\frac{r}{4} + \frac{\alpha}{2}\right) C_{m+1}^{n+1}$$
$$= \left(\frac{\alpha}{2} - \frac{r}{4}\right) C_{m-1}^n + \left(1 - \alpha - \frac{Q\Delta t}{2}\right) C_m^n + \left(\frac{\alpha}{2} + \frac{r}{4}\right) C_{m+1}^n + n(x,t)\, pC_a \Delta t. \tag{18}$$

We solve the Crank-Nicolson equations by linear equations, which are unconditionally stable [30].

IV. Numerical Experiments and Results

Assuming that the volume of a bus is $V = 75$ (m^3), each passenger's breathing rate is assumed to be $p = 0.12$ (L/s), the diffusion coefficient of carbon dioxide is $D = 0.732$ (m^2/s), the airflow velocity is $u = 0.025$ (m/s), the carbon dioxide fraction included in the breathed air is $C_a = 0.04$, and the rate of change of the carbon dioxide at the frontend and the backend of the bus is no change, which is $C_F = C_B = 0$. The probability of airborne infection risk for susceptible individuals on the bus with constant ventilation rates is considered. Their parameter settings are followed.

Table I displays the number of passengers in each row. The bus employs an air vent rate of 75, says $Q = 0.1$ (L/s). $C_0 = 0.1$ is the initial value of the carbon dioxide concentration in the air on a bus (ppm). We achieve the approximate solutions presented in Figure 3 by employing

the Forward-Time Centered-Space (FTCS) method (3) to (17) and the Crank-Nicolson (CN) method (3) to (16) and (18) indicated in Figures 4 to 12. An example of a seating plan for a 48-seat bus at a time of 50 to 60 minutes is shown in Figure 2.

TABLE I
THE NUMBER OF PASSENGERS ON THE BUS IN EACH ROW

Time (min)	Row of Seats												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0-10	0	1	1	1	1	3	3	3	3	4	4	4	5
10-20	0	2	2	2	2	3	3	3	3	4	4	4	5
20-30	0	1	1	1	1	4	4	4	4	4	4	4	4
30-40	0	2	2	2	4	4	4	4	4	3	3	3	3
40-50	0	3	2	2	2	2	4	4	4	4	4	4	5
50-60	0	3	4	4	4	4	1	1	1	1	4	4	5

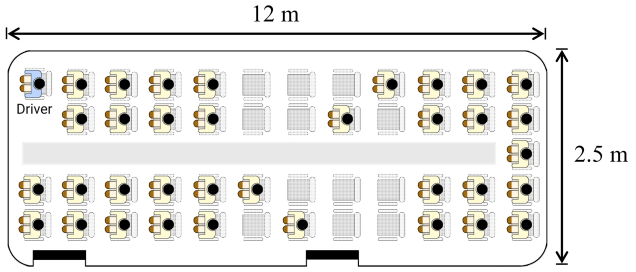

Fig. 2. Seat map depicting the number of passengers seated on the bus for 50 to 60 minutes.

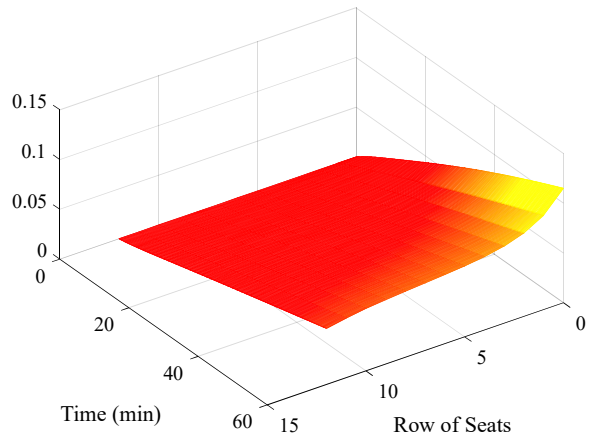

Fig. 3. The probability of airborne infection risk for susceptible individuals. Used the Forward-Time Centered-Space finite difference method.

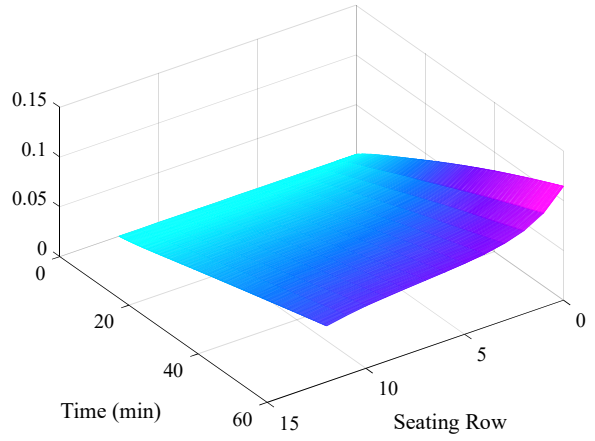

Fig. 4. The probability of airborne infection risk for susceptible individuals. Used the Crank-Nicolson implicit method.

From Figures 3 to 4, assume that time step $\Delta t = 0.1$ and step size $\Delta x = 1.0$. The result of the probability of airborne infection risk for susceptible individuals on a bus still had stability. We can observe that the airborne infection risk probability for susceptible individuals on a bus decreases as more passengers are in each row when passengers get on or off the bus at a stop.

Table II displays the comparison stability when changing the time step and seat distance of passengers on the bus. We set the value for $\Delta x = 0.70 - 1.00$ because the Department of Land Transport specifies that the distance between the seats is not less than 70 centimeters and $\Delta t = 0.10$, 0.20, 0.40, 0.50, 0.55, 0.60, 0.75, and 1.00. From the table shown, we can see that when comparing the use of two numerical methods, FTCS and Crank-Nicolson. The Crank-Nicolson method makes the obtained solution stable for all values of Δt and Δx.

TABLE II
COMPARE STABILITY WHEN CHANGING THE TIME STEP AND SEAT DISTANCE OF PASSENGERS ON THE BUS

Δt	Δx	Stability	
		FTCS	CN
0.10	0.70	S	S
	0.75	S	S
	0.80	S	S
	0.90	S	S
	1.00	S	S
0.20	0.70	S	S
	0.75	S	S
	0.80	S	S
	0.90	S	S
	1.00	S	S
0.40	0.70	U	S
	0.75	U	S
	0.80	S	S
	0.90	S	S
	1.00	S	S
0.50	0.70	U	S
	0.75	U	S
	0.80	U	S
	0.90	S	S
	1.00	S	S
0.55	0.70	U	S
	0.75	U	S
	0.80	U	S
	0.90	S	S
	1.00	S	S
0.60	0.70	U	S
	0.75	U	S
	0.80	U	S
	0.90	U	S
	1.00	S	S
0.75	0.70	U	S
	0.75	U	S
	0.80	U	S
	0.90	U	S
	1.00	U	S
1.00	0.70	U	S
	0.75	U	S
	0.80	U	S
	0.90	U	S
	1.00	U	S

S = stable, U = unstable, Δt = time step, Δx = step size or seat distance of passengers on the bus, FTCS = the Forward-Time Centered-Space method, CN = the Crank-Nicolson method.

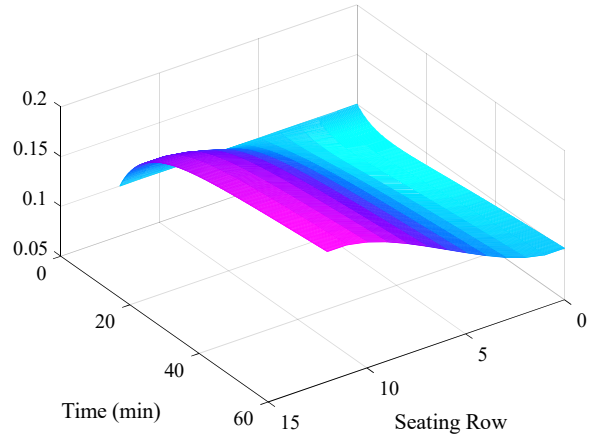

Fig. 5. The approximated carbon dioxide concentration in the air on a bus with a ventilation system where $\Delta t = 0.1$ and $T = 60$. Used the Crank-Nicolson implicit method.

TABLE III
THE CONCENTRATION OF CARBON DIOXIDE AMONG BUS PASSENGERS EVERY 10 MINUTES

Seating Row	The Concentration of Carbon Dioxide Among Bus Passengers					
	10 min	20 min	30 min	40 min	50 min	60 min
0	0.0852	0.0875	0.0914	0.0951	0.0985	0.1018
1	0.0853	0.0876	0.0916	0.0953	0.0987	0.1019
2	0.0889	0.0918	0.0958	0.0995	0.1029	0.1062
3	0.0946	0.0985	0.1028	0.1065	0.1099	0.1132
4	0.1015	0.1068	0.1114	0.1152	0.1186	0.1219
5	0.1092	0.1161	0.1210	0.1249	0.1284	0.1316
6	0.1171	0.1258	0.1311	0.1350	0.1385	0.1418
7	0.1250	0.1355	0.1411	0.1451	0.1486	0.1519
8	0.1325	0.1446	0.1505	0.1546	0.1581	0.1614
9	0.1393	0.1527	0.1589	0.1630	0.1665	0.1698
10	0.1448	0.1591	0.1655	0.1697	0.1732	0.1765
11	0.1482	0.1630	0.1695	0.1737	0.1772	0.1805
12	0.1483	0.1631	0.1696	0.1738	0.1773	0.1806

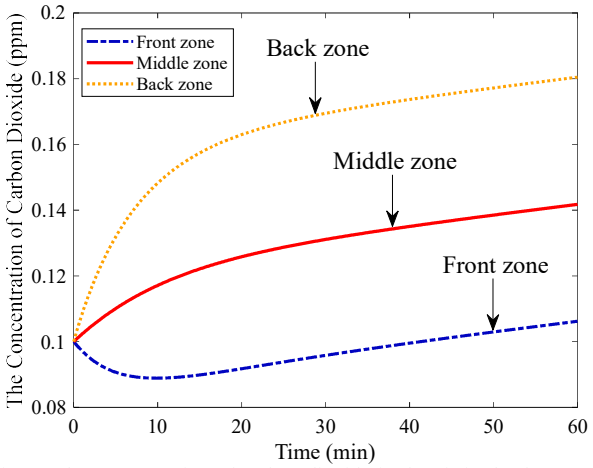

Fig. 6. The concentration of carbon dioxide in the air in the front, middle, and back zones on a bus with a ventilation where $\Delta t = 0.1$ and $T = 60$. Used the Crank-Nicolson implicit method.

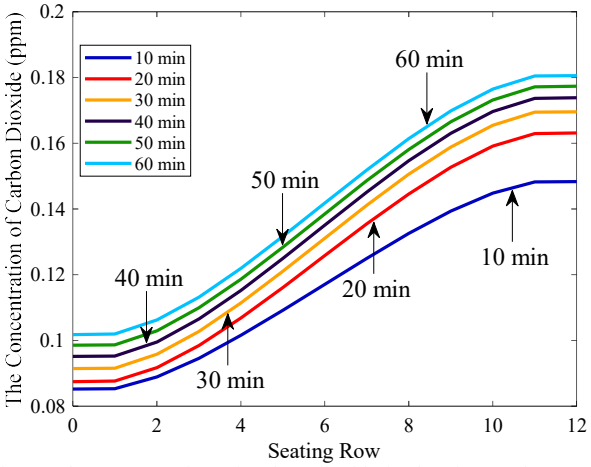

Fig. 7. The concentration of carbon dioxide in the air on a bus every 10 minutes with a ventilation system where $\Delta t = 0.1$ and $T = 60$. Used the Crank-Nicolson implicit method.

From Figure 5, we can observe that the concentration of carbon dioxide in the air on the bus decreases as there are fewer passengers in each row when passengers get off the bus at a stop. Given the large number of passengers in Figure 6, we can observe that the carbon dioxide concentration in the back zone is higher than in the other zones. The concentration of carbon dioxide in the air on a bus increased every 10 minutes with a ventilation system shown in Figure 7.

From Table III, we clearly show a comparison of the values of the concentration of carbon dioxide in bus passengers every 10 minutes. When riding a bus for a long time, the concentration of carbon dioxide increases.

TABLE IV
PHYSICAL PARAMETERS

I	θ	β	μ
1.1151	0.25	100 particles/s	87 particles/s

s = second.

Table IV shows physical characteristics. We achieve the approximate solutions illustrated in Figures 8 to 12 using (3) shown in Figure 8, (4) shown in Figure 9, (5) shown in Figure 10 and (6) shown in Figures 11 to 12.

From Figure 8 to Figure 10, we show the mesh graphs. In Figure 8, we observe that the volume of exhaled air on the bus increases with the number of passengers, which is the area at the back where a large number of passengers. From Figure 9, we see that at the beginning of the period, the concentration of airborne infectious particles gradually increases. From the 20th minute, the concentration of airborne infection particles stabilizes. Figure 10 observes that the number of airborne infection particles increases in direct variation with time.

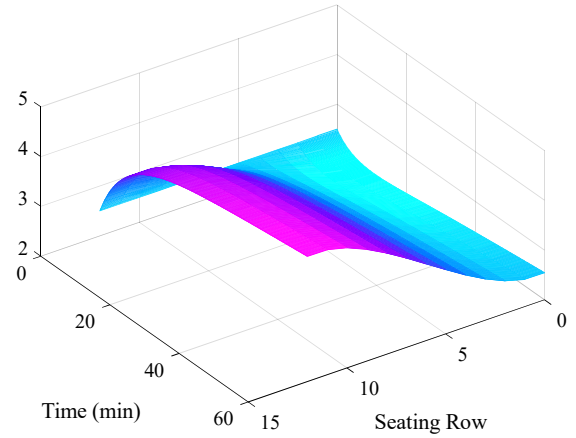

Fig. 8. The volume fraction of exhaled air on a bus where $\Delta t = 0.1$ and $T = 60$. Used the Crank-Nicolson implicit method.

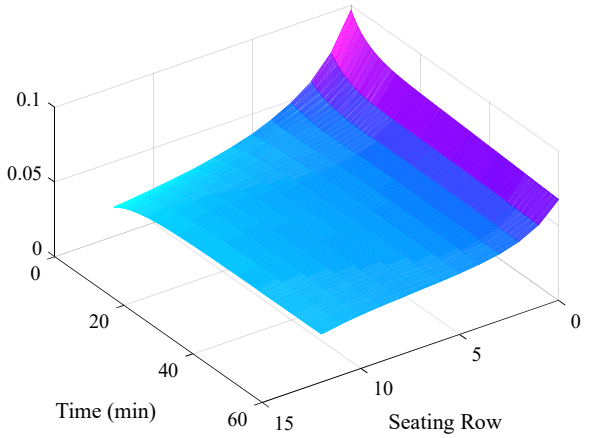

Fig. 9. The concentration of airborne infectious particles where $\Delta t = 0.1$ and $T = 60$. Used the Crank-Nicolson implicit method.

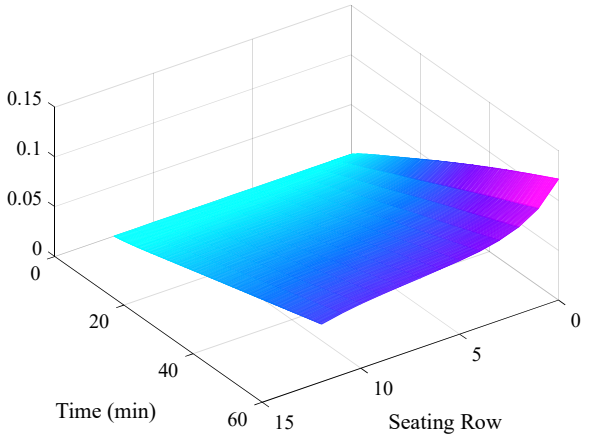

Fig. 10. The number of airborne infectious particles where $\Delta t = 0.1$ and $T = 60$. Used the Crank-Nicolson implicit method.

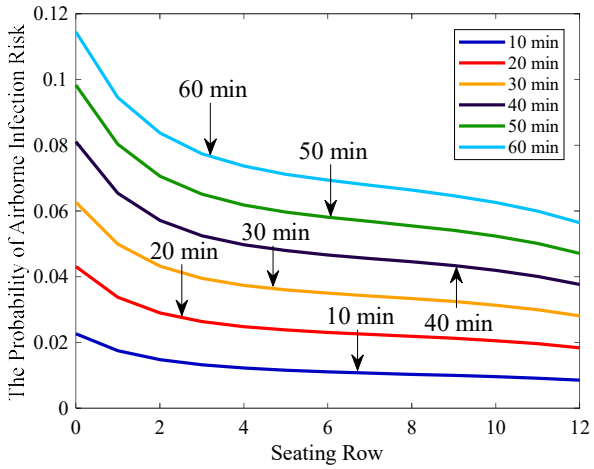

Fig. 11. The probability of airborne infection risk for susceptible individuals every 10 minutes. Used the Crank-Nicolson implicit method.

From Figure 11, when considering the risk of airborne infection for susceptible individuals every 10 minutes, the back of the vehicle has the lowest risk of airborne infection.

Table V shows the percentage comparison of bus passengers at risk of airborne infection every 10 minutes. With longer bus rides, passengers are at an increased risk of airborne infection.

TABLE V
THE PERCENTAGE OF BUS PASSENGERS WHO ARE AT RISK OF CONTRACTING AN AIRBORNE INFECTION EVERY 10 MINUTES

Row of Seats	The Percentage Risk of Airborne Infection Among Bus Passengers					
	10 min	20 min	30 min	40 min	50 min	60 min
0	2.2690	4.3080	6.2718	8.1127	9.8335	11.4525
1	1.7446	3.3722	4.9872	6.5418	8.0290	9.4567
2	1.4748	2.8962	4.3226	5.7131	7.0601	8.3680
3	1.3202	2.6338	3.9544	5.2483	6.5102	7.7432
4	1.2229	2.4784	3.7367	4.9713	6.1795	7.3644
5	1.1563	2.3782	3.5973	4.7928	5.9648	7.1169
6	1.1067	2.3062	3.4974	4.6643	5.8093	6.9365
7	1.0664	2.2470	3.4146	4.5571	5.6790	6.7846
8	1.0306	2.1904	3.3339	4.4522	5.5510	6.6351
9	0.9953	2.1287	3.2442	4.3349	5.4078	6.4672
10	0.9569	2.0547	3.1349	4.1917	5.2325	6.2617
11	0.9109	1.9606	2.9944	4.0075	5.0069	5.9969
12	0.8521	1.8367	2.8089	3.7640	4.7084	5.6461

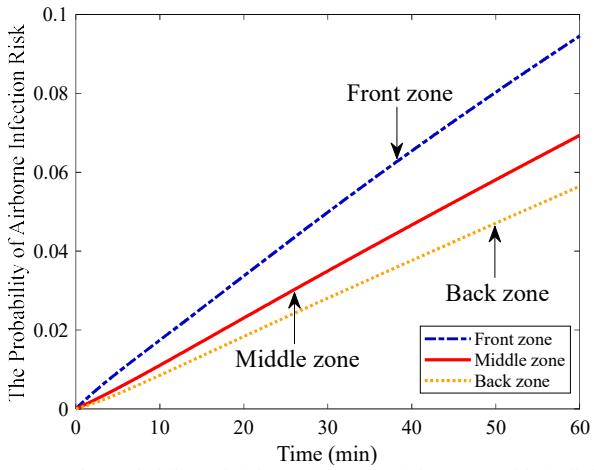

Fig. 12. The probability of airborne infection risks susceptible individuals in the front, middle, and back zones on a bus. Used the Crank-Nicolson implicit method.

Figure 12 shows that passengers who choose to sit in the back are at less risk than passengers in other zones.

V. Discussion

As shown in the demonstrated mathematical model in simulation, we estimated the carbon dioxide concentration in the air on the bus with different ventilation rates according to the number of passengers in each row in Table I. Figures 3 and 4 show the probability of airborne infection risk for susceptible individuals using the Forward-Time Centered-Space finite difference method and the Crank-Nicolson implicit method, respectively. The results obtained by both methods are similar; the probability of airborne infection risk of susceptible individuals on the bus decreases with more passengers in each row, and the FTCS method remains stable. However, from Table II, we compare the stability of the two methods when the time step and the seat distance of bus passengers change, and we found that the Crank-Nicolson method is stable for all cases, unlike the FTCS method, which is unstable in some cases. Therefore, we will show the results using the Crank-Nicolson method only. Figure 5 shows the mesh plot of the approximated carbon dioxide concentration in the air on a bus with a ventilation system at 60 minutes. It found that in the area where the carbon dioxide concentration on the bus decreases due to the number of passengers in the rows, there are few when passengers get off the bus at the bus stop. Table III shows the carbon dioxide concentration values on the bus every 10 minutes. We concluded that when the bus is on for a long time, the carbon dioxide concentration on the bus is increased. From the number of passengers in each row in Table I, there are more passengers in the back zone than in other zones. Therefore, the results in Figure 6, where we show the trend of carbon dioxide concentration on the bus in the three areas in the front, middle, and rear of the bus, show that the carbon dioxide concentration in the back zone is higher than in other zones. Similarly, the results concluded in Table III show that the carbon dioxide concentration in the air on the bus increases every 10 minutes with the ventilation system shown in Figure 7. The probability of airborne infection risk for susceptible individuals every 10 minutes in Figure 11 and the values shown in Table V found that passengers will have an increased risk of airborne infection if they spend a long time on the bus. Figure 12 shows the trend of the probability of airborne infection risk in all three zones. We found that the back zone has a lower risk of airborne infection than other zones because there are more bus passengers.

Using the Crank-Nicolson method is stable for all cases. The estimated carbon dioxide concentration in the air on a bus will increase or decrease with the number of passengers, and the simulations have more passengers in the back zone than in other zones; the carbon dioxide concentration is higher, and the airborne infection risk is lower. The passengers will be at an increased risk of airborne infection if they spend a long time on the bus.

VI. Conclusion

In addition, the numerical mathematical model is used to estimate the risk of airborne infection among bus passengers using the ventilation system to control the possibility of airborne infection and improve ventilation for better air quality management and control the number of passengers. Since the FTCS method has stability limitations, we use unconditionally stable implicit Crank-Nicolson to avoid such limitations. This research is better than previous research because the Crank-Nicolson method is unconditionally stable, can use grid spacing with high accuracy and grid time with the desired width or narrowness, and the calculation is fast. The Crank-Nicolson technique is stable in any situation. Since the simulations have more passengers in the back zone than in other zones, the carbon dioxide concentration is higher, and the airborne infection risk is lower. The estimated carbon dioxide concentration in the air on a bus will increase or decrease with the number of passengers. Long bus rides put passengers at higher risk of contracting an airborne infection.

This makes it easier to adapt than the previously mentioned methods because the user can simply change the spatial resolution to improve the simulation. The proposed technique allows for more precise simulation of scenarios, reducing the risk of airborne infection and improving ventilation. One of the key advantages of the ventilation system for lowering the risk of airborne infections among bus passengers is increased air quality management, which must be balanced with the number of passengers allowed to travel on the bus.

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