Double Bifurcation and Stability Analysis in a Predator-prey Ecosystem Incorporating Dinosaur Functional Responses and Spatial Diffusion

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Abstract—**This research focuses on a spatially diffusive predator-prey model incorporating the Dinosaur-type functional response, subject to homogeneous Dirichlet boundary constraints. The primary aim is to explore the emergence of nontrivial, small-amplitude positive solutions branching from the zero equilibrium and to assess their long-term stability. To substantiate the theoretical findings, computational experiments are performed. The theoretical framework is established using LyapunovCSchmidt decomposition, the application of the implicit function theorem, and linear approximation methods.**

Index Terms—**Predator-prey model, Dinosaur functional response, Double bifurcation, Stability.**

I. Introduction

ONE of the central topics in mathematical ecology [1] is the study of the dynamic interplay between predators and their prey. This field involves modeling and analyzing population fluctuations of predator and prey species over time within ecological systems. Mathematical models serve as powerful tools for investigating the complexities of predator-prey interactions and their implications for ecosystem stability and dynamics. Over the past three decades, both mathematicians and biologists have increasingly explored the intricate dynamics of these systems [2]–[7], reflecting a growing awareness of the fundamental role predator-prey relationships play in shaping ecological communities and maintaining ecosystem stability. In 1995, Hsu and Huang [8] proposed the following predatorCprey model:

$$\begin{cases} \dot{x} = x\left(r_1 - \dfrac{x}{K_1}\right) - b\,\varphi(x)\,y, \\ \dot{y} = y\left(r_2 - \dfrac{y}{K_2}\right) + c\,\varphi(x)\,y, \\ x(0) > 0, \quad y(0) > 0, \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ denote the densities of prey and predator populations, respectively. The parameters r_1, r_2, K_1, K_2, b, and c are strictly positive constants. Here, K_1 and K_2 represent the environmental carrying capacities for the prey and predator species, while r_1 and r_2 correspond to their intrinsic growth rates. The function $\varphi(x)$ captures the predator's functional responsełthat is, the per capita rate of prey consumption. The coefficient b reflects the predation pressure on prey, and c indicates the extent to which predation contributes to predator reproduction or survival, including its natural mortality adjustment.

With the progression of mathematical ecology, a wide range of ecologically relevant functional responses have been introduced, such as the Holling Types I-IV, the Beddington-DeAngelis (B-D) formulation [10], the Ivlev model [11], the Crowley-Martin (C-M) type [12], and the Hassell-Varley response [13]. The Dinosaur functional response, considered an improvement or simplification of the Ivlev-type response [14], provides a refined model for describing changes in prey population density. Unlike the Ivlev-type functional response, $(1 - e^{-kx})$, the Dinosaur reaction term more effectively captures prey behavior by accounting for their defensive adaptations when their population density exceeds a certain threshold. This adaptive strategy enhances their ability to evade or camouflage themselves. By incorporating the Dinosaur functional response, defined as $\phi(x) = xe^{-kx}$ [14], [15], system (1) can thus be reformulated as follows:

$$\begin{cases} \dot{x} = x(r_1 - \frac{x}{K_1}) - bxe^{-kx}y, \\ \dot{y} = y(r_2 - \frac{y}{K_2}) + cxe^{-kx}y, \\ x(0) > 0, y(0) > 0 \end{cases} \tag{2}$$

Taking spatial heterogeneity into consideration, the reaction-Cdiffusion counterpart of system (2) can be expressed as

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + u\left(a - u - bve^{-ku}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v + v\left(c - v + due^{-ku}\right), & x \in \Omega,\ t > 0, \\ u = v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \not\equiv 0, & x \in \Omega, \end{cases} \tag{3}$$

where $u(t,x)$ and $v(t,x)$ represent the spatiotemporal densities of prey and predator populations, respectively. The operator Δ denotes the Laplacian acting on the spatial domain Ω, a bounded region in Euclidean space with smooth boundary $\partial\Omega$. The coefficients a, b, c, d, and k are all strictly positive constants. Analyzing steady states in such reactionCdiffusion models has become a central theme in contemporary mathematical ecology. In the present work, we focus on the equilibrium solutions associated with system (3), which are governed by the following elliptic system:

$$\begin{cases} -\Delta u = u(a - u - bve^{-ku}) & x \in \Omega, \\ -\Delta v = v(c - v + due^{-ku}) & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

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Building upon the foundational work of Feng et al. [14], who utilized bifurcation and perturbation methods to analyze single bifurcations emerging near semi-trivial steady states, the present study turns its attention to the existence and stability of double bifurcation phenomena in the vicinity of the trivial (zero) equilibrium. The contributions of Feng et al. [14] concerning the existence, uniqueness, multiplicity, and stability of single bifurcating branches form the theoretical underpinning for the current investigation.

Departing from conventional approaches, this study investigates the existence and asymptotic stability of small-amplitude positive solution branches bifurcating from the trivial equilibrium of system (4). By leveraging advanced analytical tools such as LyapunovCSchmidt reduction, the implicit function theorem, and linearization techniques, we provide novel insights into population persistence and stability. The results indicate that once a critical parameter threshold is surpassed, further parameter enhancement contributes to the stabilization of population levels. These findings enhance the theoretical framework of predatorCprey interactions and introduce versatile analytical methods that can be extended to a wide range of ecological models, offering a foundation for continued exploration in spatially structured population dynamics.

The organization of this paper is as follows: Section II provides the essential background to contextualize the research motivation and methodological innovations of the present work. Sections III and IV are devoted to analyzing the existence and asymptotic stability of double bifurcation phenomena in system (4). Section V highlights the principal outcomes of numerical simulations. Finally, Section VI summarizes the main conclusions and discusses potential directions for future research.

II. Prior estimates of positive solutions

Now, we establish certain fundamental principles to facilitate the interpretation of the research's motivation and innovation in this paper. Let $E = \phi \in C^{2,\alpha}(\bar{\Omega}, \mathbb{R}) : \phi|_{\partial\Omega} = 0$, where E denotes a Banach space. Assume that $q \in C^{\alpha}(\bar{\Omega}; \mathbb{R})$ with $0 < \alpha < 1$. Consider the following eigenvalue problem:

$$-\Delta\varphi + q(x)\varphi = \lambda\varphi, \quad x \in \Omega, \quad \varphi = 0, \quad x \in \partial\Omega.$$

Let $\lambda_1(q)$ denote the principal (smallest) eigenvalue associated with this problem. Then $\lambda_1(q)$ depends continuously on the function q, is simple (i.e., algebraic multiplicity one), and its corresponding eigenfunctions do not change sign in Ω. Furthermore, if $q_1 \le q_2$ with $q_1 \not\equiv q_2$, then it holds that $\lambda_1(q_1) < \lambda_1(q_2)$. For convenience, we write $\lambda_1 := \lambda_1(0)$, and denote by φ_1 the eigenfunction corresponding to λ_1, normalized so that $\int_\Omega \varphi_1^2\,dx = 1$. It is known that φ_1 is strictly positive throughout Ω.

Consider the equation

$$(-\Delta + q)w = \Lambda w - w^2, \quad x \in \Omega, \quad w|_{\partial\Omega} = 0, \tag{5}$$

where $\Lambda \in \mathbb{R}^+$ and $q \in C^{\alpha}(\bar{\Omega}; \mathbb{R})$. A solution w is called a positive solution to equation (5) if $w > 0$ in Ω and $w|_{\partial\Omega} = 0$.

Lemma 1. *[16] If $\Lambda > \lambda_1(q)$, then system (5) possesses a unique positive solution, denoted as $\theta_\Lambda(q)$. Conversely, if $\Lambda \le \lambda_1(q)$, then $\theta_\Lambda(q) = 0$; in the case where $q = 0$, let $\theta_\Lambda(0) = \theta_\Lambda$. For a fixed q, $\theta_\Lambda(q)$ exhibits continuity and strict monotonicity throughout $(\lambda_1(q), \infty)$ with respect to Λ. Furthermore, $\theta_\Lambda(q)$ is globally asymptotically stable.*

Theorem 2. *If (u, v) is a positive solution to system (4), then:*

1. *(a, c) satisfies $a > \lambda_1$ and $c > \lambda_1 - \frac{d}{ek}$;*
2. *(u, v) satisfies $0 < u \le \theta_a < a$ and $0 \le \theta_c \le v \le \theta_{c+\frac{d}{ek}} < c + \frac{d}{ek}$, for any $x \in \Omega$.*

Proof: Let (u, v) be a positive solution to system (4). By considering the first equation of system (4) and Lemma 1, we obtain that

$$a = \lambda_1(u + bve^{-ku}) > \lambda_1(0) = \lambda_1. \tag{6}$$

Similarly, considering the second equation of system (4) and Lemma 1, we deduce that

$$c = \lambda_1(v - due^{-ku}).$$

Let $h(u) = ue^{-ku}$. It is readily observed that $h'(u) > 0$ holds for $0 \le u < 1/k$, while $h'(u) < 0$ for $u > 1/k$, indicating that the function $h(u)$ achieves its global maximum at $u = 1/k$. Moreover, we have $h(1/k) = \frac{1}{ek}$. Thus, it follows that

$$c = \lambda_1\left(v - due^{-ku}\right) > \lambda_1\left(-\frac{d}{ek}\right) = \lambda_1 - \frac{d}{ek}. \tag{7}$$

Consider the following three equations:

$$-\Delta w_1 = aw_1 - {w_1}^2,\ x \in \Omega, \quad w_1|_{\partial\Omega} = 0, \tag{8a}$$

$$-\Delta w_2 = cw_2 - {w_2}^2,\ x \in \Omega, \quad w_2|_{\partial\Omega} = 0, \tag{8b}$$

$$-\Delta w_2 = (c + \frac{d}{ek})w_2 - {w_2}^2,\ x \in \Omega, \quad w_2|_{\partial\Omega} = 0. \tag{8c}$$

Based on Lemma 1 and inequality (6), equation (8a) admits a unique strictly positive solution, denoted by θ_a. Likewise, in light of Lemma 1 and inequality (7), equation (8b) has a unique non-negative solution θ_c, while equation (8c) admits a unique positive solution $\theta_{c+\frac{d}{ek}}$.

From the first equation in system (4), we observe that

$$-\Delta u = u(a - u - bve^{-ku}) < u(a - u).$$

Utilizing the comparison principle for elliptic equations, it follows that $u \le \theta_a$. Examining the second equation of system (4), we find

$$v(c-v) \le -\Delta v = v\left(c - v + due^{-ku}\right) \le v\left(c + \frac{d}{ek} - v\right).$$

These inequalities imply that the solution v is bounded from above by $\theta_{c+\frac{d}{ek}}$ and from below by θ_c. Similarly, it holds that $\theta_c \le v \le \theta_{c+\frac{d}{ek}}$. Utilizing the maximum principle for elliptic equations, we conclude that $\theta_a < a$ and $\theta_{c+\frac{d}{ek}} < c + \frac{d}{ek}$. This completes the proof. ∎

III. Existence of double bifurcation

In various fields, the concept of double bifurcation is pivotal as systems transition from a singular state to a bifurcation that yields two states, potentially leading to increased complexity and resulting in diverse outcomes such as stability, oscillation, or chaos [17], [18]. This phenomenon plays a crucial role in understanding and predicting system behaviors.

This section is devoted to analyzing the existence of positive solutions to system (4) that bifurcate from the trivial solution. The analysis is conducted via the Lyapunov-Schmidt reduction method, following the approaches outlined in [9], [19].

Assume $p > N$. Define the Banach spaces X and Y:

$$X = [W^{2,p} \cap W_0^{1,p}]^2, \quad Y = [L^p]^2.$$

Next, define the operator $F : R \times R \times X \to Y$:

$$F(a, c, u, v) = \left(\begin{array}{c} -\Delta u - u(a-u) + buve^{-ku} \\ -\Delta v - v(c-v) - duve^{-ku} \end{array} \right).$$

Then the system (4) is equivalent to the nonlinear equation

$$F(a, c, u, v) = 0. \tag{9}$$

Let

$$L(u, v) = \left(\begin{array}{c} -\Delta u - \lambda_1 u \\ -\Delta v - \lambda_1 v \end{array} \right),$$

$$G(a, c, u, v) = \left(\begin{array}{c} (\lambda_1 - a)u \\ (\lambda_1 - c)v \end{array} \right)$$

and

$$H(u, v) = \left(\begin{array}{c} u^2 + buve^{-ku} \\ v^2 - duve^{-ku} \end{array} \right).$$

Then equation (9) is transformed into

$$L(u, v) + G(a, c, u, v) + H(u, v) = 0. \tag{10}$$

Let $N(L)$ denote the null space, $R(L)$ the range, and L^* the adjoint of the linear operator L. It can be readily verified that:

$$N(L) = N(L^*) = \text{span}\left\{(\varphi_1, 0)^T, (0, \varphi_1)^T\right\}.$$

Therefore, L is a Fredholm operator, enabling the decomposition of the spaces X and Y into distinct subspaces.

$$X = X_1 \oplus X_2, \quad Y = Y_1 \oplus Y_2, \tag{11}$$

where $X_1 = Y_1 = N(L), Y_2 = R(L)$. Define projection operators $P : Y \to Y_1$ and $Q \doteq I - P : Y \to Y_2$, where

$$P(U) = (\int_\Omega \varphi_1 u dx, \int_\Omega \varphi_1 v dx)^T \varphi_1, \forall\, U \doteq (u, v) \in Y.$$

Thus, the system denoted by (4) is effectively transformable into an equivalent nonlinear system of equations:

$$\left\{ \begin{array}{l} PF(a, c, u, v) = 0, \\ QF(a, c, u, v) = 0. \end{array} \right. \tag{12}$$

Given that $(u, v) \in X$, and in light of the space decomposition articulated in (11), the components u and v can be represented as follows:

$$u = \alpha(\varphi_1 + \phi),\ v = \beta(\varphi_1 + \psi),$$

where $\alpha,\ \beta \in R, (\phi, \psi) \in X_2$. Therefore,

$$\begin{aligned} F(a, c, u, v) = {} & L(\alpha(\varphi_1 + \phi), \beta(\varphi_1 + \psi)) \\ & + G(a, c, \alpha(\varphi_1 + \phi), \beta(\varphi_1 + \psi)) \\ & + H(\alpha(\varphi_1 + \phi), \beta(\varphi_1 + \psi)). \end{aligned}$$

Denote $K(\phi, \psi; a, c, \alpha, \beta) \doteq QF(a, c, u, v)$, then

$$\begin{aligned} K(\phi, \psi; a, c, \alpha, \beta) \doteq {} & L(\alpha(\varphi_1 + \phi), \beta(\varphi_1 + \psi)) \\ & + QG(a, c, \alpha(\varphi_1 + \phi), \beta(\varphi_1 + \psi)) \\ & + QH(\alpha(\varphi_1 + \phi), \beta(\varphi_1 + \psi)). \end{aligned}$$

Therefore, $QF(a, c, u, v) = 0$, as detailed in (12), is equivalent to the nonlinear equation:

$$K(\phi, \psi; a, c, \alpha, \beta) = 0. \tag{13}$$

Obviously $K(0, 0; \lambda_1, \lambda_1, 0, 0) = 0$. Now we calculate the product of the matrix $\left(\begin{array}{cc} 1/\alpha & 0 \\ 0 & 1/\beta \end{array} \right)$ and the Frechet derivative of $K(\phi, \psi; a, c, \alpha, \beta)$. Simple calculations yield that

$$\begin{aligned} & \left(\begin{array}{cc} 1/\alpha & 0 \\ 0 & 1/\beta \end{array} \right) K_{(\phi,\psi)}(\phi, \psi; a, c, \alpha, \beta) \\ & = L + \left(\begin{array}{cc} A_1 & A_2 \\ A_3 & A_4 \end{array} \right) + \left(\begin{array}{cc} B_1 & B_2 \\ B_3 & B_4 \end{array} \right), \end{aligned}$$

where $A_2 = A_3 = 0$ and

$$A_1 = (\lambda_1 - a)[1 - \varphi_1 \textstyle\int_\Omega \varphi_1 \bullet dx],$$

$$A_4 = (\lambda_1 - c)[1 - \varphi_1 \textstyle\int_\Omega \varphi_1 \bullet dx],$$

$$\begin{aligned} B_1 = {} & 2\alpha\phi + b\beta(\varphi_1 + \psi)[1 - k\alpha(\varphi_1 + \phi)]e^{-k\alpha(\varphi_1+\phi)} \\ & - \beta\beta\varphi_1 \textstyle\int_\Omega \varphi_1 [1 - k\alpha(\varphi_1 + \phi)]e^{-k\alpha(\varphi_1+\phi)} dx, \end{aligned}$$

$$\begin{aligned} B_2 = {} & b\beta(\varphi_1 + \phi)e^{-k\alpha(\varphi_1+\phi)} \\ & - b\beta\varphi_1 \textstyle\int_\Omega \varphi_1 (\varphi_1 + \phi)e^{-k\alpha(\varphi_1+\phi)} dx, \end{aligned}$$

$$\begin{aligned} B_3 = {} & -d\alpha(\varphi_1 + \psi)[1 - k\alpha(\varphi_1 + \phi)]e^{-k\alpha(\varphi_1+\phi)} \\ & - d\alpha\varphi_1 \textstyle\int_\Omega \varphi_1 [1 - k\alpha(\varphi_1 + \phi)]e^{-k\alpha(\varphi_1+\phi)} dx, \end{aligned}$$

$$\begin{aligned} B_4 = {} & 2\beta\psi - d\alpha(\varphi_1 + \phi)e^{-k\alpha(\varphi_1+\phi)} \\ & - d\alpha\varphi_1 \textstyle\int_\Omega \varphi_1 (\varphi_1 + \phi)e^{-k\alpha(\varphi_1+\phi)} dx. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left(\begin{array}{cc} 1/\alpha & 0 \\ 0 & 1/\beta \end{array} \right) K_{(\phi,\psi)}(0, 0; \lambda_1, \lambda_1, 0, 0) \\ & = \left(\begin{array}{cc} -\Delta - \lambda_1 & 0 \\ 0 & -\Delta - \lambda_1 \end{array} \right) = L. \end{aligned}$$

It is clear that the mapping $L : X_2 \to Y_2$ defines a homeomorphism. Applying the implicit function theorem, we confirm the existence of a constant $\delta_0 > 0$ and a continuously differentiable function

$$\bar{\phi} \doteq \bar{\phi}(a, c, \alpha, \beta), \bar{\psi} \doteq \bar{\psi}(a, c, \alpha, \beta)$$

such that when $|a - \lambda_1|, |c - \lambda_1|, |\alpha|, |\beta| < \delta_0$, they satisfy equation (13), and

$$\bar{\phi}(\lambda_1, \lambda_1, 0, 0) = 0, \quad \bar{\psi}(\lambda_1, \lambda_1, 0, 0) = 0.$$

Assume $|a - \lambda_1|, |c - \lambda_1|, |\alpha|, |\beta| < \delta_0$ and let

$$\bar{u} = \alpha(\varphi_1 + \bar{\phi}), \bar{v} = \beta(\varphi_1 + \bar{\psi}).$$

From the definition of the projection operator P, we derive the expression for $T(a, c, \alpha, \beta)$ as given by (14). Obviously $T(\lambda_1, \lambda_1, 0, 0) = 0$. Similarly, (15) holds, where

$$C_1 = \textstyle\int_\Omega \varphi_1 [-\varphi_1 - \bar{\phi} + (\lambda_1 - a)\bar{\phi}_a] dx,$$

$$C_2 = (\lambda_1 - a) \textstyle\int_\Omega \varphi_1 \bar{\phi}_c dx,$$

$$C_3 = (\lambda_1 - c) \textstyle\int_\Omega \varphi_1 \bar{\psi}_a dx,$$

$$C_4 = \textstyle\int_\Omega \varphi_1 [-\varphi_1 - \bar{\psi} + (\lambda_1 - c)\bar{\psi}_c] dx,$$

$$\begin{aligned} D_1 = {} & \textstyle\int_\Omega \varphi_1 \{2\alpha(\varphi_1 + \bar{\phi})\bar{\phi}_a \\ & + b\beta\bar{\phi}_a e^{-k\alpha(\varphi_1+\bar{\phi})}[1 - k\alpha(\varphi_1 + \bar{\phi})](\varphi_1 + \bar{\psi}) \\ & + b\beta(\varphi_1 + \bar{\phi})e^{-k\alpha(\varphi_1+\bar{\phi})}\bar{\psi}_a\} dx, \end{aligned}$$

$$D_2 = \int_\Omega \varphi_1\{2\alpha(\varphi_1+\bar{\phi})\bar{\phi}_c + b\beta\bar{\phi}_c e^{-k\alpha(\varphi_1+\bar{\phi})}[1-k\alpha(\varphi_1+\bar{\phi})](\varphi_1+\bar{\psi}) + b\beta(\varphi_1+\bar{\phi})e^{-k\alpha(\varphi_1+\bar{\phi})}\bar{\phi}_c\}dx,$$

$$D_3 = \int_\Omega \varphi_1\{2\beta(\varphi_1+\bar{\psi})\bar{\psi}_a - d\alpha\bar{\phi}_a e^{-k\alpha(\varphi_1+\bar{\phi})}[1-k\alpha(\varphi_1+\bar{\phi})](\varphi_1+\bar{\psi}) - d\alpha(\varphi_1+\bar{\phi})e^{-k\alpha(\varphi_1+\bar{\phi})}\bar{\psi}_a\}dx,$$

$$D_4 = \int_\Omega \varphi_1\{2\beta(\varphi_1+\bar{\psi})\bar{\psi}_c - d\alpha\bar{\phi}_c e^{-k\alpha(\varphi_1+\bar{\phi})}[1-k\alpha(\varphi_1+\bar{\phi})](\varphi_1+\bar{\psi}) - d\alpha(\varphi_1+\bar{\phi})e^{-k\alpha(\varphi_1+\bar{\phi})}\bar{\psi}_c\}dx.$$

Hence,

$$\begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/\beta \end{pmatrix} T_{(a,c)}(\lambda_1,\lambda_1;0,0) = \begin{pmatrix} -\varphi_1 & 0 \\ 0 & -\varphi_1 \end{pmatrix}.$$

By applying the implicit function theorem again, it is established that there exists a constant $\delta_1 > 0$ and continuously differentiable functions $\bar{a} \doteq \bar{a}(\alpha,\beta), \bar{c} \doteq \bar{c}(\alpha,\beta)$ such that $PF(\bar{a},\bar{c},\alpha(\varphi_1+\bar{\phi}(\bar{a},\bar{c},\alpha,\beta)),\beta(\varphi_1+\bar{\psi}(\bar{a},\bar{c},\alpha,\beta))) = 0$ and $\bar{a}(0,0)=\lambda_1, \bar{c}(0,0)=\lambda_1$. According to Theorem 2, for fixed values of d, e and k, when $a, c+\frac{d}{ek} > \lambda_1, 0<\alpha,\beta<\delta_1$, the expression in (16) represents a positive bifurcation solution near the zero solution of the system outlined in (4). Therefore, the main theorem of this paper can be stated as follows:

Theorem 3. *The system described in (4) displays positive bifurcation solutions in the vicinity of the zero solution. Additionally, for appropriately small positive constants α and β, the bifurcation solutions can be expressed as*

$$u(\alpha,\beta) = \alpha(\varphi_1+\bar{\phi}(\bar{a}(\alpha,\beta),\bar{c}(\alpha,\beta),\alpha,\beta)),$$
$$v(\alpha,\beta) = \beta(\varphi_1+\bar{\psi}(\bar{a}(\alpha,\beta),\bar{c}(\alpha,\beta),\alpha,\beta)).$$

Moreover, $\bar{\phi}(\lambda_1,\lambda_1,0,0)=0$ and $\bar{\psi}(\lambda_1,\lambda_1,0,0)=0$.

Corollary 4. *For sufficiently small positive values of α and β, the linear approximation of the parameters $\bar{a}(\alpha,\beta)$ and $\bar{c}(\alpha,\beta)$, as described in Theorem 3, is given by:*

$$\bar{a}(\alpha,\beta) = \lambda_1 + (\alpha+b\beta)\int_\Omega \varphi_1^3 dx + o(|\alpha|,|\beta|),$$
$$\bar{c}(\alpha,\beta) = \lambda_1 + (\beta-d\alpha)\int_\Omega \varphi_1^3 dx + o(|\alpha|,|\beta|).$$

Proof: Let

$$\bar{\phi} = \bar{\phi}(\alpha,\beta) = \bar{\phi}(\bar{a}(\alpha,\beta),\bar{c}(\alpha,\beta),\alpha,\beta),$$
$$\bar{\psi} = \bar{\psi}(\alpha,\beta) = \bar{\psi}(\bar{a}(\alpha,\beta),\bar{c}(\alpha,\beta),\alpha,\beta).$$

The positive bifurcation solution to the system (4) near the zero solution can be expressed as:

$$\bar{u}(\alpha,\beta) = \alpha(\varphi_1+\bar{\phi}(\alpha,\beta)), \quad \bar{v}(\alpha,\beta) = \alpha(\varphi_1+\bar{\psi}(\alpha,\beta)).$$

Substituting these expressions into the second equation of system (4) results in:

$$-\Delta(\varphi_1+\bar{\psi}) = (\varphi_1+\bar{\psi})[\bar{c}-\beta(\varphi_1+\bar{\psi})] + d\alpha(\varphi_1+\bar{\phi})(\varphi_1+\bar{\psi})e^{-k\alpha(\varphi+\bar{\phi})}. \tag{17}$$

By differentiating both sides of equation (17) with respect to α and β at the point $(\alpha,\beta)=(0,0)$, we obtain

$$(\Delta+\lambda_1)\frac{\partial\psi(0,0)}{\partial\alpha} + \varphi_1\left(\frac{\partial\bar{c}(0,0)}{\partial\alpha} + d\varphi_1\right) = 0, \tag{18a}$$

$$(\Delta+\lambda_1)\frac{\partial\psi(0,0)}{\partial\beta} + \varphi_1\left(\frac{\partial\bar{c}(0,0)}{\partial\beta} - \varphi_1\right) = 0. \tag{18b}$$

Multiplying both sides of equations (18a) and (18b) by φ_1, integrating over the domain Ω, and applying Green's formula, we obtain

$$\frac{\partial\bar{c}(0,0)}{\partial\alpha} = -d\int_\Omega \varphi_1^3 dx < 0 \quad \frac{\partial\bar{c}(0,0)}{\partial\beta} = \int_\Omega \varphi_1^3 dx > 0. \tag{19}$$

Similarly,

$$\frac{\partial\bar{a}(0,0)}{\partial\alpha} = \int_\Omega \varphi_1^3 dx > 0 \quad \frac{\partial\bar{a}(0,0)}{\partial\beta} = b\int_\Omega \varphi_1^3 dx > 0. \tag{20}$$

Based on equations (19) and (20), for sufficiently small positive values of α and β,

$$\bar{a}(\alpha,\beta) = \lambda_1 + (\alpha+b\beta)\int_\Omega \varphi_1^3 dx + o(|\alpha|,|\beta|), \tag{21a}$$

$$\bar{c}(\alpha,\beta) = \lambda_1 + (\beta-d\alpha)\int_\Omega \varphi_1^3 dx + o(|\alpha|,|\beta|). \tag{21b}$$

The proof is complete. ■

IV. STABILITY OF DOUBLE BIFURCATION

Stability analysis is essential for evaluating the robustness of a system's equilibrium and its behavior under perturbations. It aims to predict whether the system will remain stable, oscillate, or undergo substantial changes [20]–[23].

This section focuses on discussing the stability of positive solutions resulting from bifurcations. In general, the stability of these solutions is assessed through the linearized system.

Assuming that $\alpha=\beta\equiv s$, for a sufficiently small positive s, the positive bifurcation solution originating from the zero solution, as discussed in Theorem 3, can be parameterized in terms of s, i.e.,

$$\bar{u}(s) = s\varphi_1 + o(s), \bar{v}(s) = s\varphi_1 + o(s). \tag{22}$$

The linearized system of equation (4) at $(\bar{u}(s),\bar{v}(s))$ is expressed as (23). Note that

$$L = \begin{pmatrix} -\Delta-\lambda_1 & 0 \\ 0 & -\Delta-\lambda_1 \end{pmatrix}$$

and define the matrices $R(s)$ and $Z(s)$ as given in (24) and (25), respectively:

$$R(s) = \begin{pmatrix} \lambda_1-\bar{a}(s) & 0 \\ 0 & \lambda_1-\bar{c}(s) \end{pmatrix} \tag{24}$$

Then, system (23) is transformed into the equation $A(s)U \doteq (L+R(s)+Z(s))U=0$, where $U=(u,v)^T$. The corresponding eigenvalue problem for system (23) is given by

$$\begin{gathered}(A(s)-\mu I)(y,z)^T = 0,\\ 0 \neq (y,z) \in D(A(s)) \subset X,\end{gathered} \tag{26}$$

where I denotes the identity operator. The stability of the bifurcated positive solution $(\bar{u}(s),\bar{v}(s))$ is determined by the locations of the eigenvalues associated with equation (26) in the complex plane. Specifically, if all eigenvalues lie in the right half of the complex plane, the solution is asymptotically stable. Conversely, if any eigenvalues lie in the left half of the complex plane, the solution is unstable.

Theorem 5. *For sufficiently small $s>0$, the positive bifurcation solution $(\underline{u}(s),\underline{v}(s))$ of the system in equation (4) is asymptotically stable.*

Proof: The eigenfunction (y, z) associated with the eigenvalue problem (26) takes the following form:

$$y = \varphi_1 + w_1, z = m\varphi_1 + w_2, \quad (27)$$

where m is a complex number, and $\langle \phi_1, w_i \rangle_2 = 0, i = 1, 2$. Let $\xi = (\varphi_1, 0)^T, \eta = (0, \varphi_1)^T, w = (w_1, w_2)^T \in X_2$. The eigenvalue problem (26) can then be reformulated as

$$(L + R(s) + Z(s))(\xi + m\eta + w) - \mu(\xi + m\eta + w) = 0. \quad (28)$$

Given that $\int_\Omega \phi_1^2 \, dx = 1$, we take the inner product of both sides of equation (28) with ξ and η, respectively, and obtain

$$\lambda_1 - \bar{a}(s) + \langle Z(s)(\xi + m\eta + w), \xi \rangle_2 - \mu = 0, \quad (29a)$$

$$m(\lambda_1 - \bar{c}(s)) + \langle Z(s)(\xi + m\eta + w), \eta \rangle_2 - m\mu = 0. \quad (29b)$$

By the definitions of the projection operators P and Q as introduced in Section III, we apply Q to both sides of equation (28) to obtain

$$Lw + R(s)w + QZ(s)(\xi + m\eta + w) - \mu w = 0. \quad (30)$$

By combining equations (29a) and (30), we define the operator $J : X_2 \times C \times C \times R \to Y_2 \times C$:

$$J(w, \mu; m, s) = \begin{pmatrix} Lw + R(s)w + QZ(s)(\xi + m\eta + w) - \mu w \\ \lambda_1 - \bar{a}(s) + \langle Z(s)(\xi + m\eta + w), \xi \rangle_2 - \mu \end{pmatrix}.$$

It is clear that $J(0, 0; m, 0) = 0$. Now, let us compute the Frechét derivative of $J_{(\omega,\mu)}(\omega, \mu; m, s)$ at $(0, 0; m, 0)$. Upon calculation, we obtain

$$J_{(\omega,\mu)}(\omega, \mu; m, s) = \begin{pmatrix} L + R(s) + QZ(s) - \mu I & -\omega \\ \bar{X}\,\bar{Y} & -I \end{pmatrix},$$

where $\bar{X} = \int_\Omega (2\,\bar{u} + b\,\bar{v}\,e^{-k\,\bar{u}} - kb\,\bar{v}\,\bar{u}\,e^{-k\,\bar{u}})\varphi_1 dx$ and $\bar{Y} = \int_\Omega b\,\bar{u}\,e^{-k\,\bar{u}}\varphi_1 dx$. Therefore,

$$J_{(\omega,\mu)}(0, 0; m, 0) = \begin{pmatrix} L & 0 \\ 0 & -I \end{pmatrix}.$$

It is clear that $J_{(\omega,\mu)}(0, 0; m, 0) : X_2 \times C \to Y_2 \times C$ is a homeomorphism. By the implicit function theorem, we deduce that for sufficiently small $s > 0$, there exists a continuously differentiable function $(\omega(m, s), \mu(m, s))$ such that $J(\omega(m, s), \mu(m, s); m, s) = 0$ with $\omega(m, 0) = 0$ and $\mu(m, 0) = 0$. Since $\bar{u}'(0) = \bar{v}'(0) = \varphi_1$, we have (31) and then (32a) and (32b) hold.

From (21a) and (21b), we know that

$$\lambda_1 - \bar{a}(s) = -s(1 + b)\int_\Omega {\varphi_1}^3 dx + o(|\alpha||\beta|), \quad (33a)$$

$$\lambda_1 - \bar{c}(s) = s(d - 1)\int_\Omega {\varphi_1}^3 dx + o(|\alpha||\beta|). \quad (33b)$$

Let $\bar{u} = \frac{\mu}{s \int_\Omega {\varphi_1}^3 dx}$. From (29a), (32a) and (33a), we can deduce that

$$1 + mb - \bar{\mu} + o(1) = 0. \quad (34)$$

From (29b), (32b) and (33b), we have

$$(2 - b + \bar{\mu})m + k + 2d - 1 + o(1) = 0. \quad (35)$$

Moreover, it follows from (35) that $m = \frac{1-k-2d}{2-b+\bar{\mu}} + o(1)$. We can combine it with (34) to obtain a quadratic equation with respect to $\bar{\mu}$:

$$\bar{\mu}^2 + (b - 1)\,\bar{\mu} + (b(k - 2d) - 2) = 0. \quad (36)$$

It is easy to see that all roots of equation (36) have positive real parts. This completes the proof. ■

V. Numerical simulation

The primary objective is to present the results from numerical simulations that further validate the analytical results discussed earlier. These simulations are performed for the parabolic system defined in (3) within a one-dimensional spatial domain. For simplicity, we choose $\Omega = (0, 2\pi)$ and proceed to examine

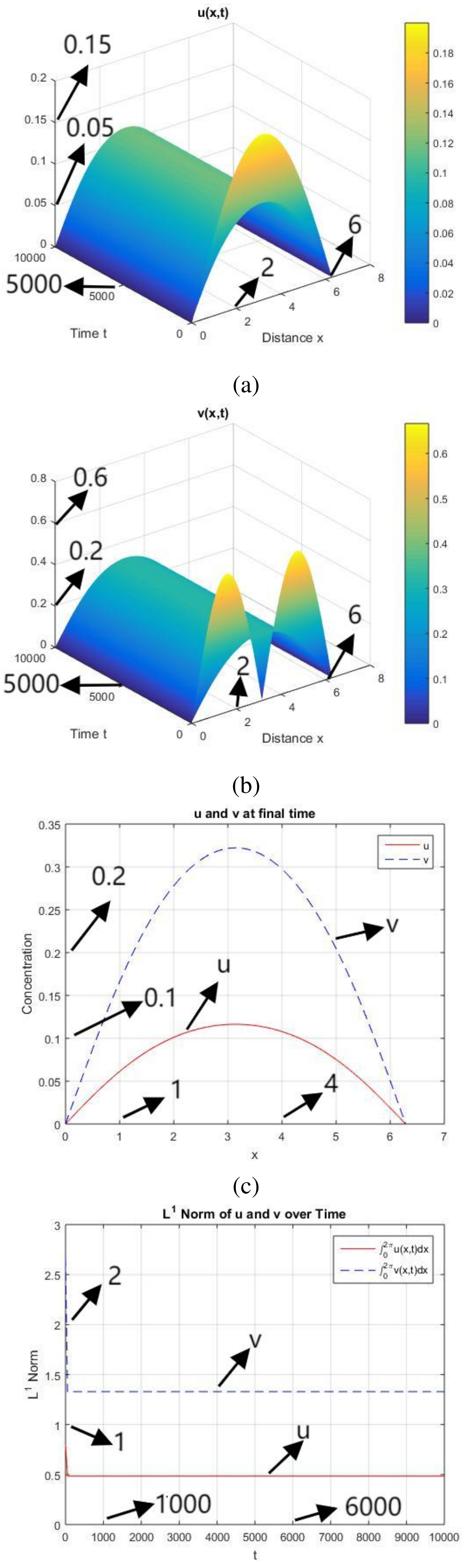
$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u(a - u - bve^{-ku}), x \in (0, 2\pi), t > 0, \\ \frac{\partial v}{\partial t} = \Delta v + v(c - v + due^{-ku}), x \in (0, 2\pi), t > 0, \\ u(0, t) = u(2\pi, t) = v(0, t) = v(2\pi, t) = 0, t > 0, \\ u(x, 0) = \frac{1}{5}\sin(\frac{1}{2}x), v(x, 0) = \frac{2}{3}|sin(\frac{1}{3}x)|, x \in (0, 2\pi). \end{cases} \quad (37)$$

The MATLAB function `pdepe` [22], [24] is well-established for solving initial-boundary value problems of parabolic-elliptic partial differential equations (PDEs) in one dimension. In this study, we carefully select appropriate parameters for equation (37) to conduct numerical simulations using the MATLAB environment and the pdepe function. The results are illustrated in figures captured at a significantly advanced time point (specifically, $T = 10000$), allowing us to consider the solution profiles as representative of steady states. For these simulations, we fix $a = 0.6, b = 1, c = 0.5$ and $d = 0.3$ while varying the other parameters k, all positive, to demonstrate diverse behaviors and outcomes of the system.

(1) Based on Corollary 4 and MATLAB with parameters $a = 0.6$, $b = 1$, $c = 0.5$, $d = 0.3$, and $k = 1$, we investigate the presence and stability of non-constant positive steady-state solution adjacent to the zero solution of system (37), which are depicted in Figure 1 (a) and (b). Figure 1(c) illustrates the $u - x$ projection of the parabolic system (37) onto a plane after a prolonged duration ($t = 10000$), evidencing the existence of positive solutions for the related elliptic system associated with (37). Furthermore, Figure 1(d) delineates the L^1 norm of the parabolic system (37) over the interval $t \in [0, 10000]$, showcasing the stability of the positive solutions for the analogous elliptic system to (37).

(2) Note that the Dinosaur functional response $\phi(x) = xe^{-kx}$. Let parameters $a = 0.6$, $b = 1$, $c = 0.5$, $d = 0.3$ and k vary. Figure 2 shows the the impact of parameter k on population densities u and v to system (37) with and $k = 1, 3, 5, 10, 30, 50$, respectively in Figure 2 (a)-(f).

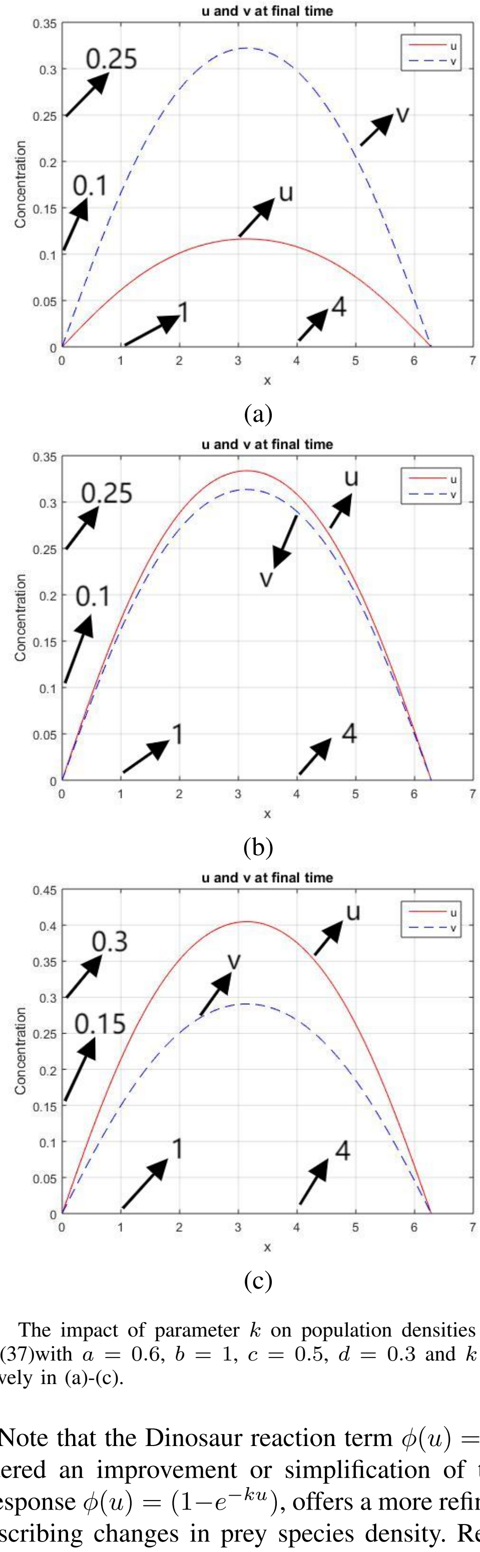
1) As the parameter k incrementally increase through the sequence $1, 3, 5$ and subsequently to 10, there is a continuous augmentation in the population density denoted by u. However, upon further escalation of the parameter k, the increment in the population density u becomes negligible.
2) Variations in the parameter k have an imperceptible impact on the population density v.
3) When the parameter k is minimal, the population density v surpasses that of u. Conversely, as k increases, the population density u exceeds v. The pivotal value for the parameter is established at $k = 5$ in Figure 2 (3).



(a) (b) (c) (d)

Fig. 1. Numerical simulation diagrams of population densities u and v to system (37) with $a = 0.6$, $b = 1$, $c = 0.5$, $d = 0.3$ and $k = 1$.

4) Upon reaching a specific threshold, such as $k = 10$ in Figure 2 (d), an increment in the parameter k results in the stabilization of population densities u and v, maintaining them at a constant level without further variation.



(a) (b) (c)

Fig. 2. The impact of parameter k on population densities u and v to system (37)with $a = 0.6$, $b = 1$, $c = 0.5$, $d = 0.3$ and $k = 1, 5, 30$, respectively in (a)-(c).

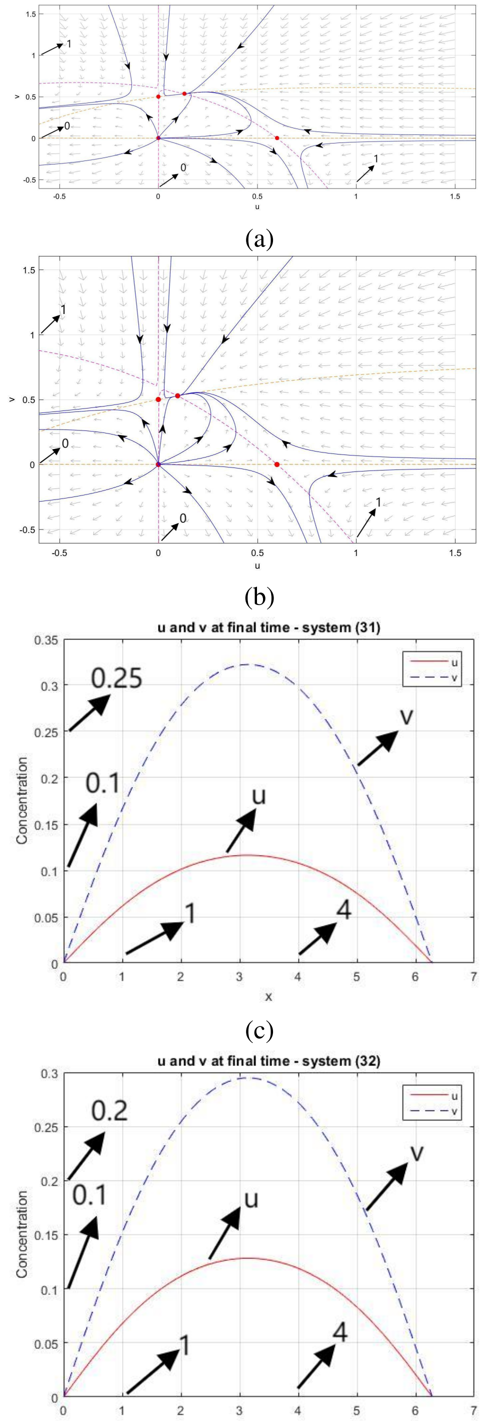
(3) Note that the Dinosaur reaction term $\phi(u) = (ue^{-ku})$, considered an improvement or simplification of the Ivlev-type response $\phi(u) = (1-e^{-ku})$, offers a more refined model for describing changes in prey species density. Replace the Dinosaur reaction term with the Ivlev-type response, and the system (37) becomes:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u(a-u) - b(1-e^{-ku})v, x \in (0, 2\pi), t > 0, \\ \frac{\partial v}{\partial t} = \Delta v + v[(c-v) + d(1-e^{-ku})], x \in (0, 2\pi), t > 0, \\ u(0,t) = u(2\pi,t) = v(0,t) = v(2\pi,t) = 0, t > 0, \\ u(x,0) = \frac{1}{5}\sin(\frac{1}{2}x), v(x,0) = \frac{2}{3}|sin(\frac{1}{3}x)|, x \in (0, 2\pi). \end{cases} \tag{38}$$

Here, we compare the dynamic behaviors between the Dinosaur reaction term in system (37) and the Ivlev-type response in system (38).

1) From a biological perspective, unlike the Ivlev-type functional response, the Dinosaur reaction term effectively captures prey behavior as they prioritize defenses against predators when their population density reaches a threshold. This adaptive strategy enhances their ability to evade or disguise themselves.
2) Figures 3 (a) and 3 (b) show that, the local time systems of (37) and (38) have the same constant equilibrium and exhibit identical properties.
3) Figures 3 (c) and 3(d) show that, both systems (37) and (38) exhibit spatially non-uniform positive equilibrium solutions near $(0, 0)$.

The results provide biologically meaningful insights into predator-prey dynamics with nonlinear functional responses [25]–[28]. The presence of double bifurcation phenomena reveals complex transitions between multiple stable states, indicating potential ecological resilience or abrupt regime shifts under specific conditions. The Dinosaur-type functional response introduces a delayed and nonlinear influence on predator growth, which may contribute to population stabilization at high prey densities. Accordingly, our study suggests that under suitable parameter regimes, both predator and prey populations can stabilize, thus maintaining ecological balance. These findings have potential implications for conservation strategies aimed at promoting long-term ecosystem stability.



(a) (b) (c) (d)

Fig. 3. Comparative analysis of dynamic behaviors: Dinosaur reaction term in (37) vs. Ivlev-type response in (38), which have the same parameters: $a = 0.6$, $b = 1$, $c = 0.5$, $d = 0.3$ and $k = 1$.

VI. Conclusions

This study explores a diffusive predator-prey model with a Dinosaur-type functional response under homogeneous Dirichlet boundary conditions. We investigate the existence of small positive solutions bifurcating from the trivial (zero) solution and analyze their asymptotic stability. To support our theoretical findings, numerical simulations are also performed. Key analytical techniques, such as Lyapunov-Schmidt reduction, the implicit function theorem, and linearization methods, are employed. These methods for examining bifurcating solution existence and stability in system (4) can be broadly applied to various mathematical ecological models.

Additionally, we revisit the key findings from item (2) 4) in Section V: Once the parameter k exceeds a specific threshold—e.g., $k = 10$ in Figure 2(d)—further increases in k lead to the stabilization of the population densities u and v, maintaining them at constant levels without further variation. This observation, derived from numerical simulations, can be regarded as a conjectural insight. In future work, we will investigate the theoretical underpinnings of this behavior in greater detail.

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$$T(a,c,\alpha,\beta) \doteq PF(a,c,\bar{u},\bar{v}) \\ = PG(a,c,\bar{u},\bar{v}) + PH(a,c,\bar{u},\bar{v}) \\ = \begin{pmatrix} \int_\Omega (\lambda_1 - a)\varphi_1 \bar{u} dx \\ \int_\Omega (\lambda_1 - c)\varphi_1 \bar{v} dx \end{pmatrix} \varphi_1 + \begin{pmatrix} \int_\Omega \varphi_1 (\bar{u}^2 + b\bar{u}\bar{v}e^{-k\bar{u}}) dx \\ \int_\Omega \varphi_1 (v^2 - d\bar{u}\bar{v}e^{-k\bar{u}}) dx \end{pmatrix} \varphi_1. \tag{14}$$

$$\begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix} T_{(a,c)}(a,c;\alpha,\beta) = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \varphi_1 + \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \varphi_1, \tag{15}$$

$$(\bar{a}(\alpha,\beta), \bar{c}(\alpha,\beta), \alpha(\varphi_1 + \bar{\phi}(\bar{a}(\alpha,\beta), \bar{c}(\alpha,\beta), \alpha,\beta)), \beta(\varphi_1 + \bar{\psi}(\bar{a}(\alpha,\beta), \bar{c}(\alpha,\beta), \alpha,\beta))) \tag{16}$$

$$\begin{cases} [-\Delta - a(s) + 2\,\bar{u}(s) + b\,\bar{v}(s)e^{-k\,\bar{u}(s)} - kb\,\bar{v}(s)\,\bar{u}(s)e^{-k\,\bar{u}(s)}]u + b\,\bar{u}(s)e^{-k\,\bar{u}(s)}v = 0, \\ [(k - \bar{u}(s))d\,\bar{v}(s)e^{-k\,\bar{u}(s)}]u + [-\Delta - \bar{c}(s) + 2\,\bar{v}(s) - b\,\bar{u}(s)e^{-k\,\bar{u}(s)}]v = 0. \end{cases} \tag{23}$$

$$Z(s) = \begin{pmatrix} 2\,\bar{u}(s) + b\,\bar{v}(s)e^{-k\,\bar{u}(s)} - kb\,\bar{v}(s)\,\bar{u}(s)e^{-k\,\bar{u}(s)} & b\,\bar{u}(s)e^{-k\,\bar{u}(s)} \\ (k - \bar{u}(s))d\,\bar{v}(s)e^{-k\,\bar{u}(s)} & 2\,\bar{v}(s) - b\,\bar{u}(s)e^{-k\,\bar{u}(s)} \end{pmatrix} \tag{25}$$

$$Z(s)(\xi + m\eta + w) \\ = \begin{pmatrix} 2\,\bar{u}(s) + b\,\bar{v}(s)e^{-k\,\bar{u}(s)} - kb\,\bar{v}(s)\,\bar{u}(s)e^{-k\,\bar{u}(s)} & b\,\bar{u}(s)e^{-k\,\bar{u}(s)} \\ (k - \bar{u}(s))d\,\bar{v}(s)e^{-k\,\bar{u}(s)} & 2\,\bar{v}(s) - b\,\bar{u}(s)e^{-k\,\bar{u}(s)} \end{pmatrix} \begin{pmatrix} \phi_1 + w_1 \\ m\phi_1 + w_2 \end{pmatrix} \\ = \begin{pmatrix} (2\,\bar{u}(s) + b\,\bar{v}(s)e^{-k\,\bar{u}(s)} - kb\,\bar{v}(s)\,\bar{u}(s)e^{-k\,\bar{u}(s)})(\phi_1 + w_1) + b\,\bar{u}(s)e^{-k\,\bar{u}(s)}(m\phi_1 + w_2) \\ (k - \bar{u}(s))d\,\bar{v}(s)e^{-k\,\bar{u}(s)}\phi_1 + w_1) + (2\,\bar{v}(s) - b\,\bar{u}(s)e^{-k\,\bar{u}(s)})(m\phi_1 + w_2) \end{pmatrix} \\ = \begin{pmatrix} s(2 + b + mb){\varphi_1}^2 + o(s) \\ s(2m - bm + kd){\varphi_1}^2 + o(s) \end{pmatrix}, \tag{31}$$

$$\langle Z(s)(\xi + m\eta + w), \xi \rangle_2 = s(2 + b + mb) \int_\Omega {\varphi_1}^3 dx + o(s), \tag{32a}$$

$$\langle Z(s)(\xi + m\eta + w), \eta \rangle_2 = s(2m - bm + kd) \int_\Omega {\varphi_1}^3 dx + o(s). \tag{32b}$$