Detecting a Five-petal Flower Shaped Hyperchaotic Attractor in the Circular Restricted Three Body Problem with Control and Its Synchronization

P. Muthukumar, *Member, IAENG,* and J. Murugan

***Abstract*—In this study, a novel 5-D hyperchaotic system is constructed by employing the circular restricted three body problem (CRTBP). The existence of hyperchaos is confirmed by Lyapunov exponents and shows that they have a symmetrical structure. The system exhibits a five-petal flower shaped hyperchaotic attractor. The synchronization and control of two identical hyperchaotic systems are accomplished via a tracking control system. The simulation results showed the effectiveness of the tracking and synchronization control systems. An innovative property of the proposed system has been identified and shows its ability to generate an infinitely many different-shaped chaotic attractors with finite wings by varying only the system's parameters.**

***Index Terms*—Hyperchaos, Lyapunov exponents, Tracking control, Synchronization**

I. Introduction

CHAOS is a nonlinear activity that is extremely complex. In [1], the physical manifestation of chaos and its practical implications have been examined. The applications of chaos in image processing [2], secure communication [3], and nonlinear circuits [4], [5], [6] have been extensively studied over the past three decades. In contrast to a chaotic system, a hyperchaotic system is characterized by the presence of multiple positive Lyapunov exponents and the occurrence of more intricate dynamics and behaviors. Furthermore, hyperchaotic systems were found to have better dynamics and behaviors than chaotic systems because of the high possibility of simultaneous exponential development of their system's states in multiple directions. It has been identified in real-life systems such as financial systems [7] memristive systems, and digital industrial supply chain systems [8]. Within the domain of secure s-box generation [9], it performs a more crucial function.

The dynamic characteristics of various physical systems or processes have been extensively examined in the contemporary nonlinear dynamics literature. Chaotic flow conditions have been seen in practical systems such as vortex spin-torque oscillators [10], double pendulums [11], stretch-twist-fold flow systems [12], and four-disk dynamos [13]. Regulating chaos in a higher-dimensional system is challenging. Lyapunov stability theory is a proficient method for controlling chaotic systems. The study in [14] examined an active control strategy utilizing Lyapunov stability theory to attain synchronization between two chaotic biological oscillators. In [15], a singular controller was developed for the global asymptotic synchronization of a unidirectionally coupled identical 3D autonomous chaotic system. A nonlinear feedback controller was developed for a brushless DC motor in [16].

The study of chaotic system synchronization has received a lot of attention in the field of nonlinear research. However, synchronizing chaotic systems is difficult due to their sensitivity to initial conditions. Pecora and Carroll [17] first presented the concept of synchronizing two identical chaotic systems with different starting conditions. However, several ways for synchronizing chaotic (hyperchaotic) systems have been devised. Tracking control has been used to synchronize 4D hyperchaotic systems [18], dislocated hybrid synchronization [19], and a chaotic multi-agent supply chain network [20]. In [21], researchers looked into multi-scale synchronization of King Cobra chaotic systems. To synchronize hyperchaotic Lorenz-type systems, a fuzzy-based sliding mode observer technique [22], [23] was used. Adaptive control was used to manage the hyperchaotic [24] system. [25] investigates the predictive control and synchronization of an uncertain disturbed chaotic permanent-magnet synchronous generator. In [26], chaos management and fractional inverse matrix projective difference synchronization on parallel chaotic systems were studied. A tracking control technique has been proposed in [27] for the synchronization of integer order-fractional order chaotic systems. Additionally, an integral sliding mode control [28], [29] has been implemented to synchronize multi-stable hyperchaotic two-scroll systems. Furthermore, chaotic system control and synchronization are heavily influenced by its potential applications in secure communication [30], [31], [32], [33], [34] and signal encryption-decryption [35].

A novel five-dimensional hyperchaotic Circular Restricted Three-Body Problem (CRTBP) is formulated, leading to the identification of a five-petal flower-shaped hyperchaotic attractor. The Lyapunov exponents of the suggested system exhibit a symmetrical form upon analysis. The proposed hyperchaotic CRTBP is governed and synchronized by employing a tracking control technique. The system's unique feature has been identified. This research illustrates that modifying the system's parameters produces an unlimited array of chaotic attractors with finite wings of various shapes.

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P. Muthukumar is a senior assistant professor in the PG and Research Department of Mathematics, Gobi Arts & Science College (Bharathiar University), Gobichettipalayam- 638 453, Erode, Tamil Nadu, India. (Corresponding author, e-mail: muthukumardgl@gmail.com).

J. Murugan is a senior assistant professor and research scholar in the PG and Research Department of Mathematics, Gobi Arts & Science College (Bharathiar University), Gobichettipalayam- 638 453, Erode, Tamil Nadu, India.(e-mail: muruganjk23@gmail.com).

The subsequent sections of this paper are as follows: Section 2 examines innovative hyperchaotic systems and their dynamic characteristics. Section 3 delineates the construction of a tracking controller designed for controlling chaotic trajectories toward an unstable equilibrium. Section 4 establishes the methodology for synchronizing chaotic systems through the application of tracking controllers and corresponding numerical simulations to validate the effectiveness of the theoretical analysis. Section 5 concludes the paper.

II. System description and their dynamical properties

The restricted three body problem [36]: Two bodies revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction and a third body moves in the plane defined by the two revolving bodies, which is attracted by two bodies but not influencing their motion. Therefore, *the restricted problem of three bodies is to describe the motion of this third body*.

Consider the equations of motion of CRTBP [37], [36], which is described by

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x \\ \ddot{y} + 2\dot{x} &= \Omega_y \end{aligned} \tag{1}$$

where $\Omega = \frac{1}{2}(x^2+y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}, r_1^2 = (x-\mu_2)^2 + y^2$ and $r_2^2 = (x+\mu_1)^2 + y^2$. Here r_1 and r_2 correspond to distances between the third body and primaries (two revolving bodies); μ_1 and μ_2 are the mass parameters of the primaries. The dependent variables x and y refer to the rotating system. For instance, if they are constant then it is moving on a circle with constant velocity.

The circular restricted three body problem (1) can be written as a system of four first order differential equations[37]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + 2x_4 - \frac{\mu_1 x_1}{r_1^3} - \frac{\mu_2 x_1}{r_2^3} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_3 - 2x_2 - \frac{\mu_1 x_3}{r_1^3} - \frac{\mu_2 x_3}{r_2^3} \end{aligned} \tag{2}$$

where $x = x_1, \dot{x_1} = x_2, y = x_3, \dot{y} = x_4$. Note that, $\frac{\mu_1}{r_1^3}$ and $\frac{\mu_2}{r_2^3}$ are always positive because r_1, r_2 and μ_1, μ_2 are represents the distances and mass parameters respectively. The mass parameters μ_1 and μ_2 of primaries are connected by the relation $\mu_1 + \mu_2 = 1$ and whose distance is also unity[36].

For system (2), we have

$$\nabla . V = \frac{\partial \dot{x_1}}{\partial x_1} + \frac{\partial \dot{x_2}}{\partial x_2} + \frac{\partial \dot{x_3}}{\partial x_3} + \frac{\partial \dot{x_4}}{\partial x_4} = 0.$$

As a result, the dynamical system (2) is conservative. Furthermore, system (2) is unstable and has one trivial equilibrium point.

Motivated by the aforementioned circular restricted three body problem and its properties, a novel hyperchaotic system with a new variable x_5 is introduced without altering the variables x_1, x_2, x_3, x_4 in (2), which is described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + 2x_4 - \frac{\mu_1 x_1}{r_1^3} - \frac{\mu_2 x_1}{r_2^3} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_3 - 2x_2 - \frac{\mu_1 x_3}{r_1^3} - \frac{\mu_2 x_3}{r_2^3} \\ \dot{x}_5 &= -2cx_5 \end{aligned} \tag{3}$$

where $(x_1, x_2, x_3, x_4, x_5) \in R^5$ and $c > 0$ is the parameter of the new system (3).

The new system (3) has one trivial equilibrium point $O(0,0,0,0,0)$ and it is unstable. Further, the system (3) is symmetric about the transformation $(x_1, x_2, x_3, x_4, x_5) \rightarrow (-x_1, -x_2, -x_3, -x_4, -x_5)$.

For system (3), one can have

$$\nabla . V = \frac{\partial \dot{x_1}}{\partial x_1} + \frac{\partial \dot{x_2}}{\partial x_2} + \frac{\partial \dot{x_3}}{\partial x_3} + \frac{\partial \dot{x_4}}{\partial x_4} + \frac{\partial \dot{x_5}}{\partial x_5} = -2c < 0.$$

When $c > 0$, the dynamical system (3) is dissipative. Let $a = \frac{\mu_1}{r_1^3}$ and $b = \frac{\mu_2}{r_2^3}$. Throughout this manuscript, assume that $0 < a + b < 1$ since the total masses of primaries is unity and whose distance is also unity. For every $a, b, c > 0$, the system (3) has two positive Lyapnuov exponents, three negative Lyapnuov exponents and the sum of all the Lyapnuov exponents is $-2c$. Hence, the new system (3) exhibits hyperchaos. Fig. 1 depicts the Lyapnuov exponents of the system (3) for random values of a, b and $c = 1$.

For example, the value of the Lyapnuov exponents of the system (3) for $a = 0.5, b = 0.01, c = 1$ are $L_1 = 0.0138, L_2 = 0.0143, L_3 = -0.0143, L_4 = -0.0138$ and $L_5 = -2$, which is represented in Fig. $1(A)$. The sum of the Lyapnuov exponents is -2. It is worth noting that the first four Lyapnuov exponents of the system (3) form a symmetrical shape curve at 0 as seen in Fig. 1.

For $a = 0.5, b = 0.01$ and $c = 1$, the system (3) exhibits hyperchaos and their phase portraits are shown in Fig. 2. The five-petal flower shaped hyperchaotic attractor shown in x_1-x_3 phase portrait of the Fig. 2. Further, the time history of uncontrolled hyperchaotic system (3) is depicted in Fig. 3.

III. Control design of proposed CRTBP hyperchaotic system

In this section, a tracking controller is designed to track the states of the hyperchaotic system.
The system (3) with control inputs can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 + u_1 \\ \dot{x}_2 &= x_1 + 2x_4 - ax_1 - bx_1 + u_2 \\ \dot{x}_3 &= x_4 + u_3 \\ \dot{x}_4 &= x_3 - 2x_2 - ax_3 - bx_3 + u_4 \\ \dot{x}_5 &= -2cx_5 + u_5 \end{aligned} \tag{4}$$

where $a = \frac{\mu_1}{r_1^3}$, $b = \frac{\mu_2}{r_2^3}$ and $U = (u_1, u_2, u_3, u_4, u_5)$ is the control inputs. The system parameters a and b were thought to be unknown and would be estimated later. The design approach is based on satisfying the error equation shown below.

$$\dot{e}_i + k_i = 0, \qquad i = 1, 2, 3, 4, 5. \tag{5}$$

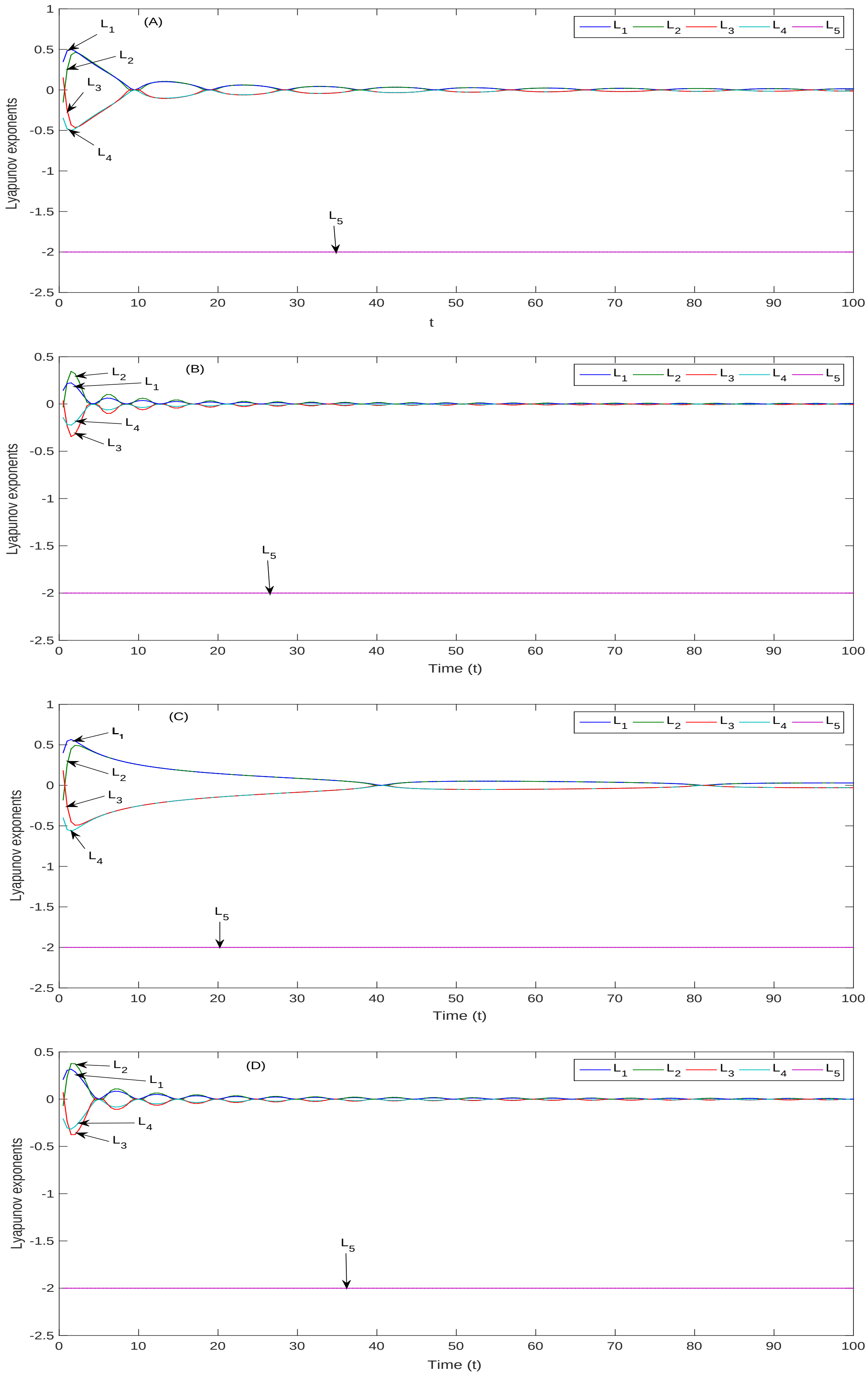

Fig. 1. Lyapunov exponents of the hyperchaotic system (3): (A) $a = 0.5, b = 0.01$, (B) $a = 0.05, b = 0.5$, (C) $a = 0.005, b = 0.001$ and (D) $a = 0.25, b = 0.15$ for fixed value of $c = 1$.

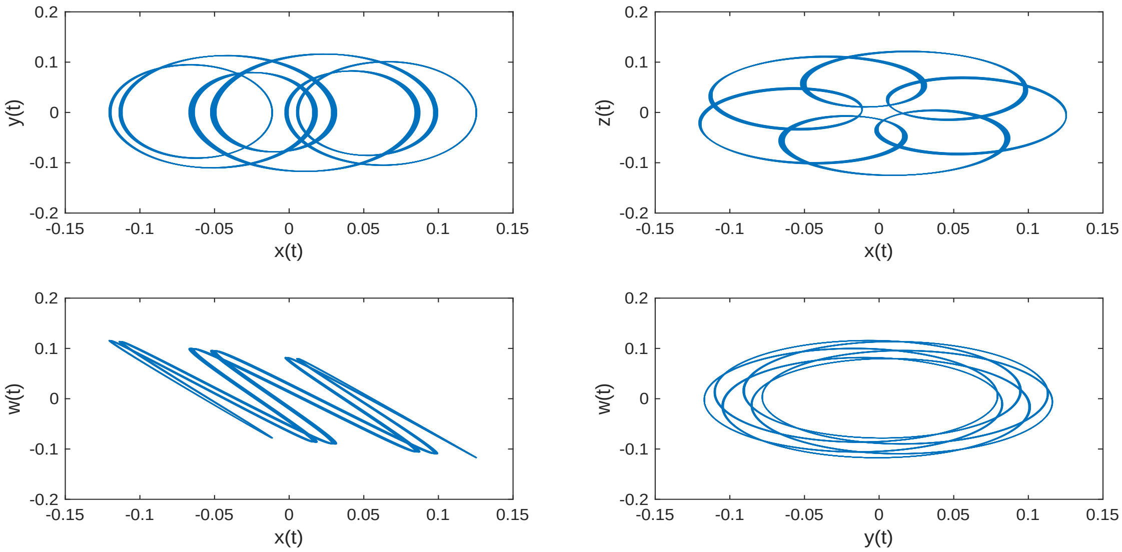

Fig. 2. Different phase portraits of the system (3) with parameters $a = 0.5, b = 0.01$ and $c = 1$.

where $e_i = r_i - d_i$, d_i is the desired state outputs of the system states x_i and k_i's are feedback gains.
The following result can be derived by substituting (4) in (5) and solving for U.

$$\begin{aligned} u_1 &= \dot{d}_1 + k_1 e_1 - x_2 \\ u_2 &= \dot{d}_2 + k_2 e_2 - x_1 - 2x_4 + a_n x_1 + b_n x_1 \\ u_3 &= \dot{d}_3 + k_3 e_3 - x_4 \\ u_4 &= \dot{d}_4 + k_4 e_4 - x_3 + 2x_2 + a_n x_3 + b_n x_3 \\ u_5 &= \dot{d}_5 + k_5 e_5 + 2c_n x_5 \end{aligned} \tag{6}$$

where a_n, b_n and c_n are the estimates of a, b and c.
When (6) is substituted in (4), the controlled system is

$$\begin{aligned} \dot{x}_1 &= \dot{d}_1 + k_1 e_1 \\ \dot{x}_2 &= \dot{d}_2 + k_2 e_2 - (a - a_n)x_1 - (b - b_n)x_1 \\ \dot{x}_3 &= \dot{d}_3 + k_3 e_3 \\ \dot{x}_4 &= \dot{d}_4 + k_4 e_4 - (a - a_n)x_3 - (b - b_n)x_3 \\ \dot{x}_5 &= \dot{d}_5 + k_5 e_5 - 2(c - c_n)x_5 \end{aligned} \tag{7}$$

The following error dynamical system is derived by substituting $x_i = d_i - e_i$ and $\dot{x}_i = \dot{d}_i - \dot{e}_i$ in (7)

$$\begin{aligned} \dot{e}_1 &= -k_1 e_1 \\ \dot{e}_2 &= -k_2 e_2 + a_e x_1 + b_e x_1 \\ \dot{e}_3 &= -k_3 e_3 \\ \dot{e}_4 &= -k_4 e_4 + a_e x_3 + b_e x_3 \\ \dot{e}_5 &= -k_5 e_5 + 2c_e x_5 \end{aligned} \tag{8}$$

where $a_e = a - a_n$, $b_e = b - b_n$ and $c_e = c - c_n$ are the errors between actual and estimated values of a, b and c.
To stabilize the error dynamics of the system, the estimated values should converge to the unknown real parameters a, b and c.

Theorem 3.1: For all initial states $X(0) = (x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) \in R^5$, the hyperchaotic system (4) will approach global and exponential asymptotical stabilized according to the tracking control law (6).

Proof: Consider the Lyapunov candidate function

$$V_1 = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + a_e^2 + b_e^2 + c_e^2) \tag{9}$$

Here V_1 is a positive definite function on R^5. It is enough to prove that $\dot{V}_1$ is negative definite.

$$\begin{aligned} \dot{V}_1 &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 + a_e\dot{a}_e \\ &\quad + b_e\dot{b}_e + c_e\dot{c}_e \qquad (10) \\ &= e_1(-k_1 e_1) + e_2(-k_2 e_2 + a_e x_1 + b_e x_1) \\ &\quad + e_3(-k_3 e_3) + e_4(-k_4 e_4 + a_e x_3 + b_e x_3) \\ &\quad + e_5(-k_5 e_5 + 2c_e x_5) + a_e(-\dot{a}_n) \\ &\quad + b_e(-\dot{b}_n) + c_e(-\dot{c}_n) \\ &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_5^2 \\ &\quad + a_e(e_2 r_1 - e_1 e_2 + e_4 r_3 - e_3 e_4 - \dot{a}_n) \\ &\quad + b_e(e_2 r_1 - e_1 e_2 + e_4 r_3 - e_3 e_4 - \dot{b}_n) \\ &\quad + c_e(2e_5 r_5 - e_5^2 - \dot{c}_n) \end{aligned}$$

If $\dot{a}_n = e_2 r_1 - e_1 e_2 + e_4 r_3 - e_3 e_4$,
$\dot{b}_n = e_2 r_1 - e_1 e_2 + e_4 r_3 - e_3 e_4$ and $\dot{c}_n = 2e_5 r_5 - e_5^2$ then

$$\begin{aligned} \dot{V}_1 &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_5^2 \\ &= -e^T P_1 e \end{aligned} \tag{11}$$

where $P_1 = \begin{pmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 \\ 0 & 0 & 0 & 0 & k_5 \end{pmatrix}$

Thus P_1 is positive definite, then $\dot{V}_1$ is negative definite for all $k_i > 0$ on R^5.
By Lyapunov stability theory, $x_1(t) \to 0, x_2(t) \to 0, x_3(t) \to 0, x_4(t) \to 0, x_5(t) \to 0$ exponentially as $t \to \infty$. Consequently, the system (4) is globally and exponentially stable. ■

A. Numerical simulations

For $k_1 = 5,\ k_2 = k_3 = 1,\ k_4 = 1.5$ and $k_5 = 2$, (6) becomes

$$\begin{aligned} u_1 &= \dot{d}_1 + 5e_1 - x_2 \\ u_2 &= \dot{d}_2 + e_2 - 5.5x_1 - 2x_4 \\ u_3 &= \dot{d}_3 + e_3 - x_4 \\ u_4 &= \dot{d}_4 + 1.5e_4 + 2x_2 - 5.5x_3 \\ u_5 &= \dot{d}_5 + 2e_5 + 0.99x_5 \end{aligned} \tag{12}$$

The controlled hyperchaotic system (4) provides

$$\begin{aligned} \dot{x}_1 &= -5x_1 \\ \dot{x}_2 &= -5.01x_1 - x_2 \\ \dot{x}_3 &= -x_3 \\ \dot{x}_4 &= -5.01x_3 - 1.5x_4 \\ \dot{x}_5 &= 0.99x_5 \end{aligned} \tag{13}$$

The time history of the controlled hyperchaotic system is depicted in Fig. 4.

IV. Synchronization of proposed CRTBP hyperchaotic systems

This section intended to synchronize two identical hyperchaotic systems. The system (3) is uncontrolled (master system) and its state outputs should be tracked or synchronized by the following slave system.

$$\begin{aligned} \dot{y}_1 &= y_2 + u_1 \\ \dot{y}_2 &= y_1 + 2y_4 - ay_1 - by_1 + u_2 \\ \dot{y}_3 &= y_4 + u_3 \\ \dot{y}_4 &= y_3 - 2y_2 - ay_3 - by_3 + u_4 \\ \dot{y}_5 &= -2cy_5 + u_5 \end{aligned} \tag{14}$$

where $U = (u_1, u_2, u_3, u_4, u_5)$ is the control inputs to be designed later. The synchronization error is defined as

$$e_i = y_i - x_i, \quad (i = 1, 2, 3, 4, 5) \tag{15}$$

The following error dynamical system is obtained by substituting (3) and (14) in (15)

$$\begin{aligned} \dot{e}_1 &= e_2 + u_1 \\ \dot{e}_2 &= e_1 + 2e_4 - ae_1 - be_1 + u_2 \\ \dot{e}_3 &= e_4 + u_3 \\ \dot{e}_4 &= e_3 - 2e_2 - ae_3 - be_3 + u_4 \\ \dot{e}_5 &= -2ce_5 + u_5 \end{aligned} \tag{16}$$

To stabilize the error dynamical system (16), the designed control inputs were developed to satisfy the following stable dynamics.

$$\dot{e}_i + k_i e_i = 0, i = 1, 2, 3, 4, 5 \tag{17}$$

Theorem 4.1: With the following tracking control law, the systems (3) and (14) will approach global and exponential asymptotical synchronization.

$$\begin{aligned} u_1 &= -k_1e_1 - e_2 \\ u_2 &= -k_2e_2 - e_1 - 2e_4 + a_ne_1 + b_ne_1 \\ u_3 &= -k_3e_3 - e_4 \\ u_4 &= -k_4e_4 + 2e_2 - e_3 + a_ne_3 + b_ne_3 \\ u_5 &= -k_5e_5 + 2c_ne_5 \end{aligned} \tag{18}$$

where $k_i's,\ i = 1, 2, 3, 4, 5$ are the the feedback gains which will be estimated in order to achieve synchronization.

Proof: Substitute (18) in (16), we get

$$\begin{aligned} \dot{e}_1 &= -k_1e_1 \\ \dot{e}_2 &= -k_2e_2 - a_ee_1 - b_ee_1 \\ \dot{e}_3 &= -k_3e_3 \\ \dot{e}_4 &= -k_4e_4 - a_ee_3 - b_ee_3 \\ \dot{e}_5 &= -k_5e_5 - 2c_ee_5 \end{aligned} \tag{19}$$

where $a_e = a - a_n,\ b_e = b - b_n,\ c_e = c - c_n$
Consider the Lyapunov candidate function as

$$V_2 = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + a_e^2 + b_e^2 + c_e^2) \tag{20}$$

The time derivative of V_2 becomes,

$$\begin{aligned} \dot{V}_2 &= e_1\dot{e}_1 + e_2\dot{e}_2 \\ &\quad + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 \\ &\quad + a_e\dot{a}_e + b_e\dot{b}_e + c_e\dot{c}_e \\ &= e_1(-k_1e_1) + e_2(-k_2e_2 - a_ee_1 - b_ee_1) \\ &\quad + e_3(-k_3e_3) + e_4(-k_4e_4 - a_ee_3 - b_ee_3) \\ &\quad + e_5(-k_5e_5 - 2c_ee_5) + a_e(-\dot{a}_n) \\ &\quad + b_e(-\dot{b}_n) + c_e(-\dot{c}_e) \\ &= -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 - k_5e_5^2 \\ &\quad + a_e(-e_1e_2 - e_3e_4 + \dot{a}_n) + b_e(-e_1e_2 - e_3e_4 \\ &\quad + \dot{b}_n) + c_e(-2e_5^2 - \dot{c}_n) \end{aligned}$$

if $a_e = e_1e_2 + e_3e_4 - \dot{a}_n;\ b_e = e_1e_2 + e_3e_4 - \dot{b}_n;\ c_e = 2e_5^2 + \dot{c}_n$ then

$$\begin{aligned} \dot{V}_2 &= -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 - k_5e_5^2 \\ \dot{V}_2 &= -e^T P_2 e \end{aligned} \tag{21}$$

where $P_2 = \begin{pmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 \\ 0 & 0 & 0 & 0 & k_5 \end{pmatrix}$

Thus P_2 is positive definite, then $\dot{V}_2$ is negative definite for all $k_i > 0$ on R^5.
By Lyapunov stability theory, $e_1(t) \to 0, e_2(t) \to 0, e_3(t) \to 0, e_4(t) \to 0, e_5(t) \to 0$ exponentially as $t \to \infty$. As a result, the system (3) and (14) are globally and exponentially synchronized. ■

A. Numerical simulations

For $k_1 = 5,\ k_2 = k_3 = 1,\ k_4 = 1.5$ and $k_5 = 1$, (18) gives

$$\begin{aligned} u_1 &= -5e_1 - e_2 \\ u_2 &= 9.01e_1 - e_2 - 2e_4 \\ u_3 &= -e_3 - e_4 \\ u_4 &= 2e_2 + 9.01e_3 - e_4 \\ u_5 &= e_5 \end{aligned} \tag{22}$$

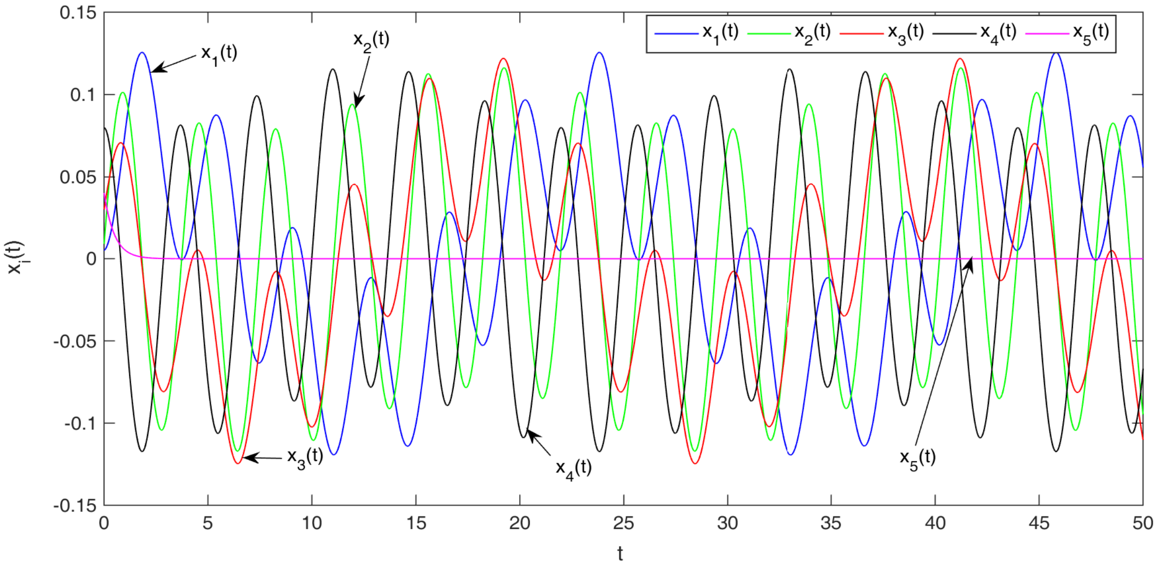

Fig. 3. The time response of uncontrolled hyperchaotic system (3).

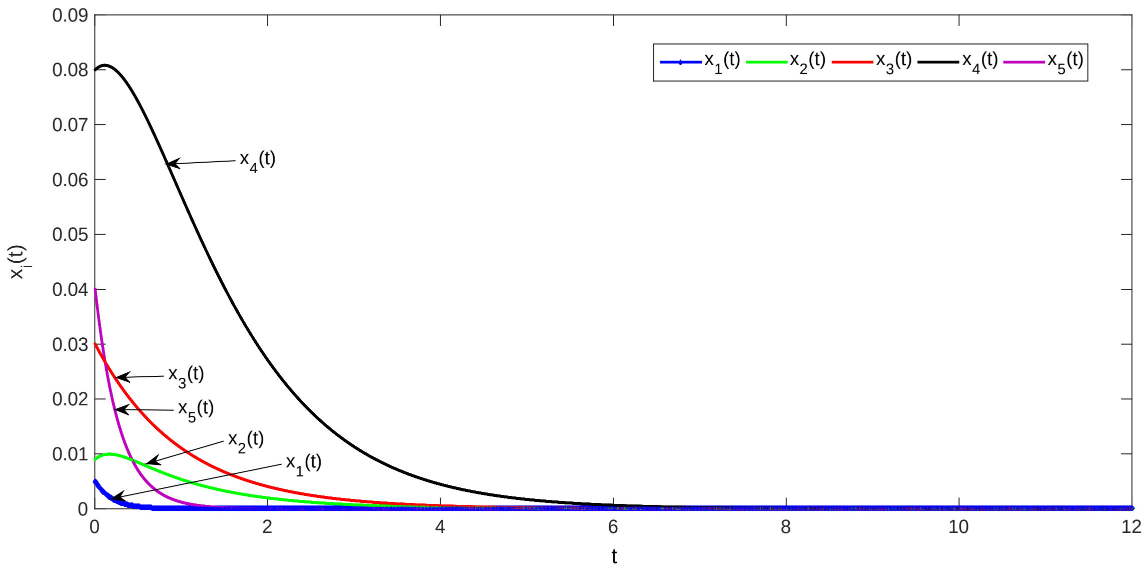

Fig. 4. The stabilization of hyperchaotic system (4)

The error system (19) can be written as

$$\begin{aligned} \dot{e}_1 &= -5e_1 \\ \dot{e}_2 &= 9.5e_1 - e_2 \\ \dot{e}_3 &= -e_3 \\ \dot{e}_4 &= 9.5e_3 - e_4 \\ \dot{e}_5 &= -e_5 \end{aligned} \tag{23}$$

The synchronized master and slave systems is depicted in Fig. 5 and their corresponding time variation of error system (23) using tracking controller is interpreted in Fig. 6.

Remark 4.2: By changing the system's parameters a and b, the proposed 5-D hyperchaotic CRTBP generates distinct chaotic attractors with a lot of wings. When compared to any other chaotic systems that are currently in existence, this is a very interesting and distinctive property of the proposed system. For instance, the APPENDIX displays a few dissimilar chaotic attractors that correspond to the system (3) for various values of a and b such that $0 < a + b < 500$ when $c = 1$. Finally, we draw the conclude that the special and unique feature of the proposed system is its ability to generate an infinitely many different-shaped chaotic attractors with finite wings by varying only the system's parameters.

V. Conclusions

A fascinating novel five-petal flower shaped 5D hyperchaotic system in CRTBP has been proposed in this paper. The necessary conditions for stabilization and synchronization of the proposed uncontrolled system have been established by a suitable tracking controller. Lyapunov exponents and numerical simulations have been

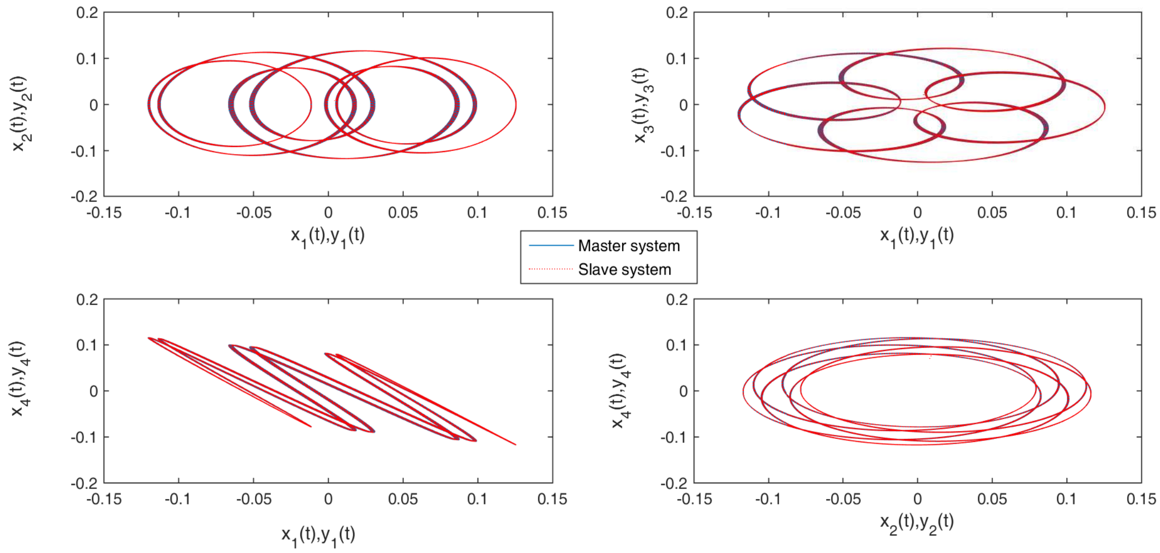

Fig. 5. The different phase portraits of synchronized master and slave systems.

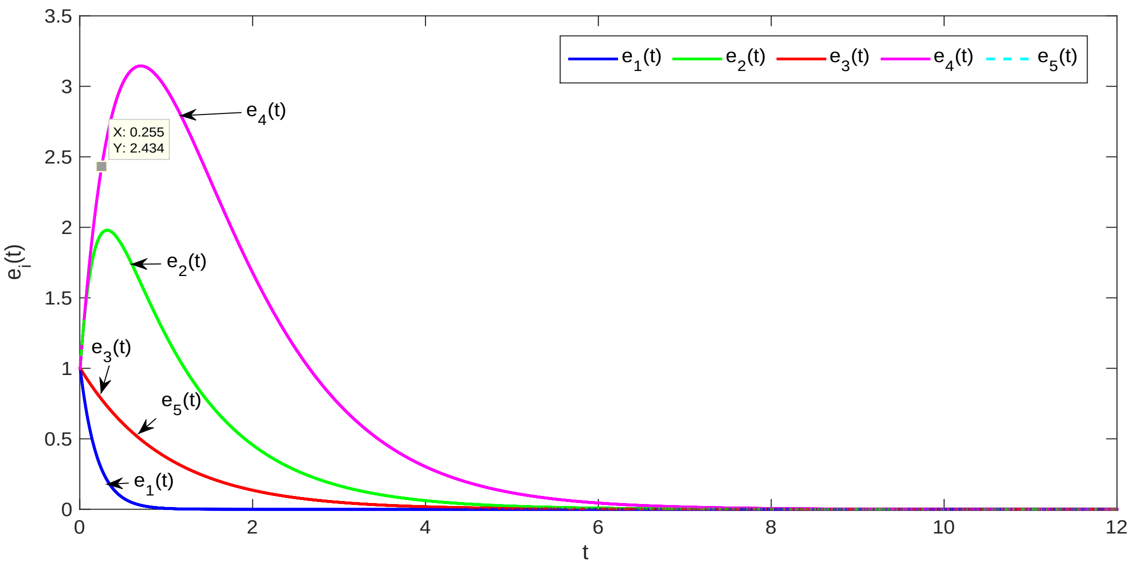

Fig. 6. Time history of synchronization error system (19)

demonstrated to validate the efficacy of the proposed theory. Additionally, the special feature of the proposed system has been made visible by adjusting the parameters of the system.

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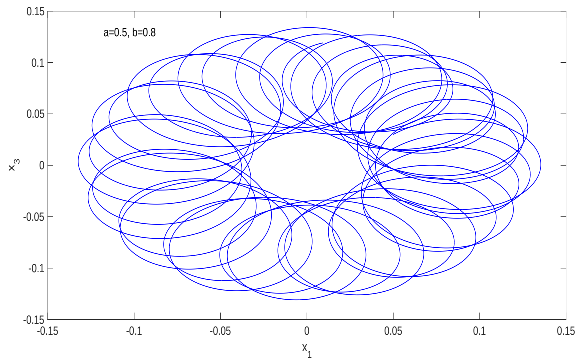
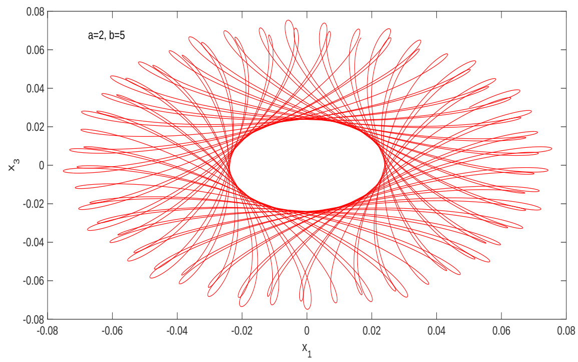
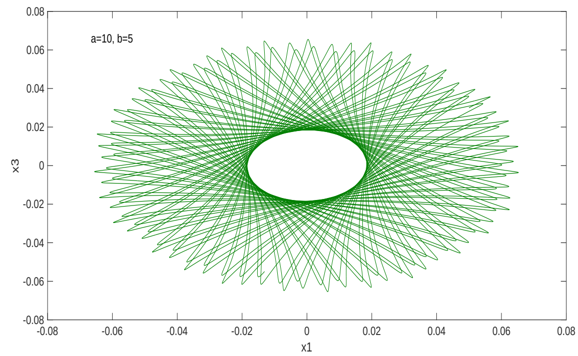
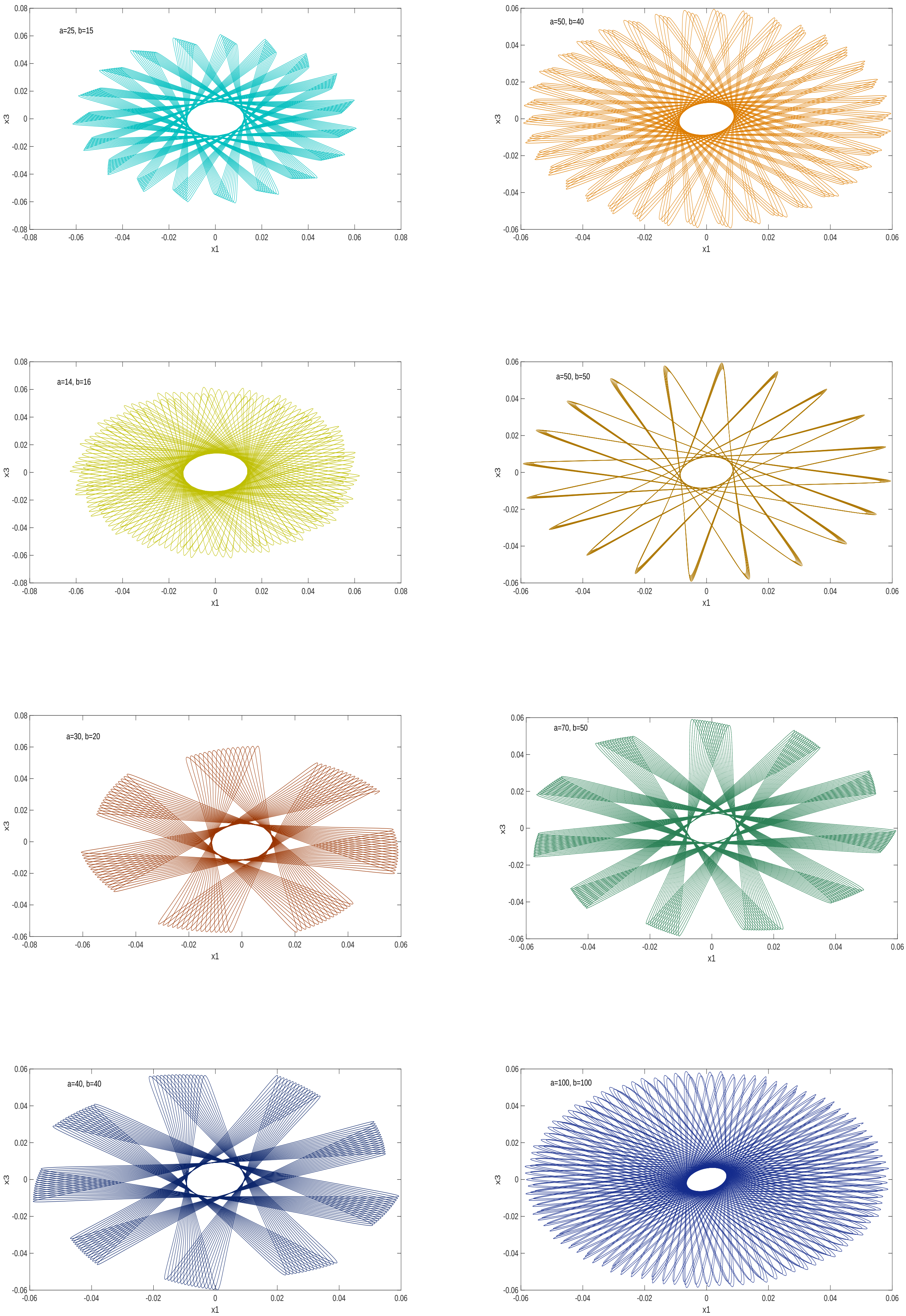
APPENDIX





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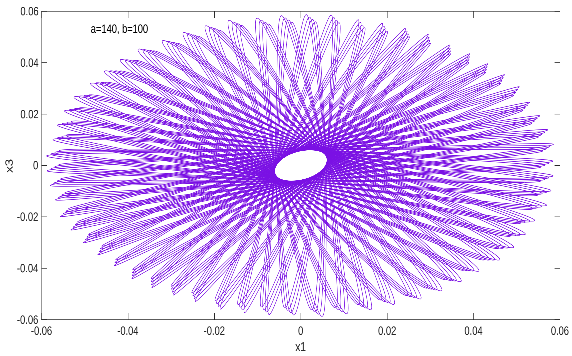
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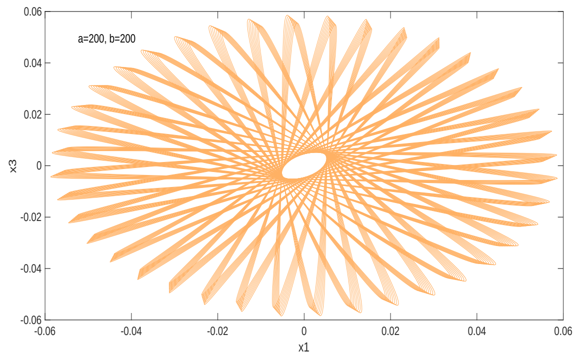
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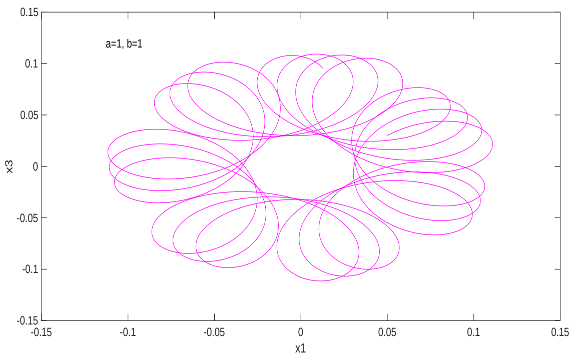
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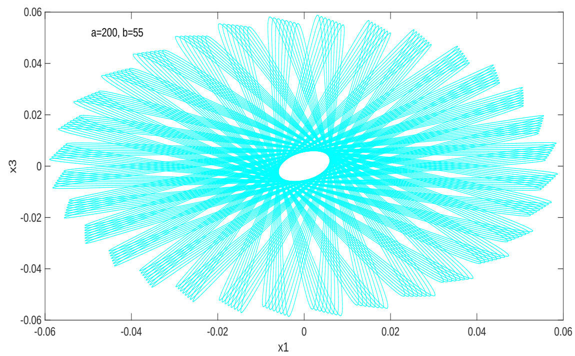
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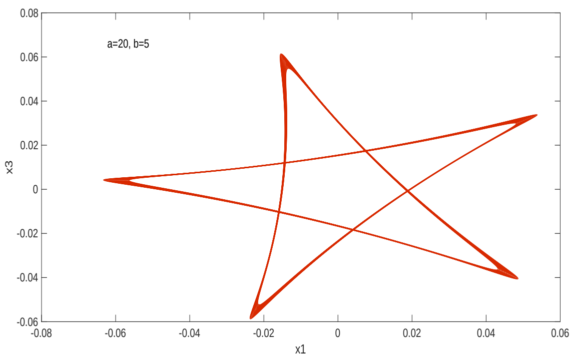
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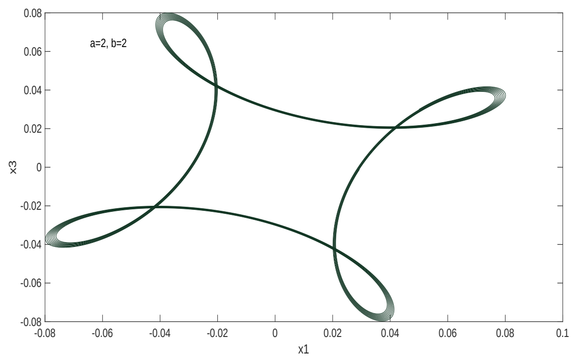
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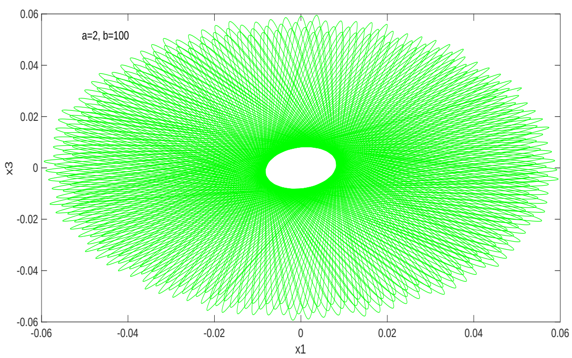
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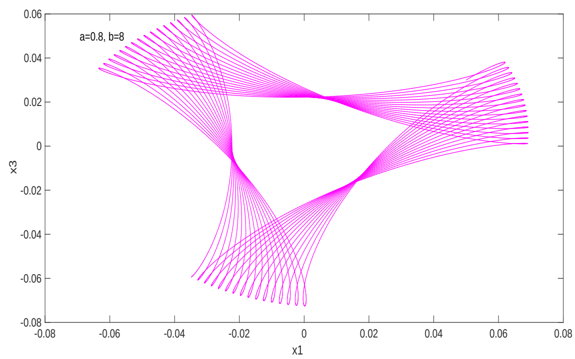
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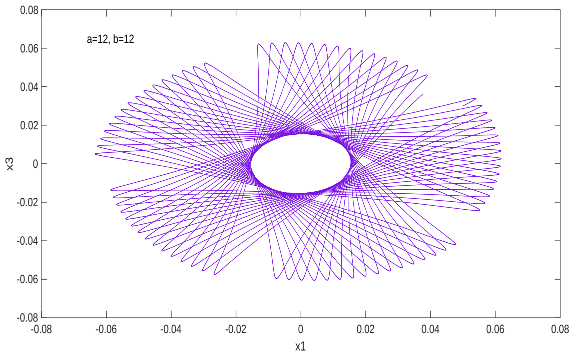

Dr. P. Muthukumar is an Assistant Professor (Senior Grade) in the PG & Research Department of Mathematics at Gobi Arts & Science College, Bharathiar University, Coimbatore, Tamil Nadu, India. He received his M.Sc. in Mathematics from Madurai Kamaraj University, India, in 2006, the M.Phil. degree in Mathematics from Gandhigram Rural Institute-Deemed University, India in 2007. Subsequently, he served as a Lecturer in Mathematics for four years. He received his Ph.D. degree in Mathematics from Gandhigram Rural Institute-Deemed University, Gandhigram, India in 2015. He was a Postdoctoral Fellow (PDF) at the Institute of Mathematical Sciences, University of Malaya, Malaysia during 2015 to 2016. He is a member of several academic associations, such as the International Association of Engineers (IAENG) in Hong Kong, China, and the Cryptology Research Society of India (CRSI) in Kolkata, India. Also, he serves as a member of the editorial board and technical reviewer in preferred SCI journals. He has published more than 20 original research articles with high impact factor tier-1 journals indexed by Scopus, SCI, and Web of Science. His important research areas are stability analysis, dynamical systems, fractional calculus, chaos, synchronization, number theory and cryptography.

Mr. J. Murugan is an Assistant Professor (Senior Grade) in the PG & Research Department of Mathematics at Gobi Arts & Science College, Bharathiar University, Coimbatore, Tamil Nadu, India. He received his M.Sc. degree in Mathematics from the Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai, Tamil Nadu, India, in 2007, the M.Phil. degree in Mathematics from the School of Mathematics & Statistics, Pondicherry University, India, in 2008. He qualified for the National Eligibility Test and State Eligibility Test in 2017 and is currently pursuing Ph.D in Bharathiar University under the supervision of Dr. P. Muthukumar. He has teaching experience of more than 10 years in the field of Mathematics and a research interest in the fields of dynamical systems, control systems, and Chaos synchronization.