A Characterization of the n-C-$\mathcal{X}$ Tilting Pair

Mengmeng Fan, Dajun Liu

Abstract—**The concept of a tilting pair occupies a central position in tilting theory. For a balanced pair $(\mathcal{X},\mathcal{Y})$, we introduce the notions of $*$-selforthogonal module C and n-C-$\mathcal{X}$ tilting pair with respect to $(\mathcal{X},\mathcal{Y})$. In this paper, we mainly prove key properties of the former and establish equivalent characterizations for the latter, thereby generalizing the main results of classical tilting pairs.**

Index Terms—**balanced pair; $*$-selforthogonal module; n-C-$\mathcal{X}$ tilting pair; $*$-acyclic complex.**

I. Introduction

The concept of balanced pairs was introduced by Chen [1], who established sufficient conditions under which a balanced pair of subcategories induces a triangle equivalence between the homotopy categories of complexes. Later, in 2016, Li, Wang, and Hang [2] extended this notion to Abelian categories by introducing cotorsion pairs relative to balanced pairs. Subsequently, Dan and Yang [3] further explored the relationships among balanced pairs, special approximations, and cotorsion pairs. Building on these developments, Estrada, Prez, and Zhu [4] explored in 2018 the connection between balanced pairs and cotorsion triplets, providing a new characterization of Abelian categories with enough projectives and injectives. Further advancements were made in 2022 by Zhang, Liu, and Wei [5], who studied the interrelations of balanced pairs across three Abelian categories while also introducing the concept of relative tilting modules. Recently, in 2024, Xu and Fu [6] introduced the notion of ideal balanced pairs and presented their equivalent characterizations. Under certain conditions, they also established a one-to-one correspondence between ideal balanced pairs and balanced pairs.

In recent years, balanced pairs have been widely applied in relative homological algebra and tilting theory.

The concept of tilting pairs was first introduced by Miyashita [7] in 2001, marking the beginning of a significant research direction in homological algebra. Building on this foundation, Wei and Xi [8] established several key propositions and provided a concise characterization of tilting pairs. Further developments were made by Liao and Chen [9], who expanded the theory by linking tilting pairs to contravariant finite subcategories and cotorsion pairs, revealing deeper structural connections. Gorenstein tilting pairs were introduced by Liu and Wei [10] in 2020, who demonstrated that classical Gorenstein cotilting and tilting modules are special cases within this broader framework. Later, Zhang, Ma, and Zhao [11] investigated the study of tilting pairs and Wakamatsu tilting subcategories over triangular matrix algebras, achieving this by constructing equivalence classes and identifying specific generators.

Briefly, this paper offers several conclusions about $*$-selforthogonal modules, explores the properties of n-C-$\mathcal{X}$ tilting pairs, and delivers a crucial characterization of n-C-$\mathcal{X}$ tilting pairs. Specifically, if C is $*$-selforthogonal and $C \in \mathcal{X}_{\mathrm{Pres}^n_{C}\mathcal{Q}}(T)$, then (C,T) is an n-C-$\mathcal{X}$ tilting pair if and only if $\mathcal{X}_{\mathrm{Pres}^n_{C}\mathcal{Q}}(T) = T^{*\perp}\bigcap{}_C\mathcal{Q}$. These findings generalize classical results on n-tilting pairs (see Corollary 4.1 for detailed proofs and extensions).

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II. Preliminaries

In this paper, we consistently maintain the standing assumption that subcategories are closed under isomorphisms. Let R be a fixed Artin k-algebra. By convention, all modules always mean finitely generated left R-modules and we denote by $R-\mathrm{mod}$ the category of all finitely generated left $R-$modules.

Let $\mathcal{A}$ be an Abelian category. Assume that $\mathcal{X}$ is a subcategory of $\mathcal{A}$ and $C(\mathcal{A})$ is the category of cochain complexes over $\mathcal{A}$. A complex $A^\bullet$ in $C(\mathcal{A})$ is called right (respectively, left) $\mathcal{X}$-acyclic if the complex $\mathrm{Hom}_{\mathcal{A}}(X, A^\bullet)$) (respectively, $\mathrm{Hom}_{\mathcal{A}}(A^\bullet, X)$) is acyclic, for any $X \in \mathcal{X}$. Assume that $\mathcal{X}$ is a subcategory of $R-\mathrm{mod}$. For any left $R-\mathrm{mod}$ M, there exists a complex $X^\bullet = \cdots \to X_2 \to X_1 \to X_0 \to M \to 0$ with $X_i \in \mathcal{X}$ for $i \geq 0$. If it is exact by applying the functor $\mathrm{Hom}_R(X,-)$ for any $X \in \mathcal{X}$, then, we call the complex a $\mathcal{X}$-resolution of M. A subcategory $\mathcal{B}$ of $R-\mathrm{mod}$ is said to be contravariantly finite, if for any left $R-\mathrm{mod}$ A, it has a right $\mathcal{B}$- approximation, i.e., there is a homomorphism $f : B \to A$ for some $B \in \mathcal{B}$ such that $\mathrm{Hom}_R(B', f)$ is surjective for any $B' \in \mathcal{B}$. Dually, we have the definition of the $\mathcal{X}$-coresolution and covariantly finite subcategory.

Definition 2.1: ([5, Definition 2.1]) A pair $(\mathcal{X},\mathcal{Y})$ of subcategories in an Abelian category $\mathcal{A}$ is called a balanced pair if the following conditions are satisfied:

(1) $\mathcal{X}$ is contravariantly finite in $\mathcal{A}$ and $\mathcal{Y}$ is covariantly finite in $\mathcal{A}$.

(2) For any object $M \in \mathcal{A}$,there exists a $\mathcal{X}$-resolution $X^\bullet \to M$ of M such that it is left $\mathcal{Y}$- acyclic.

(3) For any object $N \in \mathcal{A}$,there exists a $\mathcal{Y}$-coresolution $N \to Y^\bullet$ of N such that it is right $\mathcal{X}$- acyclic.

If a complex $A^\bullet$ is both right $\mathcal{X}$-acyclic and left $\mathcal{Y}$-acyclic, then we call $*$-acyclic.

Let $\mathcal{A}$ be an Abelian category. A contravariantly finite subcategory $\mathcal{B}$ of $\mathcal{A}$ is admissible provided that each right $\mathcal{B}$-approximation is surjective. Dually, a covariantly finite subcategory $\mathcal{C}$ of $\mathcal{A}$ is coadmissible provided that each left $\mathcal{C}$-approximation is monomorphism. If $\mathcal{X}$ is admissible or $\mathcal{Y}$ is coadmissible, we will call that a balanced pair $(\mathcal{X},\mathcal{Y})$ is admissiable.

Throughout this paper, we always assume that $(\mathcal{X}, \mathcal{Y})$ is an admissible balanced pair in $R-\mathrm{mod}$. We denote the projective (or injective) modules by $\mathrm{Proj}R$ (or $\mathrm{Inj}R$). As is well-known, $(\mathrm{Proj}R, \mathrm{Inj}R)$ is a classical balanced pair. Consider a $*$-acyclic complex $0 \to A \overset{i_*}{\to} B \overset{\pi_*}{\to} C \to 0$, where i_* and π_* are respectively called $*$-monomorphism and $*$-epimorphism. Therefore, it is easy to know that $*$-monomorphism must be monomorphism and $*$-epimorphism must be epimorphism, that is, a $*$-acyclic complex must be an exact sequence.

Lemma 2.1: ([2]) Let $(\mathcal{X}, \mathcal{Y})$ be a balanced pair and $M, N \in \mathcal{A}$. For any $i \in \mathbb{Z}$, there exists an isomorphism of Abelian groups $\mathrm{Ext}^i_{\mathcal{X}}(M, N) \cong \mathrm{Ext}^i_{\mathcal{Y}}(M, N)$. We denote both Abelian groups by $\mathrm{Ext}^i_*(M, N)$.

Lemma 2.2: (1) The pushout of $*$-acyclic complexes are still $*$-acyclic complexes.

(2) The pullback of $*$-acyclic complexes are still $*$-acyclic complexes.

Proof: (1) Assume that $0 \to A \to B \to C \to 0$ and $0 \to A \to D \to E \to 0$ are $*$-acyclic complexes, then we have the following pushout diagram, referring to Fig. 1.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\beta"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & E \arrow[r, equal] \arrow[d] & E \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Fig. 1. The pushout diagram of morphisms α and β

Applying the functor $\mathrm{Hom}_R(X, -), \forall X \in \mathcal{X}$ to Fig. 1, we want to prove that $0 \to \mathrm{Hom}_R(X, D) \to \mathrm{Hom}_R(X, F) \to \mathrm{Hom}_R(X, C) \to 0$ is exact. That is, for all $f : X \to C$, there exists a morphism $g : X \to F$. Since the sequence $0 \to \mathrm{Hom}_R(X, A) \to \mathrm{Hom}_R(X, B) \to \mathrm{Hom}_R(X, C) \to 0$ is exact, there exists a morphism $g' : X \to B$. Let $f' : B \to F$, then we can construct a morphism $g : X \to F$ such that $g = f'g'$. Consequently, the middle row is exact by applying the functor $\mathrm{Hom}_R(X, -)$. Subsequently, by the Snake Lemma, we obtain the middle column is still exact when this functor is likewise applied. Therefore, all rows and columns in Fig. 1 are also exact by applying the functor $\mathrm{Hom}_R(-, Y), \forall Y \in \mathcal{Y}$ since the balanced pair $(\mathcal{X}, \mathcal{Y})$ is admissible [1, Corollary 2.3].

In other case, assume that the sequences $0 \to A \to B \to C \to 0$ and $0 \to B \to D \to E \to 0$ are $*$-acyclic complexes. Consequently, we have another pushout diagram, as shown in Fig. 2.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r, "\mu"] \arrow[d, "\upsilon"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & E \arrow[r, equal] \arrow[d] & E \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Fig. 2. The pushout diagram of morphisms μ and υ

Similar to the proof of (1), we are able to deduce that all rows and columns in Fig. 2 are $*$-acyclic complexes.

(2) Given that $0 \to A \to B \to C \to 0$ and $0 \to D \to E \to C \to 0$ are $*$-acyclic complexes, we have the subsequent pullback diagram, as illustrated in Fig. 3.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & D \arrow[r, equal] \arrow[d] & D \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "\delta"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Fig. 3. The pullback diagram of morphisms δ and γ

By applying the functor $\mathrm{Hom}_R(-, Y), \forall Y \in \mathcal{Y}$ to Fig. 3, we aim to demonstrate that the sequence $0 \to \mathrm{Hom}_R(E, Y) \to \mathrm{Hom}_R(F, Y) \to \mathrm{Hom}_R(A, Y) \to 0$ is exact. This means for all $s : A \to Y$, there exists a morphism $t : F \to Y$. Since the sequence $0 \to \mathrm{Hom}_R(C, Y) \to \mathrm{Hom}_R(B, Y) \to \mathrm{Hom}_R(A, Y) \to 0$ is exact, we can find a morphism $t' : B \to Y$. Let $s' : F \to B$, and consequently, there exists a morphism $t : F \to Y$ such that $t = t's'$. Therefore, the middle row remains exact by applying the functor $\mathrm{Hom}_R(-, Y)$. Furthermore, by the Snake Lemma, We obtain the middle column is still exact by applying the functor $\mathrm{Hom}_R(-, Y)$. Also, all rows and columns in Fig. 3 are also exact when the functor $\mathrm{Hom}_R(X, -), \forall X \in \mathcal{X}$

is applied since the balanced pair $(\mathcal{X}, \mathcal{Y})$ is admissible [1, Corollary 2.3].

In other case, let $0 \to A \to B \to C \to 0$ and $0 \to D \to E \to B \to 0$ are $*$-acyclic complexes, we can construct another pullback diagram, as shown in Fig. 4.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & D \arrow[r, equal] \arrow[d] & D \arrow[d] & & \\
0 \arrow[r] & F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "\rho"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\pi"] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Fig. 4. The pullback diagram of morphisms π and ρ

Similar to the proof of (2), we can conclude that all rows and columns in Fig. 4 are $*$-acyclic complexes, thereby completing the proof. ☐

Let C be an $R-$mod, and $\mathrm{Add}C$ denote the subcategory consisting of all direct summands of finite direct sums of copies of C. We use $\overset{\vee}{\mathrm{Add}}_* C$ to represent the category cosisting of all moudles M for which there exists a $*-$acyclic complex $0 \to M \to C_0 \to \cdots \to C_m \to 0$ for some integer m with each $C_i \in \mathrm{Add}C$. Let $M \in \overset{\vee}{\mathrm{Add}}_* C$, we define $\mathrm{codim}_{*C}(M)$ as the minimal integer m such that there exists a $*$-acyclic complex $0 \to M \to C_0 \to \cdots \to C_m \to 0$ with each $C_i \in \mathrm{Add}C$. Additionally, $(\overset{\vee}{\mathrm{Add}}_* C)_n$ denotes the subcategory of all $M \in \overset{\vee}{\mathrm{Add}}_* C$ with $\mathrm{codim}_{*C}(M) \leqslant n$. Dually, $\overset{\wedge}{\mathrm{Add}}_* C$ is the category of all moudles M for which there exists a $*$-acyclic complex $0 \to C_m \to \cdots \to C_0 \to M \to 0$ for some integer m with each $C_i \in \mathrm{Add}C$. The term $\dim_{*C}(M)$ is used to denote the minimal integer m for which there exists a $*$-acyclic complex $0 \to C_m \to \cdots \to C_0 \to\to M \to 0$ with each $C_i \in \mathrm{Add}C$. Similarly, $(\overset{\wedge}{\mathrm{Add}}_* C)_n$ denotes the category of all $M \in \overset{\wedge}{\mathrm{Add}}_* C$ with $\dim_{*C}(M) \leqslant n$.

We denote by $C^{*\perp}$ (resp., ${}^{\perp}C^*$) the subcategory of all $R-$modules M such that $\mathrm{Ext}_*^{i\geqslant 1}(C, M) = 0$, (resp., $\mathrm{Ext}_*^{i\geqslant 1}(M, C) = 0$). An $R-$module C is referred to as $*$-selforthogonal if $\mathrm{Ext}_*^{i\geqslant 1}(C, C) = 0$. ${}_C\mathcal{Q}$ (resp., $\mathcal{Q}_C$) is denoted as the full subcategory of $C^{*\perp}$ (resp., ${}^{\perp}C^*$) consisting of all $R-$modules M such that there exists a $*$-acyclic complex $\cdots \overset{f_2}{\to} C_1 \overset{f_1}{\to} C_0 \overset{f_0}{\to} M \to 0$ (resp., $0 \to M \overset{f_0}{\to} C_0 \overset{f_1}{\to} C_1 \overset{f_2}{\to} \cdots$) with each $C_i \in \mathrm{Add}C$ and each $\mathrm{Im}\, f_i \in C^{*\perp}$ (resp., $\mathrm{Im}\, f_i \in {}^{\perp}C^*$). It is obvious that $\overset{\wedge}{\mathrm{Add}}_* C \subseteq {}_C\mathcal{Q} \subseteq C^{*\perp}$ (resp., $\overset{\vee}{\mathrm{Add}}_* C \subseteq \mathcal{Q}_C \subseteq {}^{\perp}C^*$).

Remark 2.1: (1) If $(\mathcal{X}, \mathcal{Y}) = (\mathrm{Proj}R, \mathrm{Inj}R)$, in fact, $*$-selforthogonal modules are exactly selforthogonal modules [8].

(2) If $(\mathcal{X}, \mathcal{Y}) = (\mathcal{GP}(R), \mathcal{GI}(R))$, then $*$-selforthogonal moudles are objects in Gorenstein orthogonal classes [10].

Let $T \in R-$mod. For every $n \geq 1$, we denoted by $\mathcal{X}\mathrm{pres}^n_{C\mathcal{Q}}(T)$ the category of all $R-$modules M such that there exists a $*$-acyclic complex $0 \to X \overset{f_n}{\to} T_n \overset{f_{n-1}}{\to} \cdots \overset{f_1}{\to} T_1 \to M \to 0$ with $X \in {}_C\mathcal{Q}$ and each $T_i \in \mathrm{Add}T$. Apparently, $\mathcal{X}\mathrm{pres}^1_{C\mathcal{Q}}(T)$ is closed under direct summands.

III. $*$-SELFORTHOGONAL MODULES

In this section, we primarily introduce two crucial theorems concerning $*$-selforthogonal modules. Firstly, Let us recall the concept of $\mathcal{C}$ being closed under $*$-extensions: if for any $*$-acyclic complex $0 \to L \to M \to N \to 0$ with $L, N \in \mathcal{C}$, we can imply $M \in \mathcal{C}$.

Lemma 3.1: Let C be $*$-selforthogonal. Then:

(1) $\mathcal{Q}_C$ is closed under $*$-extensions, kernels of $*$-epimorphisms, finite direct sums and direct summands.

$(1)'$ ${}_C\mathcal{Q}$ is closed under $*$-extensions, cokernels of $*$-monomorphisms, finite direct sums and direct summands.

(2) $\mathrm{Ext}_*^{i\geqslant 1}(U, V) = 0$ for any $U \in \overset{\vee}{\mathrm{Add}}_* C$ and $V \in C^{*\perp}$.

$(2)'$ $\mathrm{Ext}_*^{i\geqslant 1}(U, V) = 0$ for any $U \in {}^{\perp}C^*$ and $V \in \overset{\wedge}{\mathrm{Add}}_* C$.

(3) $(\overset{\vee}{\mathrm{Add}}_* C)_n = \{X \in \mathcal{Q}_C \mid \mathrm{Ext}_*^{n+1}(Y, X) = 0 \text{ for all } Y \in {}^{\perp}C^*\} = \{X \in \mathcal{Q}_C \mid \mathrm{Ext}_*^{n+1}(Y, X) = 0 \text{ for all } Y \in \mathcal{Q}_C\}$.

$(3)'$ $(\overset{\wedge}{\mathrm{Add}}_* C)_n = \{X \in {}_C\mathcal{Q} \mid \mathrm{Ext}_*^{n+1}(X, Y) = 0 \text{ for all } Y \in C^{*\perp}\} = \{X \in {}_C\mathcal{Q} \mid \mathrm{Ext}_*^{n+1}(X, Y) = 0 \text{ for all } Y \in {}_C\mathcal{Q}\}$.

(4) $(\overset{\vee}{\mathrm{Add}}_* C)_n$ is closed under $*$-extensions, finite direct sums and direct summands.

$(4)'$ $(\overset{\wedge}{\mathrm{Add}}_* C)_n$ is closed under $*$-extensions, finite direct sums and direct summands.

(5) Given any $*$-acyclic complex $0 \to U \to V \to W \to 0$, if $V, W \in {}_C\mathcal{Q}$ and $\mathrm{Ext}_*^1(C, U) = 0$, then $U \in {}_C\mathcal{Q}$.

$(5)'$ Given any $*$-acyclic complex $0 \to U \to V \to W \to 0$, if $U, V \in \mathcal{Q}_C$ and $\mathrm{Ext}_*^1(W, C) = 0$, then $W \in \mathcal{Q}_C$.

Proof: Clearly, the statement corresponding to $(i)'$ is dual of (i) for all values of $1 \leq i \leq 5$, thus proving one case is sufficient.

(1) Firstly, we prove that $\mathcal{Q}_C$ is closed under $*$-extensions.

Let $0 \to L \to M \to N \to 0$ be a $*$-acyclic complex with $L, N \in \mathcal{Q}_C$, we aim to show that $M \in \mathcal{Q}_C$. By assumption, there exist two $*$-acyclic complexes $0 \to L \to C_L \to A \to 0$ and $0 \to N \to C_N \to B \to 0$ with $C_L, C_N \in \mathrm{Add}C$ and $A, B \in \mathcal{Q}_C$. Consider the following pushout diagram, referring to Fig. 5.

According to Lemma 2.2, all rows and columns in Fig. 5 are $*$-acyclic complexes. Since $N \in \mathcal{Q}_C \subseteq {}^{\perp}C^*$, it follows that $\mathrm{Ext}_*^1(N, C_L) = 0$, i.e., $U \cong C_L \oplus N$. Consequently, we have the following pushout diagram, referring to Fig. 6.

Since both $0 \to N \to C_N \to B \to 0$ and $0 \to C_L \to C_L \to 0$ are $*$-acyclic complexes, the middle column is also a $*$-acyclic complex. Since the upper row is a $*$-acyclic complex, it follows by Lemma 2.2 that all rows and columns in Fig. 6 are $*$-acyclic complexes. It follows that $V \in {}^{\perp}C^*$,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_L \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & N \arrow[r] & 0 \\
 & A \arrow[r, equal] \arrow[d] & A \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Fig. 5. The pushout diagram of morphisms α and β

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & C_L \oplus N \arrow[r, "\mu"] \arrow[d, "\upsilon"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & C_L \oplus C_N \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Fig. 6. The pushout diagram of morphisms μ and υ

since $A, B \in {}^{\perp}C^*$ and ${}^{\perp}C^*$ is closed under $*$-extensions. Observing that $A, B \in \mathcal{Q}_C$ and repeating the process, it is straightforward to conclude that $M \in \mathcal{Q}_C$. Hence $\mathcal{Q}_C$ is closed under $*$-extensions.

Secondly, we need to prove that $\mathcal{Q}_C$ is closed under kernels of $*$-epimorphisms.

Conside a $*$-acyclic complex $0 \to L \to M \to N \to 0$ with $M, N \in \mathcal{Q}_C$, our goal is to show that $L \in \mathcal{Q}_C$. By assumption, there exists a $*$-acyclic complex $0 \to M \to C_0 \to B \to 0$ with $C_0 \in \mathrm{Add}C$ and $B \in \mathcal{Q}_C$. Based on this, we can construct the following pushout diagram, referring to Fig. 7.

Similarly, the right column and the middle row in Fig. 7 are also $*$-acyclic complexes by Lemma 2.2. It follows that $A \in \mathcal{Q}_C$ since $N, B \in \mathcal{Q}_C$ and $\mathcal{Q}_C$ is closed under $*$-extensions. Consequently, we obtain that $L \in \mathcal{Q}_C$. That is, $\mathcal{Q}_C$ is closed under kernels of $*$-epimorphisms.

Finally, we prove that $\mathcal{Q}_C$ is closed under finite direct sums and direct summands.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & M \arrow[r, "f"] \arrow[d, "g"] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & C_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Fig. 7. The pushout diagram of morphisms f and g

Assume that $M \cong L \oplus N$. Consider the $*$-acyclic complex $0 \to L \to M \to N \to 0$ with $M \in \mathcal{Q}_C$, we want to show that $L \in \mathcal{Q}_C$. The sequence $0 \to M \cong L \oplus N \to L \oplus A \to B \to 0$ is a direct sum of two $*$-acyclic complexes by the Fig. 7, hence it is a $*$-acyclic complex. Since $M, B \in \mathcal{Q}_C$, we obtain that $L \oplus A \in \mathcal{Q}_C$ and therefore $A \in {}^{\perp}C^*$. By repeating this process with the sequence $0 \to A \to L \oplus A \to L \to 0$, we obtain that $L \in \mathcal{Q}_C$. Therefore $\mathcal{Q}_C$ is closed under finite direct sums and direct summands.

(2) If $U \in \overset{\vee}{\mathrm{Add}}_* C$, then there exists a $*-$acyclic complex $0 \to U \to C_0 \to \cdots \to C_n \to 0$ with each $C_i \in \mathrm{Add}C$. Thus, for any $V \in C^{*\perp}$, by applying the functor $\mathrm{Hom}_R(-, V)$, we get $\mathrm{Ext}^i_*(U, V) \cong \mathrm{Ext}^{i+n}_*(C_n, V) = 0$ for any $i \geq 1$ by dimension shift.

(3) For any $X \in (\overset{\vee}{\mathrm{Add}}_* C)_n$, there exists a $*$-acyclic complex $0 \to X \to C_0 \to \cdots \to C_n \to 0$ with each $C_i \in \mathrm{Add}C$. Consequently, for any $Y \in {}^{\perp}C^*$, by applying the functor $\mathrm{Hom}_R(Y, -)$, we get $\mathrm{Ext}^{n+1}_*(Y, X) \cong \mathrm{Ext}^1_*(Y, C_n) = 0$ for any $n \geq 1$ by dimension shift.

On the other hand, for any $X \in \mathcal{Q}_C$, then there exists a $*$-acyclic complex $0 \to X \xrightarrow{f_0} C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots$ with each $C_i \in \mathrm{Add}C$ and $\mathrm{Im}f_i \in {}^{\perp}C^*$. If $\mathrm{Ext}^{n+1}_*(Y, X) = 0$ for any $Y \in {}^{\perp}C^*$, then by applying the functor $\mathrm{Hom}_R(\mathrm{Im}f_{n+1}, -)$, we obtain that $\mathrm{Ext}^1_*(\mathrm{Im}f_{n+1}, \mathrm{Im}f_n) \cong \mathrm{Ext}^{n+1}_*(\mathrm{Im}f_{n+1}, X) = 0$ by dimension shift. It follows that the sequence $0 \to \mathrm{Im}f_n \to C_n \to \mathrm{Im}f_{n+1} \to 0$ is split, that is, $C_n \cong \mathrm{Im}f_n \oplus \mathrm{Im}f_{n+1}$. Hence $\mathrm{Im}f_n \in \mathrm{Add}C$, i.e., $X \in (\overset{\vee}{\mathrm{Add}}_* C)_n$.

(4) Let $0 \to L \to M \to N \to 0$ be a $*$-acyclic complex. If $L, N \in (\overset{\vee}{\mathrm{Add}}_* C)_n$, then it follows from (1) that $M \in \mathcal{Q}_C$. Moreover, by (3), we have $\mathrm{Ext}^{n+1}_*(Y, M) = 0$ for all $Y \in \mathcal{Q}_C$. Consequently, by applying (3) again, we obtain that $M \in (\overset{\vee}{\mathrm{Add}}_* C)_n$. The remainder of the proof follows similarly.

(5) Since $W \in {}_C\mathcal{Q}$, then we have a $*$-acyclic complex $0 \to W' \to C' \to W \to 0$ with $W' \in {}_C\mathcal{Q}$ and $C' \in \mathrm{Add}C$. Note that $\mathrm{Ext}^1_*(C, U) = 0$. So we can construct the following

pullback diagram, referring to Fig. 8.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & W' \arrow[r, equal] \arrow[d] & W' \arrow[d] & \\
0 \arrow[r] & U \arrow[r] \arrow[d, equal] & U \oplus C' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & U \arrow[r] & V \arrow[r, "\delta"] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Fig. 8. The pullback diagram of morphisms δ and γ

It is evident that all rows and columns are $*$-acyclic complexes in Fig. 8. Since $V, W' \in {}_C\mathcal{Q}$ and ${}_C\mathcal{Q}$ is closed under $*$-extensions, then we have that $U \oplus C' \in {}_C\mathcal{Q}$. Now by $(1)'$, we conclude that $U \in {}_C\mathcal{Q}$. $\square$

Let $\mathcal{Q} \subseteq R-\mathrm{mod}$ be a subcategory closed under finite direct sums and summands. Assume that $M \in \mathcal{Q}$, then M is called a relative generator of $\mathcal{Q}$ if for any $Q \in \mathcal{Q}$, there exists a $*$-acyclic complex $0 \to Q' \to M_Q \to Q \to 0$ with $M_Q \in \mathrm{Add}M$ and $Q, Q\prime \in \mathcal{Q}$. Dually, M is called a relative cogenerator of $\mathcal{Q}$ if for any $Q \in \mathcal{Q}$, there exists a $*$-acyclic complex $0 \to Q \to M_Q \to Q' \to 0$ with $M_Q \in \mathrm{Add}M$ and $Q' \in \mathcal{Q}$.

For example, ${}_C\mathcal{Q}$ and $\overset{\wedge}{\mathrm{Add}}_*C$ both have a $*$-selforthogonal relative generator C, while $\mathcal{Q}_C$ and $\overset{\vee}{\mathrm{Add}}_*C$ both have a $*$-selforthogonal relative cogenerator C'.

In other words, the lemma mentioned below is useful in obtaining the $*-$acyclic complex we are aiming for within certain-specific subcategories, hence we offer a more thorough proof.

Lemma 3.2: Assume that $\mathcal{Q} \subseteq R-\mathrm{mod}$ contains a $*$-selforthogonal relative generator C and is closed under $*$-extensions, finite direct sums and summands. Let X be an $R-\mathrm{mod}$ such that there exists a $*$-acyclic complex $0 \to X \to N_m \to \cdots \to N_1 \to Z \to 0$ for some m and Z, with each $N_i \in \mathcal{Q}$. Then:

(1) There exists a $*$-acyclic complex $0 \to U_m \to V_m \to X \to 0$ for some $U_m \in \mathcal{Q}$, and for some V_m such that there exists a $*$-acyclic complex $0 \to V_m \to C_m \to \cdots \to C_1 \to Z \to 0$ with each $C_i \in \mathrm{Add}C$.

(2) If, moreover, $Z \in \mathcal{Q}$ too, then there exists a $*$-acyclic complex $0 \to U \to V \to X \to 0$ for some $U \in \mathcal{Q}$, and for some $V \in (\overset{\vee}{\mathrm{Add}}_*C)_m$.

(3) There exists a $*$-acyclic complex $0 \to X \to U \to V \to 0$ for some $U \in \mathcal{Q}$, and for some V such that there exists a $*$-acyclic complex $0 \to V \to C_{m-1} \to \cdots \to C_1 \to Z \to 0$ with each $C_i \in \mathrm{Add}C$. If moreover, $U \in \mathcal{Q}$, then V can be taken in $(\overset{\vee}{\mathrm{Add}}_*C)_{m-1}$.

Proof: (1) We prove the statement by induction on m. If $m = 1$, then we have a $*$-acyclic complex $0 \to X \to N_1 \to Z \to 0$ with $N_1 \in \mathcal{Q}$ for some $X \in R-\mathrm{mod}$. Since $N_1 \in \mathcal{Q}$, there exists a $*$-acyclic complex $0 \to U_1 \to C' \to N_1 \to 0$ with $U_1 \in \mathcal{Q}$ and $C' \in \mathrm{Add}C$. Consequently, we can construct the following pullback diagram, referring to Fig. 9.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & U_1 \arrow[r, equal] \arrow[d] & U_1 \arrow[d] & & \\
0 \arrow[r] & V_1 \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d] & N_1 \arrow[r] \arrow[d] & Z \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Fig. 9. The pullback diagram of morphisms α and β

We have the middle row is also a $*$-acyclic complex by Lemma 2.2. Therefore, the left column is just the desired $*$-acyclic complex by the snake lemma.

Assume that the result holds for $m-1$. We will show that the conclusion holds for m.

Firstly, by the induction assumption, there exists a $*$-acyclic complex $0 \to U'_{m-1} \to V'_{m-1} \to X' \to 0$ for some $U'_{m-1} \in \mathcal{Q}$, and for some V'_{m-1} such that there exists a $*$-acyclic complex $0 \to V'_{m-1} \to C'_{m-1} \to \cdots \to C'_1 \to Z \to 0$ with each $C'_i \in \mathrm{Add}C$. Secondly, define $X' = \mathrm{Coker}(X \to N_m)$, we can construct the following pullback diagram, referring to Fig. 10.

It is evident that both the middle column and the middle row are $*$-acyclic complexes. Since $N_m, U'_{m-1} \in \mathcal{Q}$ and $\mathcal{Q}$ is closed under $*$-extensions, it follows that $Y \in \mathcal{Q}$. Consequently, there exists a $*$-acyclic complex $0 \to U_m \to C_Y \to Y \to 0$ with $C_Y \in \mathrm{Add}C$ and $U_m \in \mathcal{Q}$. So we have the following pullback diagram, referring to Fig. 11.

It is obvious that the middle row is a $*$-acyclic complex by Lemma 2.2, and it is straightforward to verify that the left column is represents the desired $*$-acyclic complex by the snake lemma.

(2) If $Z \in \mathcal{Q}$, then there exists a $*$-acyclic complex $0 \to Z' \to C_0 \to Z \to 0$ with $C_0 \in \mathrm{Add}C$ and $Z' \in \mathcal{Q}$. Define $X_1 = \mathrm{Ker}(N_1 \to Z)$. We have the following pullback diagram, referring to Fig. 12.

Again we observe that both the middle column and the middle row are $*$-acyclic complexes. It follows that there exists a $*$-acyclic complex $0 \to X \to N_m \to \cdots \to N_2 \to Y' \to C_0 \to 0$. Since $N_1, Z' \in \mathcal{Q}$, by Lemma 3.1, it follows from the middle column that $Y' \in \mathcal{Q}$. Hence by (1), we

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & U'_{m-1} \arrow[r, equal] \arrow[d] & U'_{m-1} \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & V'_{m-1} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X \arrow[r] & N_m \arrow[r, "\delta"] \arrow[d] & X' \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}$$

Fig. 10. The pullback diagram of morphisms δ and γ

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& U_m \arrow[r, equal] \arrow[d] & U_m \arrow[d] & & \\
0 \arrow[r] & V_m \arrow[r] \arrow[d] & C_Y \arrow[r] \arrow[d, "\rho"] & V'_{m-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "\pi"] \arrow[d] & Y \arrow[r] \arrow[d] & V'_{m-1} \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

Fig. 11. The pullback diagram of morphisms π and ρ

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & Z' \arrow[r, equal] \arrow[d] & Z' \arrow[d] & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X_1 \arrow[r] & N_1 \arrow[r, "\delta"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}$$

Fig. 12. The pullback diagram of morphisms δ and γ

obtain a $*$-acyclic complex $0 \to U \to V \to X \to 0$ for some $U \in \mathcal{Q}$ and for some V such that there exists a $*$-acyclic complex $0 \to V \to C_m \to \cdots \to C_1 \to C_0 \to 0$ with each $C_{\rm i} \in \mathrm{Add}C$, i.e., $V \in (\check{\mathrm{Add}}_* C)_m$.

(3) By (1), there exists a $*$-acyclic complex $0 \to U' \to V' \to X \to 0$ for some $U' \in \mathcal{Q}$. And for some V' such that there exists a $*$-acyclic complex $0 \to V' \to C_m \to \cdots \to C_1 \to Z \to 0$ with each $C_i \in \mathrm{Add}C$. Denote $V = \mathrm{Coker}(V' \to C_m)$. Consequently we have the following pushout diagram, referring to Fig. 13.

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U' \arrow[r] \arrow[d, equal] & V' \arrow[r, "\mu"] \arrow[d, "\upsilon"] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U' \arrow[r] & C_m \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & 0 \\
& & V \arrow[r, equal] \arrow[d] & V \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

Fig. 13. The pushout diagram of morphisms μ and υ

The middle row is also a $*$-acyclic complex, so we can get the right column is just the desired $*$-acyclic complex.

The rest of proof is similar to (2), so we omit it. $\square$

The following is a dual statement of Lemma 3.2.

Lemma 3.3: Assume that C is $*$-selforthogonal and n is an integer. Let X be an $R-\mathrm{mod}$ such that there exists a $*$-acyclic complex $0 \to Z \to N_1 \to \cdots \to N_m \to X \to 0$ for some m and Z with each $N_i \in (\check{\mathrm{Add}}_* C)_n$. Then there exists a $*$-acyclic complex $0 \to X \to V \to U \to 0$ for some $U \in (\check{\mathrm{Add}}_* C)_{n-1}$ and for some V such that there exists a $*$-acyclic complex $0 \to Z \to C_1 \to \cdots \to C_m \to V \to 0$ with each $C_i \in \mathrm{Add}C$.

IV. n-C-$\mathcal{X}$ TILTING PAIRS

In this section, we introduce the concept of n-C-$\mathcal{X}$ tilting pair and explore related theorems and propositions. Firstly, we recall the concept of n-tilting pairs.

Definition 4.1: Let R be an Artin algebra. A pair (C, T) is said to be n-tilting provided

(1) C is selforthogonal;
(2) T is selforthogonal;
(3) $T \in (\hat{\mathrm{Add}}_R C)_n$;
(4) $C \in (\check{\mathrm{Add}}_R T)_n$.

Definition 4.2: Let R be an Artin algebra. A pair (C, T) is said to be n-C-$\mathcal{X}$ tilting provided

(1) C is $*$-selforthogonal;
(2) T is $*$-selforthogonal;

(3) $T \in (\overset{\wedge}{\mathrm{Add}}_* C)_n$;

(4) $C \in (\overset{\vee}{\mathrm{Add}}_* T)_n$.

From now on, we assume that C is $*-$selforthogonal and $n \geqslant 1$ is a fixed integer. The following lemma tells that how to compare the relationship between ${}_T\mathcal{Q}$ and ${}_C\mathcal{Q}$.

Lemma 4.1: Let T be $*$-selforthogonal.

(1) If $C \in (\overset{\vee}{\mathrm{Add}}_* T)_n$, then $T^{*\perp} \subseteq C^{*\perp}$.

(2) If $T \in {}_C\mathcal{Q}$ and $C \in (\overset{\vee}{\mathrm{Add}}_* T)_n$, then ${}_T\mathcal{Q} \subseteq {}_C\mathcal{Q}$.

In particular, if (C,T) is n-C-$\mathcal{X}$ tilting pair, then $T^{*\perp} \subseteq C^{*\perp}$ and ${}_T\mathcal{Q} \subseteq {}_C\mathcal{Q}$.

Proof: (1) Suppose that C belongs to $(\overset{\vee}{\mathrm{Add}}_* T)_n$. Consequently, there exists a $*$-acyclic complex $0 \to C \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with each $T_i \in \mathrm{Add}T$. Given that $\mathrm{Ext}_*^{i\geq 1}(T, N) = 0$ for any $N \in T^{*\perp}$, we apply the contravariant functor $\mathrm{Hom}_R(-, N)$, and we get $\mathrm{Ext}_*^i(C, N) \cong \mathrm{Ext}_*^{i+n}(T_n, N) = 0$ for any $i \geqslant 0$ by demension shift. Hence $T^{*\perp} \subseteq C^{*\perp}$.

(2) Let $M \in {}_T\mathcal{Q}$, then there exists a $*$-acyclic complex $\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0$ with each $T_i \in \mathrm{Add}T$ and $Kerf_i \in {}_T\mathcal{Q}$. Since $T \in {}_C\mathcal{Q}$, we can apply Lemma 3.2 to the $*$-acyclic complex $0 \to Kerf_0 \to T_0 \to M \to 0$. Consequently, we obtain a $*$-acyclic complex

$$0 \to H_0 \to M_0 \to Kerf_0 \to 0 \qquad (*_1)$$

with $H_0 \in {}_C\mathcal{Q}$ and M_0 satisfying the following $*$-acyclic complex

$$0 \to M_0 \to C_0 \to M \to 0 \qquad (*_2)$$

where $C_0 \in \mathrm{Add}C$. Since $H_0 \in {}_C\mathcal{Q} \subseteq C^{*\perp}$, $Kerf_0 \in C^{*\perp}$ by (1) and $C^{*\perp}$ is closed under $*$-extensions, therefore, we have $M_0 \in C^{*\perp}$. At this point, we can consider the following pullback diagram, referring to Fig. 14.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Kerf_1 \arrow[r, equal] \arrow[d] & Kerf_1 \arrow[d] & \\
0 \arrow[r] & H_0 \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & H_0 \arrow[r] & M_0 \arrow[r, "\delta"] \arrow[d] & Kerf_0 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Fig. 14. The pullback diagram of morphisms δ and γ

Based on Lemma 2.2, we ascertain that all sequences are $*$-acyclic complexes. Since $H_0, T_1 \in {}_C\mathcal{Q}$ and ${}_C\mathcal{Q}$ is closed under $*$-extensions, so we obtain that $Y \in {}_C\mathcal{Q}$ from the middle row. Thus, there exists a $*$-acyclic complex $0 \to H_1 \to C_1 \to Y \to 0$ with $H_1 \in {}_C\mathcal{Q}$ and $C_1 \in \mathrm{Add}C$. This leads us to consider the following pullback diagram, as illustrated in Fig. 15.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & H_1 \arrow[r, equal] \arrow[d] & H_1 \arrow[d] & & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d, "\rho"] & M_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Kerf_1 \arrow[r, "\pi"] \arrow[d] & Y \arrow[r] \arrow[d] & M_0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Fig. 15. The pullback diagram of morphisms π and ρ

Since the middle column and the below row are $*$-acyclic complexes, we infer that the middle row is a $*$-acyclic complex. Hence, there exists a $*$-acyclic complex $0 \to M_1 \to C_1 \to M_0 \to 0$ where $C_1 \in \mathrm{Add}C$. From this, we derive another $*$-acyclic complex

$$0 \to H_1 \to M_1 \to Kerf_1 \to 0 \qquad (*_3)$$

with $H_1 \in {}_C\mathcal{Q}$. And M_1 satisfying a $*$-acyclic complex

$$0 \to M_1 \to C_1 \to M_0 \to 0 \qquad (*_4)$$

where $C_1 \in \mathrm{Add}C$.

By repeating $(*_1)$ to $(*_4)$, we get a $*$-acyclic complex $\cdots \xrightarrow{g_2} C_1 \xrightarrow{g_1} C_0 \xrightarrow{g_0} M \to 0$ with each $C_i \in \mathrm{Add}C$ and $Kerg_i = M_i \in C^{*\perp}$. Thus, we conclude that $M \in {}_C\mathcal{Q}$. $\square$

Proposition 4.1: Assume that (C, T) is an n-C-$\mathcal{X}$ tilting pair. The following are equivalent for an $R-$mod M :

(1) $M \in {}_T\mathcal{Q}$;

(2) $M \in T^{*\perp} \bigcap {}_C\mathcal{Q}$;

(3) $M \in \mathcal{X}_{\mathrm{Pres}^n_{{}_C\mathcal{Q}}}(T)$.

Proof: (1) $\Rightarrow$ (2). Since (C, T) is an n-C-$\mathcal{X}$ tilting pair, we obtain that $M \in {}_T\mathcal{Q} \subseteq {}_C\mathcal{Q}$ from Lemma 4.1. So $M \in T^{*\perp} \bigcap {}_C\mathcal{Q}$, as $M \in {}_T\mathcal{Q} \subseteq T^{*\perp}$.

(2) $\Rightarrow$ (3). Firstly, we need to prove if $M \in T^{*\perp} \bigcap {}_C\mathcal{Q}$, then $M \in \mathcal{X}_{\mathrm{Pres}^1_{{}_C\mathcal{Q}}}(T)$. In fact, since $M \in {}_C\mathcal{Q}$, then there exists a $*$-acyclic complex $0 \to M_1 \to C_0 \to M \to 0$ with $M_1 \in {}_C\mathcal{Q}$ and $C_0 \in \mathrm{Add}C$. Given that (C, T) is an n-C-$\mathcal{X}$ tilting pair, we obtain that there exists a $*$-acyclic complex $0 \to C_0 \to T' \to X \to 0$ with $T' \in \mathrm{Add}T$ and $X \in \overset{\vee}{\mathrm{Add}}_* T$. This allows us to consider the following pushout diagram, referring to Fig. 16.

It is obvious that the middle row and the right column are $*$-acyclic complexes. Since $M \in T^{*\perp}$ and $X \in \overset{\vee}{\mathrm{Add}}_* T$, we get that $\mathrm{Ext}_*^{i\geq 1}(X, M) = 0$ by Lemma 3.1(2), then we deduce from the right column that $Y \cong M \oplus X$. Furthermore, Since $Y \in \mathcal{X}_{\mathrm{Pres}^1_{{}_C\mathcal{Q}}}(T)$ by the middle row, it necessarily follows that $M \in \mathcal{X}_{\mathrm{Pres}^1_{{}_C\mathcal{Q}}}(T)$, too.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d, equal] & C_0 \arrow[r, "\mu"] \arrow[d, "\upsilon"] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & T' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
 & & X \arrow[r, equal] \arrow[d] & X \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Fig. 16. The pushout diagram of morphisms μ and υ

Secondly, take $M \in T^{*\perp} \bigcap {}_C\mathcal{Q}$, we have a $*$-acyclic complex $0 \to M' \to T_M \to M \to 0$ with $M' \in T^{*\perp}$ and $T_M \in \mathrm{Add}T$, as $M \in \mathcal{X}_{\mathrm{Pres}^1_{C\mathcal{Q}}}(T)$. Since $T^{*\perp} \subseteq C^{*\perp}$ and $\mathrm{Add}T \subseteq {}_C\mathcal{Q}$, it follows that $M' \in T^{*\perp} \bigcap {}_C\mathcal{Q}$ by Lemma 3.1(5). Now repeating the process to M', and so on, we conclude that $M \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$.

(3) $\Rightarrow$ (2). For any $M \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$, then there exists a $*$-acyclic complex $0 \to X \to T_n \to \cdots \to T_1 \to M \to 0$ where $X \in {}_C\mathcal{Q}$ and each $T_i \in \mathrm{Add}T$. It is evident that $M \in {}_C\mathcal{Q}$ by Lemma 3.1(1)$'$. Hence we just have to prove that $M \in T^{*\perp}$. Since T is $*$-selforthogonal and by applying the functor $\mathrm{Hom}_R(T,-)$, we have that $\mathrm{Ext}^i_*(T,M) \cong \mathrm{Ext}^{i+n}_*(T,X)$ for all $i \geqslant 1$ by demension shift. Since T is n-C-$\mathcal{X}$ tilting, then there exists a $*$-acyclic complex $0 \to C_n \to \cdots \to C_0 \to T \to 0$. By applying the functor $\mathrm{Hom}_R(-,X)$, we also obtain that $\mathrm{Ext}^{i+n}_*(T,X) \cong \mathrm{Ext}^i_*(C_n,X) = 0$ for all $i \geqslant 1$ by demension shift. From the above results we get $\mathrm{Ext}^{i\geq 1}_*(T,M) = 0$, that is, $M \in T^{*\perp}$.

(2) $\Rightarrow$ (1). In the process of the proof from (2) to (3), we notice that there exists a $*$-acyclic complex $\cdots T_1 \xrightarrow{f_1} T_0 \to M \to 0$ with each $T_i \in \mathrm{Add}T$ and each $\mathrm{Im}\, f_i \in T^{*\perp}$ for any $M \in T^{*\perp} \bigcap {}_C\mathcal{Q}$. It follows that $M \in {}_T\mathcal{Q}$ by the definition. □

It is straightforward to derive the following result.

Proposition 4.2: If $\mathcal{X}_{\mathrm{Pres}^m_{C\mathcal{Q}}}(T) = T^{*\perp} \bigcap {}_C\mathcal{Q}$ for some $m \geqslant 1$, then $T \in T^{*\perp} \bigcap {}_C\mathcal{Q}$. □

Proposition 4.3: If $\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T) = T^{*\perp} \bigcap {}_C\mathcal{Q}$, then $\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T) = \mathcal{X}_{\mathrm{Pres}^{n+1}_{C\mathcal{Q}}}(T)$.

Proof: Clearly, we only need to prove that for any $M \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$, there exists a $*$-acyclic complex $0 \to L \to T_M \to M \to 0$ with $L \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$ and $T_M \in \mathrm{Add}T$. In fact, if $M \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$, there exists a $*$-acyclic complex $0 \to N \to T_1 \to M \to 0$ with $N \in \mathcal{X}_{\mathrm{Pres}^{n-1}_{C\mathcal{Q}}}(T)$ and $T_1 \in \mathrm{Add}T$. Since $M, T_1 \in {}_C\mathcal{Q}$ and ${}_C\mathcal{Q}$ is closed under cokernels of $*$-monomorphisms, we have $N \in {}_C\mathcal{Q}$. Since $M \in {}_C\mathcal{Q}$, we can obtain another $*$-acyclic complex $0 \to L \to T_M \to M \to 0$ with $L \in {}_C\mathcal{Q}$ and $T_M \in \mathrm{Add}T$. It follows that T is $*$-selforthogonal by Proposition 4.2, and $M \in T^{*\perp}$. Moreover, we have $L \in T^{*\perp}$ since $T^{*\perp}$ is closed under kernels of $*$-epimorphisms. Now consider the following pullback diagram, referring to Fig. 17.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N \arrow[r, equal] \arrow[d] & N \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & L \arrow[r] & T_M \arrow[r, "\delta"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Fig. 17. The pullback diagram of morphisms δ and γ

It is manifest that the middle row and the middle column are $*$-acyclic complexes. Given that $T_M, N \in {}_C\mathcal{Q}$ and noticing that ${}_C\mathcal{Q}$ is also closed under $*$-extensions, it follows that $Y \in {}_C\mathcal{Q}$. Furthermore, since $L \in T^{*\perp}$, we deduce from the middle row that $Y \cong L \oplus T_1$. As ${}_C\mathcal{Q}$ is closed under direct summands, it necessarily follows that $L \in {}_C\mathcal{Q}$. Consequently, we conclude that $L \in T^{*\perp} \bigcap {}_C\mathcal{Q} = \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$. □

Proposition 4.4: If $\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T) = T^{*\perp} \bigcap {}_C\mathcal{Q}$, then $\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T) = \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T))$.

Proof: For any $M \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T))$, we have a $*$-acyclic complex $0 \to X \to P_n \to \cdots \to P_1 \to M \to 0$ with each $P_i \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$ and $X \in {}_C\mathcal{Q}$. It is evident that $\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T) = T^{*\perp} \bigcap {}_C\mathcal{Q}$ is closed under $*$-extensions, finite direct sums and summands. In addition, we assert that T is $*$-selforthogonal by Proposition 4.2. Hence the category $\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$ satisfies assumptions of Lemma 3.2, then we are able to construct a $*$-acyclic complex $0 \to U \to V \to X \to 0$ for some $U \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$ and for some V such that there exists a $*$-acyclic complex $0 \to V \to T'_n \to \cdots \to T'_1 \to M \to 0$ with each $T'_i \in \mathrm{Add}T$. Since $U, X \in {}_C\mathcal{Q}$, ${}_C\mathcal{Q}$ is also closed under $*$-extensions, we have that $V \in {}_C\mathcal{Q}$. Consequently, we establish that $M \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$. This is, $\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)) \subseteq \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$. The other inclusion is obvious, since $\mathrm{Add}T \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$.

Proposition 4.5: Assume that $\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T) = T^{*\perp} \bigcap {}_C\mathcal{Q}$. If $C \in \mathrm{Copres}^n(\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T))$, then $C \in (\overset{\vee}{\mathrm{Add}}_* T)_n$.

Proof: If $C \in \mathrm{Copres}^n(\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T))$, then there exists a $*$-acyclic complex $0 \to C \to P_n \to \cdots \to P_1 \to P_0 \to 0$ with $P_i \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$ for $1 \leqslant i \leqslant n$. We obtain that $P_0 \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$ by assumptions and Proposition 4.4. By applying Lemma 3.2 to $\mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$, we get a $*$-acyclic complex $0 \to U \to V \to C \to 0$ for some $U \in \mathcal{X}_{\mathrm{Pres}^n_{C\mathcal{Q}}}(T)$ and for some $V \in (\overset{\vee}{\mathrm{Add}}_* T)_n$. Note that $U \in C^{*\perp}$ by assumption. Hence $V \cong C \oplus U$. Finally, we get that $C \in (\overset{\vee}{\mathrm{Add}}_{*}T)_n$ by Lemma 3.1(4). □

Proposition 4.6: Assume that $\mathcal{X}_{\mathrm{Pres}^n_{C}\mathcal{Q}}(T) = T^{*\perp} \bigcap {}_C\mathcal{Q}$. If $C \in \mathrm{Copres}^n(\mathcal{X}_{\mathrm{Pres}^n_{C}\mathcal{Q}}(T))$, then $T \in (\overset{\wedge}{\mathrm{Add}}_{*}C)_n$.

Proof: Firstly, we prove that $\mathrm{Ext}^{n+1}_{*}(T, M) = 0$ for any $M \in {}_C\mathcal{Q}$.

For each $M \in {}_C\mathcal{Q}$, then there exists a $*$-acyclic complex $0 \to Z \to C_1 \to \cdots \to C_n \to M \to 0$ with $Z \in {}_C\mathcal{Q}$ and each $C_i \in \mathrm{Add}C$. Then we have that each $C_i \in (\overset{\vee}{\mathrm{Add}}_{*}T)_n$ by Proposition 4.5. Thus, we obtain a $*$-acyclic complex $0 \to M \to V \to U \to 0$ for some $U \in (\overset{\vee}{\mathrm{Add}}_{*}T)_{n-1}$ and for some V such that there exists a $*$-acyclic complex $0 \to Z \to T'_1 \to \cdots \to T'_n \to V \to 0$ with each $T'_i \in \mathrm{Add}T$ by Lemma 3.3. It is easy to see that $V \in \mathcal{X}_{\mathrm{Pres}^n_{C}\mathcal{Q}}(T) = T^{*\perp} \bigcap {}_C\mathcal{Q}$. So we conclude that $\mathrm{Ext}^{n+1}_{*}(T, M) \cong \mathrm{Ext}^{n}_{*}(T, U) = 0$ by dimension shift and Lemma 3.1(3).

Since $T \in {}_C\mathcal{Q}$ by Proposition 4.2, then there exists a $*$-acyclic complex $0 \to X_{n+1} \to C_n \xrightarrow{f_n} C_{n-1} \to \cdots \xrightarrow{f_1} C_0 \to T \to 0$ with $X_{n+1} \in {}_C\mathcal{Q}$ and each $C_i \in \mathrm{Add}C$. We also have that $\mathrm{Ext}^1_{*}(\mathrm{Im}f_n, X_{n+1}) \cong \mathrm{Ext}^{n+1}_{*}(T, X_{n+1}) = 0$ by dimension shift and the above arguments. It follows that $\mathrm{Im}\, f_n \in \mathrm{Add}C$. Thus, we get that $T \in (\overset{\wedge}{\mathrm{Add}}_{*}C)_n$. □

The primary outcome highlighted in this study introduces a streamlined depiction of the n-C-$\mathcal{X}$ tilting pair, which stands as the core discovery of our research.

Theorem 4.1: Assume that C is $*$-selforthogonal and $C \in \mathrm{Copres}^n(\mathcal{X}_{\mathrm{Pres}^n_{C}\mathcal{Q}}(T))$. Then (C, T) is an n-C-$\mathcal{X}$ tilting pair if and only if $\mathcal{X}_{\mathrm{Pres}^n_{C}\mathcal{Q}}(T) = T^{*\perp} \bigcap {}_C\mathcal{Q}$.

Proof: The only if part follows from Propositions 4.1 and the if part follows from Proposition 4.2,4.5,4.6. □

Corollary 4.1: ([8, Theorem 3.10]) Assume that C is selforthogonal and $C \in \mathrm{Copres}^n(\mathrm{Pres}^n_{C}\mathcal{X}(T))$. Then (C, T) is an n-tilting pair if and only if $T^{\perp} \cap {}_C\mathcal{X} = \mathrm{Pres}^n_{C}\mathcal{X}(T)$.

Proof: Take $(\mathcal{X}, \mathcal{Y}) = (\mathrm{Proj}R, \mathrm{Inj}R)$. By remark 2.1 (1), we know that $*$-selforthogonal moudles are exactly selforthogonal moudles, ${}_C\mathcal{Q} = {}_C\mathcal{X}$, $T^{*\perp} = T^{\perp}$, and $\mathcal{X}_{\mathrm{Pres}^n_{C}\mathcal{Q}}(T) = \mathrm{Pres}^n_{C}\mathcal{X}(T)$. Hence, $\mathrm{Copres}^n(\mathcal{X}_{\mathrm{Pres}^n_{C}\mathcal{Q}}(T)) = \mathrm{Copres}^n(\mathrm{Pres}^n_{C}\mathcal{X}(T))$. Then, the result holds by Theorem 4.1 immediately.

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