Dynamic Analysis of the System with Multiple Failure Modes and Imperfect Repairs

Wurulnisa Abduwaki, Ehmet Kasim

Abstract—**In traditional repairable system reliability analyses, it is commonly assumed that system components possess only a single failure mode and can be restored to a new condition or replaced directly after repair. However, this is often not the case. Many large components may possess multiple failure modes and cannot be fully repaired to their original state. Therefore, this paper investigates repairable systems with multiple failure modes and incomplete repair characteristics. The well-posedness of the system and the asymptotic behavior of the time-dependent solutions are examined using the C_0-semigroup theory and the spectral theory of linear operators. That is, it is proven that there exists a unique positive time-dependent solution for the system that satisfies probabilistic properties, and this time-dependent solution converges exponentially to its steady-state solution.**

Index Terms—**Multiple Failure Modes, Imperfect Repair, Well-posedness, Asymptotic Behavior, Reliability Indices.**

I. Introduction

TRADITIONAL repairable system reliability analyses often assume that system components have only a single failure mode (SFM) and that failed components may be repaired and returned to a like-new condition or directly replaced. However, this is not always the case. Many large components can have multiple failure modes (MFMs) and cannot be completely repaired to a new condition. Some components, even when they fail, may not need to be replaced immediately but can continue to be used through multiple imperfect repairs (IRs), which do not restore them to a like-new condition and gradually reduce their usability with each repair until they are eventually no longer usable. This repair strategy can effectively reduce costs. Therefore, the study of repairable systems with MFMs and IRs characteristics is not only of important theoretical significance but also has significant application value.

Most engineering systems are known to be affected by MFM due to their complex structures and failure behaviors. Different failure modes have different effects on system failure behavior and repair strategies. As a result, the assumption of an SFM is no longer sufficient to deal with the complexity of engineering applications, and such simplifying assumptions may even lead to significant differences in system availability. In system reliability engineering, evaluating the impact of various failure modes on system availability is crucial. Dhillon [1] first introduced the concept of MFMs when he examined the availability of a two-component standby system. Since then, this research direction has received increasing attention. Chung [2] proposed a reliability model for a $k/n(G)$ system with M mutually exclusive failures and common cause failures. The repair times of the failed components follow an arbitrary distribution. He obtained the Laplace transform for the transient availability and the steady-state availability (SSA). Moustafa [3], [4] studied a $k/n(G)$ voting system with M failure modes, where both the distribution of component lifetimes and repair times follow an exponential distribution. He derived the system's SSA, reliability, and mean time to failure considering repair and no-repair scenarios. Subsequently, scholars such as Jain and Sharma [5], Nikolov [6], and Qiu and Cui [7] have also studied various repairable systems with MFM and derived some reliability indices, such as system availability.

Although scholars often assume that failed components in repairable systems can be completely repaired, this is not always the case. Some components cannot be restored to a new condition due to IRs, but they can be repaired multiple times until they become unusable. Subramanian and Natarajan [8] first investigated a 2-unit cold standby system with the "IR" property. They derived the transient availability and reliability of the system by assuming that each component could undergo k IRs, each with a different distribution of component failure times. Biswas and Sarkar [9] studied a repairable system that passes through k IRs before replacement or complete repair, under the condition that both the life and repair time distributions of the components obey an exponential distribution. They obtained the system availability using Fourier transforms and compared it with its availability under a complete repair policy. Hajeeh [10] investigated two types of repairable systems, both based on the assumption that the lifetime and repair time distributions of components follow an exponential distribution. In model 1, the system is replaced by a new system after n IRs; in model 2, the system is imperfectly repaired after a failure with probability p and replaced by a new system with probability $(1-p)$. He first finds the steady-state probability of the system and derives the SSA of the system. Subsequently, scholars such as Muhammad et al. [11], Hajeeh [12], and Madhu and Pratap [13] have also studied the quantitative reliability indicators for different repairable systems with IRs.

Since some components wear out and age over time, it is usually unrealistic to assume that they can be repaired to an "as good as new" condition. Barlow and Hunter [14] first proposed the "minimal repair model", which does not change the systems's lifetime. Brown and Proschan [15] investigated the reliability model with IR, where the probability that a system can be "repaired as new" is p, and the probability of a "minimal repair" is $(1-p)$. They put forward the following assumptions for the gradually deteriorating repairable

Manuscript received February 7, 2025; revised May 18, 2025.

This work was supported by the Natural Science Foundation of Xinjiang Uygur Autonomous Region (No.2022D01C46).

W. Abdiwaki is a postgraduate student of College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, P.R. China (e-mail: wurnisa@stu.xju.edu.cn).

E. Kasim is an associate professor of College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, P.R. China (Corresponding author to provide e-mail: ehmetkasim@xju.edu.cn).

system: After repair, the system can operate for increasingly shorter periods, while the consecutive repair times become longer and longer, until the system can no longer function and cannot be repaired again. Lin [16] introduced the geometric process to characterize this stochastic phenomenon. Since then, scholars such as Wang and Pham [17], Chen et al. [18], Dong et al. [19], Haugen et al. [20], and Brito et al. [21] have investigated the availability and other reliability indices of various repairable systems described by the geometric process under IR strategies.

With growing interest in repairable systems featuring multiple failure modes and imperfect repairs, researchers have increasingly explored systems combining these characteristics. Hajeeh [22] expanded on prior work [11] by incorporating assumptions from [3] to derive the steady-state availability of repairable systems with MFMs, assuming exponential distributions for both component lifetimes and repair times. Nikolov [23] extended the findings from the literature [22] to examine repairable systems with MFM and IRs. In this system, the lifetime distribution of components follows an exponential distribution, while the repair time follows a general distribution. Components may fail under one of $M+1$ different failure modes; if the failure is one of the first M types, the component can be imperfectly repaired, resulting in the system operating in a degraded state. If the $(M+1)$-th failure occurs, the component is irreparable and must be replaced with a new one, which will operate normally as if no failure had occurred. Due to IRs, the failure rate of the component gradually increases, while the repair rate decreases over time. Regardless of the type of failure, after n repairs, the component will be replaced with a new one. Under the assumption that the reliability model of the system has a unique time-dependent solution (TDS) that converges to the steady-state solution (SSS), Nikolov established the mathematical model of the system, used the supplementary variable method to develop and derived the Laplace transforms of the system's SSA and instantaneous availabilities. Qiu et al. [24] studied single-component systems with IRs and MFMs, analyzed the instantaneous availabilities and SSA under continuous and periodic inspection conditions, and determined the optimal imperfect repair strategy and inspection strategy. In addition to the findings above, no other literature has addressed the dynamic analysis of repairable systems with MFMs and IRs as established by Nikolov [23]. In this paper, we build upon the work in [25]–[27] to conduct a dynamic analysis of the above reliability model. Specifically, we investigate the well-posedness of the model and the asymptotic behavior of the TDS. The results of this paper not only validate the two hypotheses proposed by Nikolov [23] but also provide more comprehensive insights into the system's behavior.

The rest of the paper is organized as follows: Section 2 introduces the model and reformulates it as an abstract Cauchy problem (ACP) in a Banach space. Section 3 examines the well-posedness of the system. Section 4 delves into the asymptotic behavior of the TDS. Finally, Section 5 utilizes numerical examples to investigate how each parameter influences the system's instantaneous reliability indices.

II. The Mathematical Model of the System

The system consists of a single component and a repairman. The system is functional when the component is operational, and it malfunctions when the component fails. The mathematical model of the system is based on the following assumptions.

1: The system can fail in one of $M+1$ different failure modes, where the probability of having $M+1$ failures is η_m, with $\sum_{m=1}^{M+1} \eta_m = 1$.

2: The operating life of the system (component) obeys an exponential distribution with parameter λ_i, and the repair time follows a general distribution. The repair rate after the m-th failure at the i-th occurrence is denoted by $\mu_{i,m}(x)$, which satisfies $\mu_{i,m}(x) \geq 0$, $\int_0^\infty \mu_{i,m}(x)dx = \infty$ $(1 \leq i \leq n-1,\ 1 \leq m \leq M+1)$. $\mu_{n,l}(x)$ denotes the repair rate after the n-th failure satisfying $\mu_{n,l}(x) \geq 0$, $\int_0^\infty \mu_{n,l}(x)dx = \infty$.

3: After the first failure with a failure rate λ_1, the system is immediately halted for repair. It is assumed that no more failures will occur during the repair process. If the system fails due to the first m $(1 \leq m \leq M)$ failure modes, it can be repaired, but it is imperfect. Upon completion of the repair, it is put back into operation in state that is "better

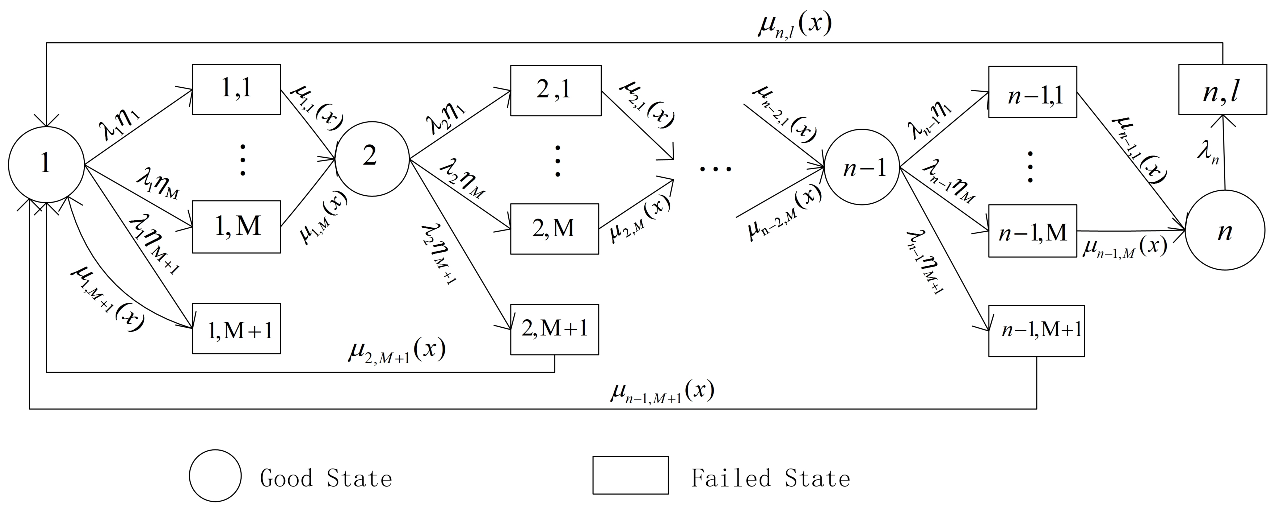


Fig. 1: State transition diagram of the system

than old, but worse than new". The $(M+1)$-st failure mode is considered irreparable, meaning that the system must be replaced and is then 'as good as new'.

4: An imperfectly repairable system has a failure rate $\lambda_2 > \lambda_1$, and can fail in one of $M+1$ different failure modes. After each failure, the system is repaired and put back into operation in the same manner as after the first cycle. In each subsequent cycle I, the system behaves similarly with a failure rate $\lambda_i > \lambda_{i-1}$, an m-st $(1 \le m \le M)$ failure, and imperfect repair. If the $(M+1)$-th failure occurs, the system is replaced. The maximum number of such cycles is n, i.e., after the n-th failure, the system is replaced regardless of the failure type.

5: The new system (or component) is put into operation at time $t=0$. All random variables are mutually independent. The replacement time of the failed component is negligible, and the replaced component is identical to the new one.

Based on the above description, the state transfer diagram of the system is shown in Fig. 1:

According to Nikolov [23], the following partial differential integral equations describe the mathematical model of the system with MFM and IR:

$$\frac{dP_1(t)}{dt} = -\lambda_1 P_1(t) + \sum_{i=1}^{n-1}\int_0^\infty \mu_{i,M+1}(x)P_{i,M+1}(x,t)dx + \int_0^\infty \mu_{n,l}(x)P_{n,l}(x,t)dx \tag{1}$$

$$\frac{\partial P_{i,m}(x,t)}{\partial t} + \frac{\partial P_{i,m}(x,t)}{\partial x} = -\mu_{i,m}(x)P_{i,m}(x,t), \quad 1 \le i \le n-1,\ 1 \le m \le M+1 \tag{2}$$

$$\frac{dP_i(t)}{dt} = -\lambda_i P_i(t) + \sum_{m=1}^{M}\int_0^\infty \mu_{i-1,m}(x)P_{i-1,m}(x,t)dx, \quad 2 \le i \le n \tag{3}$$

$$\frac{\partial P_{n,l}(x,t)}{\partial t} + \frac{\partial P_{n,l}(x,t)}{\partial x} = -\mu_{n,l}(x)P_{n,l}(x,t) \tag{4}$$

with the boundary and the initial conditions:

$$P_{i,m}(0,t) = \lambda_i \eta_m P_i(t),\ 1 \le i \le n-1,\ 1 \le m \le M+1 \tag{5}$$

$$P_{n,l}(0,t) = \lambda_n P_n(t) \tag{6}$$

$$P_1(0)=1, \quad P_{i,m}(x,0) = P_{i+1}(0) = P_{n,l}(x,0) = 0, \quad 1 \le i \le n-1,\ 1 \le m \le M+1, \quad x \in (0,\infty) \tag{7}$$

where, $P_1(t)$ denotes the probability that at time t, the system is running as 'new'; $P_{i,m}(t,x)$ denotes the probability that at time t, the system is being repaired after the i-th $(1 \le i \le n-1)$ occurrence of the m $(1 \le m \le M+1)$ type failure and that the component being repaired has consumed repair time x; $P_i(t)$ denotes the probability that at time t, the system is operating in a deteriorated state after the $i-1$st $(2 \le i \le n)$ repair; $P_{n,l}(t,x)$ denotes the probability that at time t, the system is being repaired after the nth failure and the repair time consumed by the component being repaired is x.

In the following we transform the system represented in (1)-(7) into an ACP in the Banach space X. For convenience, we introduce the following notation:

$$\Gamma = diag\Big(\Gamma_1, \Gamma_2, \cdots, \Gamma_{n-1}, \Gamma_n\Big)$$

where, $2 \le i \le n-1$

$$\Gamma_1 = \begin{pmatrix} e^{-x} & \\ \lambda_1\eta_1 e^{-x} & \\ \vdots & \mathbf{0} \\ \lambda_1\eta_{M+1}e^{-x} & \end{pmatrix}_{M+2\times M+2}$$

$$\Gamma_i = \begin{pmatrix} 0 & \\ \lambda_i\eta_1 e^{-x} & \\ \vdots & \mathbf{0} \\ \lambda_i\eta_{M+1}e^{-x} & \end{pmatrix}_{M+2\times M+2}$$

$$\Gamma_n = \begin{pmatrix} 0 & 0 \\ \lambda_n e^{-x} & 0 \end{pmatrix}_{2\times 2}$$

The following Banach space X is to be considered as a state space:

$$X = \left\{ P \;\middle|\; \begin{array}{c} P \in \overbrace{\mathbb{R}\times\ldots\times\mathbb{R}}^{n} \\ \times \overbrace{L^1[0,\infty)\times\ldots\times L^1[0,\infty)}^{(n-1)\times(M+1)+1} \\ \|P\| = \sum\limits_{i=1}^{n}|P_i| + \sum\limits_{i=1}^{n-1}\sum\limits_{m=1}^{M+1}\|P_{i,m}\|_{L^1[0,\infty)} \\ +\|P_{n,l}\|_{L^1[0,\infty)} \end{array} \right\}$$

In the following we will define the operator and its domain:

$$D(\Phi) = \left\{ P \in X \;\middle|\; \begin{array}{c} \frac{dP_{i,m}(x)}{dx}, \frac{dP_{n,l}(x)}{dx} \in L^1[0,\infty) \\ (1 \le i \le n-1,\ 1 \le m \le M+1), \\ P_{i,m}(x), P_{n,l}(x)\ are\ absolutely \\ continuous\ functions\ and \\ satisfy\ P(0) = \int_0^\infty \Gamma P(x)dx \end{array} \right\}$$

If we define for $P \in D(\Phi)$

$$\Phi P(x) = \begin{pmatrix} \Phi_1 & & & \\ & \ddots & & \\ & & \Phi_{n-1} & \\ & & & \Phi_n \end{pmatrix} \begin{pmatrix} P_1 \\ P_{1,1}(x) \\ \vdots \\ P_{1,M+1}(x) \\ \vdots \\ P_{n-1} \\ P_{n-1,1}(x) \\ \vdots \\ P_{n-1,M+1}(x) \\ P_n \\ P_{n,l}(x) \end{pmatrix}$$

where, $(1 \le i \le n-1)$

$$\Phi_i = diag\Big(-\lambda_i, -\frac{d}{dx} - \mu_{i,1}(x), \cdots, -\frac{d}{dx} - \mu_{i,M+1}(x)\Big)$$

$$\Phi_n = diag\Big(-\lambda_n, -\frac{d}{dx} - \mu_{n,l}(x)\Big)$$

and define for $P \in X$

$$\Omega P(x) = \begin{pmatrix} \sum_{i=1}^{n-1} \int_0^\infty \mu_{i,M+1}(x) P_{i,M+1}(x)dx \\ \mathbf{0}_{1\times(M+1)} \\ \sum_{m=1}^{M+1} \int_0^\infty \mu_{1,m}(x) P_{1,m}(x)dx \\ \mathbf{0}_{1\times(M+1)} \\ \vdots \\ \sum_{m=1}^{M+1} \int_0^\infty \mu_{n-1,m}(x) P_{n-1,m}(x)dx \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \int_0^\infty \mu_{n,l}(x) P_{n,l}(x)dx \\ \mathbf{0} \end{pmatrix}_{1\times\{n+1+(n-1)\times(M+1)\}}$$

$$D(\Omega) = X.$$

Then the above Eqs. (1)–(7) can be rewritten as an ACP in X:

$$\begin{cases} \frac{dP(t)}{dt} &= (\Phi + \Omega)P(t), \quad \forall t \in (0,\infty) \\ P(0) &= (1, 0, \cdots, 0) \end{cases} \tag{8}$$

where $\Phi + \Omega$ is the main operator of the system.

III. Well-posedness of the System (8)

Theorem 3.1. If

$$\mathcal{M} = \max_{x\in[0,\infty)} \left\{ \sup_{\substack{1\le i\le n-1 \\ 1\le m\le M+1}} \mu_{i,m}(x),\ \mu_{n,l}(x) \right\} < \infty$$

then $\Phi + \Omega$ generates a positive contraction C_0 - semigroup $\mathrm{T}(t)$.

Proof. First, we estimate $\|(\gamma I - \Phi)^{-1}\|$. For any given $y \in X$ we consider the equation $(\gamma I - \Phi)P = y$, i.e,

$$(\gamma + \lambda_i)P_i = y_i,\ 1 \le i \le n \tag{9}$$

$$(\gamma + \frac{d}{dx} + \mu_{i,m}(x))P_{i,m}(x) = y_{i,m}(x), \quad 1 \le i \le n-1,\ 1 \le m \le M+1 \tag{10}$$

$$(\gamma + \frac{d}{dx} + \mu_{n,l}(x))P_{n,l}(x) = y_{n,l}(x) \tag{11}$$

$$P_{i,m}(0) = \lambda_i \eta_m P_i, \quad 1 \le i \le n-1,\ 1 \le m \le M+1 \tag{12}$$

$$P_{n,l}(0) = \lambda_n P_n \tag{13}$$

By solving Eqs. (9)-(11) we have

$$P_i = \frac{1}{\gamma + \lambda_i} y_i \tag{14}$$

$$P_{i,m}(x) = P_{i,m}(0) e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} + e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \int_0^x y_{i,m}(\xi) e^{\gamma\xi - \int_0^\xi \mu_{i,m}(\tau)d\tau} d\xi \tag{15}$$

$$P_{n,l}(x) = P_{n,l}(0) e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} + e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \int_0^x y_{n,l}(\xi) e^{\gamma\xi - \int_0^\xi \mu_{n,l}(\tau)d\tau} d\xi \tag{16}$$

Combining Eqs. (12)-(13) with Eqs. (15)-(16) and using Eq. (14) we deduce

$$P_{i,m}(x) = \frac{\lambda_i \eta_m}{\gamma + \lambda_i} e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} y_i + e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \int_0^x y_{i,m}(\xi) e^{\gamma\xi - \int_0^\xi \mu_{i,m}(\tau)d\tau} d\xi \tag{17}$$

$$P_{n,l}(x) = \frac{\lambda_n}{\gamma + \lambda_n} e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} y_n + e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \int_0^x y_{n,l}(\xi) e^{\gamma\xi - \int_0^\xi \mu_{n,l}(\tau)d\tau} d\xi \tag{18}$$

Using Eq. (14), Eq. (17), and Eq. (18), together with Fubini theorem, we can estimate (with no loss of generality, assume $\gamma > 0$)

$$\begin{aligned}
\|P\| &= \sum_{i=1}^n |P_i| + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \|P_{i,m}\|_{L^1[0,\infty)} + \|P_{n,l}\|_{L^1[0,\infty)} \\
&\le \sum_{i=1}^n \frac{1}{\gamma + \lambda_i} |y_i| \\
&\quad + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty \frac{\lambda_i \eta_m}{\gamma + \lambda_i} e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} |y_i| \\
&\quad + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \\
&\quad \times \int_0^x |y_{i,m}(\xi)| e^{\gamma\xi + \int_0^\xi \mu_{i,m}(\tau)d\tau} d\xi dx \\
&\quad + \int_0^\infty \frac{\lambda_n}{\gamma + \lambda_n} e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} |y_n| \\
&\quad + \int_0^\infty e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \\
&\quad \times \int_0^x |y_{n,l}(\xi)| e^{\gamma\xi + \int_0^\xi \mu_{n,l}(\tau)d\tau} d\xi dx \\
&\le \sum_{i=1}^n \frac{1}{\gamma + \lambda_i} |y_i| \\
&\quad + \left(\sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \frac{\lambda_i \eta_m}{\gamma + \lambda_i} |y_i| + \frac{\lambda_n}{\gamma + \lambda_n} |y_n| \right) \int_0^\infty e^{-\gamma\xi} dx \\
&\quad + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty |y_{i,m}(\xi)| e^{\gamma\xi + \int_0^\xi \mu_{i,m}(\tau)d\tau} \\
&\quad \times \int_\xi^\infty e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} dx d\xi \\
&\quad + \int_0^\infty |y_{n,l}(\xi)| e^{\gamma\xi + \int_0^\xi \mu_{n,l}(\tau)d\tau} \\
&\quad \times \int_\xi^\infty e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} dx d\xi \\
&= \sum_{i=1}^n \frac{1}{\gamma + \lambda_i} |y_i| + \sum_{i=1}^n \frac{\lambda_i}{\gamma(\gamma + \lambda_i)} |y_i| \\
&\quad + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty |y_{i,m}(\xi)| e^{\gamma\xi} \\
&\quad \times \int_\xi^\infty e^{-\gamma x - \int_\xi^x \mu_{i,m}(\tau)d\tau} dx d\xi \\
&\quad + \int_0^\infty |y_{n,l}(\xi)| e^{\gamma\xi} \int_\xi^\infty e^{-\gamma x - \int_\xi^x \mu_{n,l}(\tau)d\tau} dx d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n} \frac{1}{\gamma+\lambda_i}|y_i| + \sum_{i=1}^{n} \frac{\lambda_i}{\gamma(\gamma+\lambda_i)}|y_i| \\
&\quad + \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty |y_{i,m}(\xi)|e^{\gamma\xi}\int_\xi^\infty e^{-\gamma x}dxd\xi \\
&\quad + \int_0^\infty |y_{n,l}(\xi)|e^{\gamma\xi}\int_\xi^\infty e^{-\gamma x}dxd\xi \\
&= \sum_{i=1}^{n}\left[\frac{1}{\gamma+\lambda_i}+\frac{\lambda_i}{\gamma(\gamma+\lambda_i)}\right]|y_i| \\
&\quad + \frac{1}{\gamma}\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty |y_{i,m}(\xi)|d\xi + \frac{1}{\gamma}\int_0^\infty |y_{n,l}(\xi)|d\xi \\
&= \frac{1}{\gamma}\|y\| \qquad (19)
\end{aligned}
$$

Eq. (19) shows that, when $\gamma > 0$

$$(\gamma I-\Phi)^{-1}: X \to D(\Phi),\ \|(\gamma I-\Phi)^{-1}\| \leq \frac{1}{\gamma}$$

In the second step we will prove that $D(\Phi)$ is dense in X. Let

$$
\mathcal{N} = \left\{ P(x) \left| \begin{array}{l}
P_{i,m} \in C_0^1[0,\infty) \textit{ and there exist} \\
\textit{constants } c_{i,m} > 0 \textit{ such that} \\
P_{i,m}(x) = 0 \textit{ for} x \in [0, c_{i,m}], \\
(1 \leq i \leq n-1,\ 1 \leq m \leq M+1), \\
P_{n,l}(x) \in C_0^1[0,\infty) \textit{ and there exist} \\
\textit{constants } c_{n,l} > 0 \textit{ such that} \\
P_{n,l}(x) = 0 \textit{ for } x \in [0, c_{n,l}]
\end{array} \right. \right\}
$$

then by Adams [28] we know that is $\overline{\mathcal{N}} = X$. As a result, proving $\overline{D(\Phi)} = X$ suffices to show that $\mathcal{N} \subset \overline{D(\Phi)}$. In fact, if $\mathcal{N} \subset \overline{D(\Phi)}$, then $X = \overline{\mathcal{N}} \subseteq \overline{\overline{D(\Phi)}} = \overline{D(\Phi)} \subset X$ gives $\overline{D(\Phi)} = X$.

Take any $P \in \mathcal{N}$, such that $P_{i,m}(x) = 0$ for all $x \in [0, c_{i,m}]$ $(1 \leq i \leq n-1;\ 1 \leq m \leq M+1)$, such that $P_{n,l}(x) = 0$ for all $x \in [0, c_{n,l}]$. From which we deduce $P_{i,m}(x) = P_{n,l}(x) = 0$ for all $x \in [0,s]$, where $0 < s < \min\{c_{1,1},\ldots,c_{1,M+1},\ldots,c_{n-1,1},\ldots,c_{n-1,M+1},c_{n,l}\}$. Define

$$
\begin{aligned}
F^s(0) &= \Big(P_1, F^s_{1,1}(0), \ldots, F^s_{1,M+1}(0), \ldots, P_{n-1}, F^s_{n-1,1}(0), \\
&\qquad \ldots, F^s_{n-1,M+1}(0), P_n, F^s_{n,l}(0)\Big) \\
&= \Big(P_1, \lambda_1\eta_1 P_1, \ldots, \lambda_1\eta_{M+1}P_1, 0, \lambda_2\eta_1 P_2, \ldots, \lambda_2\eta_{M+1}P_2, \\
&\qquad \ldots, 0, \lambda_{n-1}\eta_1 P_{n-1}, \ldots, \lambda_{n-1}\eta_{M+1}P_{n-1}, 0, \lambda_n P_{m,l}\Big)
\end{aligned}
$$

$$
\begin{aligned}
F^s(x) = &\Big(P_1,\ F^s_{1,1}(x),\ \ldots, F^s_{1,M+1}(x), \ldots, P_{n-1}, \\
&F^s_{n-1,1}(x), \ldots, F^s_{n-1,M+1}(x),\ P_n,\ F^s_{n,l}(x)\Big)
\end{aligned}
$$

where, $1 \leq i \leq n-1,\ 1 \leq m \leq M+1$

$$
F^s_{i,m}(x) = \begin{cases} F^s_{i,m}(0)(1-\frac{x}{s})^2, & x \in [0,s) \\ P_{i,m}(x), & x \in [s,\infty) \end{cases}
$$

$$
F^s_{n,l}(x) = \begin{cases} F^s_{i,m}(0)(1-\frac{x}{s})^2, & x \in [0,s) \\ P_{n,l}(x), & x \in [s,\infty) \end{cases}
$$

$$
\int_0^\infty \Gamma F^s(x)dx = \int_0^\infty \begin{pmatrix}
e^{-x}P_1 \\ \lambda_1\eta_1 e^{-x}P_1 \\ \vdots \\ \lambda_1\eta_{M+1}e^{-x}P_1 \\ 0 \\ \lambda_2\eta_1 e^{-x}P_2 \\ \vdots \\ \lambda_2\eta_{M+1}e^{-x}P_2 \\ \vdots \\ 0 \\ \lambda_{n-1}\eta_1 e^{-x}P_{n-1} \\ \vdots \\ \lambda_{n-1}\eta_{M+1}e^{-x}P_{n-1} \\ 0 \\ \lambda_n e^{-x}P_n
\end{pmatrix} dx
$$

$$
= \begin{pmatrix}
P_1 \\ \lambda_1\eta_1 P_1 \\ \vdots \\ \lambda_1\eta_{M+1}P_1 \\ 0 \\ \lambda_2\eta_1 P_2 \\ \vdots \\ \lambda_2\eta_{M+1}P_2 \\ \vdots \\ 0 \\ \lambda_{n-1}\eta_1 P_{n-1} \\ \vdots \\ \lambda_{n-1}\eta_{M+1}P_{n-1} \\ 0 \\ \lambda_n P_n
\end{pmatrix} \int_0^\infty e^{-x}dx = F^s(0)
$$

This shows that $F^s \in D(\Phi)$. Moreover

$$
\begin{aligned}
\|P - F^s\| &= \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^s |F^s_{i,m}(0)|(1-\frac{x}{s})^2 dx \\
&\quad + \int_0^s |F^s_{n,l}(0)|(1-\frac{x}{s})^2 dx \\
&= \left[\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}|F^s_{i,m}(0)| + |F^s_{n,l}(0)|\right]\frac{1}{3}s \to 0,\ s \to 0 \quad (20)
\end{aligned}
$$

Eq. (20) shows that $\mathcal{N} \subset D(\Phi)$. Therefore, $D(\Phi)$ is dense in X.

From the first step, the second step and the Hille-Yosida theorem [30], we know that Φ generates a C_0-semigroup $\mathrm{S}(t)$. It is easy to verify Ω is bounded linear operators. So the perturbation theorem of a C_0-semigroup (see [29], Theorem 1.80) imply that $\Phi + \Omega$ generates a C_0-semigroup $\mathrm{T}(t)$.

Lastly we will prove that $\Phi + \Omega$ is a dispersive operator (see [29], Definition 1.74). For any $P \in D(\Phi)$ we take

$$
\begin{aligned}
\phi(x) = &\bigg(\frac{[P_1]^+}{P_1},\ \frac{[P_{1,1}(x)]^+}{P_{1,1}(x)},\ \ldots,\ \frac{[P_{1,M+1}(x)]^+}{P_{1,M+1}(x)},\ \ldots, \\
&\frac{[P_n]^+}{P_n},\ \frac{[P_{n,l}(x)]^+}{P_{n,l}(x)}\bigg)
\end{aligned}
$$

where

$$[P_i]^+=\begin{cases} P_i & as \;\; P_i>0 \\ 0 & as \;\; P_i\le 0\end{cases},\; 1\le i\le n$$

$$[P_{i,m}(x)]^+=\begin{cases} P_{i,m}(x) & as \;\; P_{i,m}(x)>0 \\ 0 & as \;\; P_{i,m}(x)\le 0\end{cases}$$

$$1\le i\le n-1,\; 1\le m\le M+1$$

$$[P_{n,l}(x)]^+=\begin{cases} P_{n,l}(x) & as \;\; P_{n,l}(x)>0 \\ 0 & as \;\; P_{n,l}(x)\le 0\end{cases}$$

If define $W_{i,m}=\{x\in[0,\infty)\big|\; p_{i,m}(x)>0\}$, $V_{i,m}=\{x\in[0,\infty)\big|\; P_{i,m}(x)\le 0\}$, $W_{n,l}=\{x\in[0,\infty)\big|\; P_{n,l}(x)>0\}$ and $V_{n,l}=\{x\in[0,\infty)\big|\; P_{n,l}(x)\le 0\}$ for $1\le i\le n-1$, $1\le m\le M+1$, then we calculate

$$\begin{aligned}
&\int_0^\infty \frac{dP_{i,m}(x)}{dx}\frac{[P_{i,m}(x)]^+}{P_{i,m}(x)}dx\\
&\quad=\int_{W_{i,m}}\frac{dP_{i,m}(x)}{dx}\frac{[P_{i,m}(x)]^+}{P_{i,m}(x)}dx\\
&\qquad+\int_{V_{i,m}}\frac{dP_{i,m}(x)}{dx}\frac{[P_{i,m}(x)]^+}{P_{i,m}(x)}dx\\
&\quad=\int_{W_{i,m}}\frac{dP_{i,m}(x)}{dx}\frac{[P_{i,m}(x)]^+}{P_{i,m}(x)}dx\\
&\quad=\int_0^\infty d[P_{i,m}(x)]^+=-[P_{i,m}(0)]^+
\end{aligned}$$

$$1\le i\le n-1,\; 1\le m\le M+1 \tag{21}$$

$$\int_0^\infty \frac{dP_{n,l}(x)}{dx}\frac{[P_{n,l}(x)]^+}{P_{n,l}(x)}dx=-[P_{n,l}(0)]^+ \tag{22}$$

$$\int_0^\infty \mu_{i,m}(x)P_{i,m}(x)dx\le\int_0^\infty \mu_{i,m}(x)[P_{i,m}(x)]^+dx$$

$$1\le i\le n-1,\; 1\le m\le M+1 \tag{23}$$

$$\int_0^\infty \mu_{n,l}(x)P_{n,l}(x)dx\le\int_0^\infty \mu_{n,l}(x)[P_{n,l}(x)]^+dx \tag{24}$$

For $P\in D(\Phi)$ and $\phi(x)$, by using the boundary conditions on P, Eqs. (21)-(24) and $\sum\limits_{m=1}^{M+1}\eta_m=1$ we have

$$\begin{aligned}
&\big\langle(\Phi+\Omega)P\;,\;\phi\big\rangle\\
&=\Big\{-\lambda_1P_1+\sum_{i=1}^{n-1}\int_0^\infty \mu_{i,M+1}(x)P_{i,M+1}(x)dx\\
&\qquad+\int_0^\infty \mu_{n,l}(x)P_{n,l}(x)dx\Big\}\frac{[P_1]^+}{P_1}\\
&\quad+\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty\Big\{-\frac{dP_{i,m}(x)}{dx}\\
&\qquad\qquad-\mu_{i,m}(x)P_{i,m}(x)\Big\}\frac{[P_{i,m}(x)]^+}{P_{i,m}(x)}dx\\
&\quad+\sum_{i=2}^{n}\Big\{-\lambda_iP_i\\
&\qquad+\sum_{m=1}^{M}\int_0^\infty \mu_{i-1,m}(x)P_{i-1,m}(x)dx\Big\}\frac{[P_i]^+}{P_i}\\
&\quad+\int_0^\infty\Big\{-\frac{dP_{n,l}(x)}{dx}-\mu_{n,l}(x)P_{n,l}(x)\Big\}\frac{[P_{n,l}(x)]^+}{P_{n,l}(x)}dx\\
&\le-\sum_{i=1}^{n}\lambda_i[P_i]^+\\
&\quad+\frac{[P_1]^+}{P_1}\sum_{i=1}^{n-1}\int_0^\infty \mu_{i,M+1}(x)[P_{i,M+1}(x)]^+dx\\
&\quad+\frac{[P_1]^+}{P_1}\int_0^\infty \mu_{n,l}(x)[P_{n,l}(x)]^+dx+\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\lambda_i\eta_m[P_i]^+\\
&\quad-\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty \mu_{i,m}(x)[P_{i,m}(x)]^+dx\\
&\quad+\sum_{i=2}^{n}\frac{[P_i]^+}{P_i}\sum_{m=1}^{M}\int_0^\infty \mu_{i-1,m}(x)[P_{i-1,m}(x)]^+dx\\
&\quad+\lambda_n[P_n]^+-\int_0^\infty \mu_{n,l}(x)[P_{n,l}(x)]^+dx\\
&=\left(\frac{[P_1]^+}{P_1}-1\right)\sum_{i=1}^{n-1}\int_0^\infty \mu_{i,M+1}(x)[P_{i,M+1}(x)]^+dx\\
&\quad+\sum_{i=2}^{n}\left(\frac{[P_i]^+}{P_i}-1\right)\sum_{m=1}^{M}\int_0^\infty \mu_{i-1,m}(x)[P_{i-1,m}(x)]^+dx\\
&\quad+\left(\frac{[P_1]^+}{P_1}-1\right)\int_0^\infty \mu_{n,l}(x)[P_{n,l}(x)]^+dx\\
&\le 0
\end{aligned}\tag{25}$$

Eq. (25) shows that $\Phi+\Omega$ is a dispersive operator. Thus, $\Phi+\Omega$ generates a positive contraction C_0-semigroup by Phillips Theorem (see Gupur [29], Theorem 1.79). By the uniqueness of a C_0-semigroup it follows that $\mathrm{T}(t)$ is a positive contraction C_0-semigroup.

Obviously,

$$X^*=\left\{Q^*\;\middle|\;\begin{array}{c} Q^*\in\overbrace{\mathbb{R}\times\ldots\times\mathbb{R}}^{n}\\ \times\overbrace{L^\infty[0,\infty)\ldots\times L^\infty[0,\infty)}^{(n-1)\times(M+1)+1}\\ |||Q^*|||=\max\Big\{\sup\limits_{1\le i\le n}|Q_i|,\\ \sup\limits_{\substack{1\le i\le n-1\\1\le m\le M+1}}\|Q_{i,m}\|_{L^\infty[0,\infty)},\|Q_{n,l}\|_{L^\infty[0,\infty)}\Big\}\end{array}\right\}$$

the dual space of X. In X, we introduce the set

$$Y=\left\{P\in X\;\middle|\;\begin{array}{c} P(x)=\Big(P_1,P_{1,1}(x),\ldots,P_{1,M+1}(x),\\ \ldots,P_{n-1},P_{n-1,1}(x),\ldots,\\ P_{n-1,M+1}(x),P_n,P_{n,l}(x)\Big)\\ P_i\ge0\;(1\le i\le n),\;P_{i,m}(x)\ge0\\ (1\le i\le n-1,\;1\le m\le M+1),\\ P_{n,l}(x)\ge0,\;\forall x\in[0,\infty)\end{array}\right\}$$

Then, $Y\subset X$, and Theorem 3.1 ensures that $\mathrm{T}(t)Y\subset Y$. For $P\in \mathrm{D}(\Phi)\cap Y$, take $Q^*(x)=\|P\|\big(1,1,\ldots,1\big)$, then, $Q^*\in X^*$, and

$$\begin{aligned}
&\big\langle(\Phi+\Omega)P\;,\;Q^*\big\rangle\\
&=\Big\{-\lambda_1P_1+\sum_{i=1}^{n-1}\int_0^\infty \mu_{i,M+1}(x)P_{i,M+1}(x)dx\\
&\qquad+\int_0^\infty \mu_{n,l}(x)P_{n,l}(x)dx\Big\}\|P\|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty\Big\{-\frac{dP_{i,m}(x)}{dx}-\mu_{i,m}(x)P_{i,m}(x)\Big\}\|P\|dx\\
&+\sum_{i=2}^{n}\Big\{-\lambda_iP_i+\sum_{m=1}^{M}\int_0^\infty\mu_{i-1,m}(x)P_{i-1,m}(x)dx\Big\}\|P\|\\
&+\int_0^\infty\Big\{-\frac{dP_{n,l}(x)}{dx}-\mu_{n,l}(x)P_{n,l}(x)\Big\}\|P\|dx\\
=&-\sum_{i=1}^{n}\lambda_iP_i\|P\|+\|P\|\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty\mu_{i,m}(x)P_{i,m}(x)dx\\
&+\|P\|\int_0^\infty\mu_{n,l}(x)P_{n,l}(x)dx+\|P\|\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}P_{i,m}(0)\\
&-\|P\|\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty\mu_{i,m}(x)P_{i,m}(x)dx+\|P\|P_{n,l}(0)\\
&-\|P\|\int_0^\infty\mu_{n,l}(x)P_{n,l}(x)dx\\
=&-\sum_{i=1}^{n}\lambda_iP_i\|P\|+\|P\|\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty\mu_{i,m}(x)P_{i,m}(x)dx\\
&+\|P\|\int_0^\infty\mu_{n,l}(x)P_{n,l}(x)dx+\|P\|\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\lambda_i\eta_mP_i\\
&-\|P\|\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty\mu_{i,m}(x)P_{i,m}(x)dx+\|P\|\lambda_nP_n\\
&-\|P\|\int_0^\infty\mu_{n,l}(x)P_{n,l}(x)dx\\
=&\,0
\end{aligned}
\tag{26}
$$

In Eq. (26), we use $P_{i,m}\in L^1[0,\infty)\Rightarrow P_{i,m}(\infty)=0$ $(1\le i\le n-1; 1\le m\le M+1)$ and $P_{n,l}\in L^1[0,\infty)\Rightarrow P_{n,l}(\infty)=0$.

Equation (26) implies that $\Phi+\Omega$ is a conservative operator for the set

$$\theta(P)=\{Q^*\in X^*|<P,Q^*>=\|P\|^2=|||Q^*|||^2\}$$

Since the initial value $P(0)\in D(\Phi^2)\cap Y$, we have the following result by the Fattorini theorem [31].

Theorem 3.2. $\mathrm{T}(t)$ is isometric operator for $P(0)$, i.e.,

$$\|\mathrm{T}(t)P(0)\|=\|P(0)\|,\ \forall t\in[0,\infty) \tag{27}$$

We can obtain the system well-posedness from Theorems 3.1 and 3.2.

Theorem 3.3. If

$$\mathcal{M}=\max_{x\in[0,\infty)}\Big\{\sup_{\substack{0\le i\le n-1\\1\le m\le M+1}}\mu_{i,m}(x),\ \mu_{n,l}(x)\Big\}<\infty$$

then the system (8) has a unique positive TDS $P(x,t)$ satisfying

$$\|P(\cdot,t)\|=1,\ \forall t\in[0,\infty)$$

Proof. Since $P(0)\in D(\Phi^2)\cap Y$, Theorem 3.1 and Theorem 1.81 [29], show that the system (8) has a unique positive TDS $P(x,t)$ which can be expressed as

$$P(x,t)=\mathrm{T}(t)P(0),\ \forall t\in[0,\infty) \tag{28}$$

Combining Eq. (27) with Eq. (28) yields

$$
\begin{aligned}
\|P(x,t)\|&=\|\mathrm{T}(t)P(0)\|=\|P(0)\|\\
&=\|(1,0,\ldots,0)\|=1,\ \forall t\in[0,\infty)
\end{aligned}
\tag{29}
$$

IV. Asymptotic Behavior of the TDS of the System (8)

In this section, we will investigate the quasi-compactness of $\mathrm{T}(t)$.

Proposition 4.1. If $P(x,t)=(\mathrm{S}(t)\psi)(x)$ for $\psi\in X$ is a solution to the following system

$$\begin{cases}\frac{dP(t)}{dt}&=\Phi P(t),\qquad \forall t\in(0,\infty)\\ P(0)&=\psi\end{cases} \tag{30}$$

then

$$
\begin{aligned}
&P(x,t)=(\mathrm{S}(t)\psi)(x)\\
&=\begin{pmatrix}\psi_1e^{-\lambda_1t}\\ P_{1,1}(0,t-x)e^{-\int_0^\infty\mu_{1,1}(\tau)d\tau}\\ \vdots\\ P_{1,M+1}(0,t-x)e^{-\int_0^\infty\mu_{1,M+1}(\tau)d\tau}\\ \vdots\\ \psi_{n-1}e^{-\lambda_{n-1}t}\\ P_{n-1,1}(0,t-x)e^{-\int_0^\infty\mu_{n-1,1}(\tau)d\tau}\\ \vdots\\ P_{n-1,M+1}(0,t-x)e^{-\int_0^\infty\mu_{1,1}(\tau)d\tau}\\ \psi_ne^{-\lambda_nt}\\ P_{n,l}(0,t-x)e^{-\int_0^\infty\mu_{n,l}(\tau)d\tau}\end{pmatrix},x<t
\end{aligned}
$$

$$
\begin{aligned}
&P(x,t)=(\mathrm{S}(t)\psi)(x)\\
&=\begin{pmatrix}\phi_1e^{-\lambda_1t}\\ \psi_{1,1}(t-x)e^{-\int_{x-t}^x\mu_{1,1}(\tau)d\tau}\\ \vdots\\ \psi_{1,M+1}(t-x)e^{-\int_{x-t}^x\mu_{1,M+1}(\tau)d\tau}\\ \vdots\\ \phi_{n-1}e^{-\lambda_{n-1}t}\\ \psi_{n-1,1}(t-x)e^{-\int_{x-t}^x\mu_{n-1,1}(\tau)d\tau}\\ \vdots\\ \psi_{n-1,M+1}(t-x)e^{-\int_{x-t}^x\mu_{1,1}(\tau)d\tau}\\ \phi_ne^{-\lambda_nt}\\ \psi_{n,l}(t-x)e^{-\int_{x-t}^x\mu_{n,l}(\tau)d\tau}\end{pmatrix},x>t
\end{aligned}
$$

where $P_{i,m}(0,t-x)$ and $P_{n,l}(0,t-x)$ are given by Eq. (5) and Eq. (6).

Proof. $P(x,t)$ satisfys

$$\frac{dP_i(t)}{dt}=-\lambda_iP_i(t),\ 1\le i\le n \tag{31}$$

$$
\begin{aligned}
&\frac{\partial P_{i,m}(x,t)}{\partial t}+\frac{\partial P_{i,m}(x,t)}{\partial x}=-\mu_{i,m}(x)P_{i,m}(x,t)\\
&\quad 1\le i\le n-1,\ 1\le m\le M+1
\end{aligned}
\tag{32}
$$

$$\frac{\partial P_{n,l}(x,t)}{\partial t}+\frac{\partial P_{n,l}(x,t)}{\partial x}=-\mu_{n,l}(x)P_{n,l}(x,t) \tag{33}$$

$$
\begin{aligned}
&P_{i,m}(0,t)=\lambda_i\eta_mP_i(t)\\
&\quad 1\le i\le n-1,\ 1\le m\le M+1
\end{aligned}
\tag{34}
$$

$$P_{n,l}(0,t)=\lambda_nP_n(t) \tag{35}$$

$$P_i(0)=\psi_i\ (1\le i\le n),\ P_{i,m}(x,0)=\psi_{i,m}(x)$$

$$(1 \le i \le n-1,\ 1 \le m \le M+1),\ P_{n,l}(x,0) = \psi_{n,l}(x) \tag{36}$$

Take $\xi = x - t$ and define $Q_{i,m}(t) = P_{i,m}(\xi + t, t)$ $(1 \le i \le n-1;\ 1 \le m \le M+1)$ and $Q_{n,l}(t) = P_{n,l}(\xi + t, t)$, then Eqs. (32)-(33) gives

$$\frac{dQ_{i,m}(t)}{dt} = -\mu_{i,m}(\xi + t)Q_{i,m}(t) \tag{37}$$

$$\frac{dQ_{n,l}(t)}{dt} = -\mu_{n,l}(\xi + t)Q_{n,l}(t) \tag{38}$$

If $\xi < 0$ (i.e., $x < t$), then using $Q_{i,m}(-\xi) = P_{i,m}(0, -\xi) = P_{i,m}(0, t-x),\ Q_{n,l}(-\xi) = P_{n,l}(0, -\xi) = P_{n,l}(0, t-x)$ and integrating Eqs. (37)–(38) from $-\xi$ to t we have

$$\begin{aligned} P_{i,m}(x,t) &= Q_{i,m}(t) = Q_{i,m}(-\xi)e^{-\int_0^\infty \mu_{i,m}(\xi+\tau)d\tau} \\ &= P_{i,m}(0, t-x)e^{-\int_0^\infty \mu_{i,m}(\xi+\tau)d\tau} \\ &\overset{y=\xi+\tau}{=\!=\!=} P_{i,m}(0, t-x)e^{-\int_0^\infty \mu_{i,m}(y)dy} \\ &= P_{i,m}(0, t-x)e^{-\int_0^\infty \mu_{i,m}(\tau)d\tau} \end{aligned} \tag{39}$$

$$P_{n,l}(x,t) = Q_{n,l}(t) = P_{n,l}(0, t-x)e^{-\int_0^\infty \mu_{n,l}(\tau)d\tau} \tag{40}$$

If $\xi > 0$ (equivalently $x > t$), then using relations $Q_{i,m}(0) = P_{i,m}(\xi, 0) = \psi_{i,m}(x - t)$ and $Q_{n,l}(0) = P_{n,l}(\xi, 0) = \psi_{n,l}(x - t)$ and integrating Eq. (37) and Eq. (38) from 0 to t we derive

$$\begin{aligned} P_{i,m}(x,t) &= Q_{i,m}(t) = Q_{i,m}(0)e^{-\int_0^t \mu_{i,m}(\xi+\tau)d\tau} \\ &= \psi_{i,m}(t-x)e^{-\int_\xi^{\xi+\tau} \mu_{i,m}(\xi+\tau)d\tau} \\ &\overset{y=\xi+\tau}{=\!=\!=} \psi_{i,m}(t-x)e^{-\int_{x-t}^x \mu_{i,m}(y)dy} \\ &= \psi_{i,m}(t-x)e^{-\int_{x-t}^x \mu_{i,m}(\tau)d\tau} \end{aligned} \tag{41}$$

$$P_{n,l}(x,t) = Q_{n,l}(t) = \psi_{n,l}(t-x)e^{-\int_{x-t}^x \mu_{n,l}(\tau)d\tau} \tag{42}$$

From Eq. (31) and Eq. (36) we obtain

$$P_i(t) = \psi_i e^{-\lambda_i t} \tag{43}$$

Eqs. (39)–(43) show that the result of this proposition holds.

If we define two operators as follows, for $\psi \in X$:

$$(\mathcal{V}(t)\psi)(x) = \begin{cases} 0 & ,\ x \in [0,t) \\ (\mathrm{S}(t)\psi)(x) & ,\ x \in [t,\infty) \end{cases} \tag{44}$$

$$(\mathcal{W}(t)\psi)(x) = \begin{cases} (\mathrm{S}(t)\psi)(x) & ,\ x \in [0,t) \\ 0 & ,\ x \in [t,\infty) \end{cases} \tag{45}$$

then $\mathrm{S}(t)\psi = \mathcal{V}(t)\psi + \mathcal{W}(t)\psi,\ \forall \psi \in X$.

From Theorem 1.35 in [29], we deduce the following result:

Lemma 4.1. A bounded subset $G \in X$ is relatively compact if and only if the following conditions satisfy simultaneously:

$$1. \begin{cases} \lim\limits_{h\to 0} \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty |g_{i,m}(x+h) - g_{i,m}(x)|dx = 0 \\ \lim\limits_{h\to 0} \int_0^\infty |g_{n,l}(x+h) - g_{n,l}(x)|dx = 0 \end{cases}$$

$uniformly\ for\ g \in G$

$$2. \begin{cases} \lim\limits_{h\to \infty} \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_h^\infty |g_{i,m}(x)|dx = 0 \\ \lim\limits_{h\to \infty} \int_h^\infty |g_{n,l}(x)|dx = 0 \end{cases}$$

$uniformly\ for\ g \in G$

Theorem 4.1. Assume that $\mu_{i,m}(x)(1 \le i \le n-1;\ 1 \le m \le M+1)$ and $\mu_{n,l}(x)$ are Lipschitz continuous and there exists $\underline{\mu_{i,m}},\ \overline{\mu_{i,m}},\ \underline{\mu_{n,l}}$ and $\overline{\mu_{n,l}}$ such that

$$0 < \underline{\mu_{i,m}} \le \mu_{i,m}(x) \le \overline{\mu_{i,m}} < \infty$$
$$0 < \underline{\mu_{n,l}} \le \mu_{n,l}(x) \le \overline{\mu_{n,l}} < \infty,$$

then $\mathcal{W}(t)$ is a compact operator on X.

Proof. By the definition of $\mathcal{W}(t)$, it suffices to prove the condition (1) in Lemma 4.1. For bounded $\psi \in X$, we set $P(x,t) = (S(t)\psi)(x),\ x \in [0,t)$, then $P(x,t)$ is a solution of the system (8). Hence, using the result of Proposition 4.1 and Eq. (44) we get, for $x,\ h \in [0,t),\ x+h \in [0,t)$,

$$\begin{aligned} &\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^t \left|P_{i,m}(x+h,t) - P_{i,m}(x,t)\right|dx \\ &\quad + \int_0^t \left|P_{n,l}(x+h,t) - P_{n,l}(x,t)\right|dx \\ &= \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^t \Big|P_{i,m}(0,t-x-h)e^{-\int_0^{x+h}\mu_{i,m}(\tau)d\tau} \\ &\quad - P_{i,m}(0,t-x-h)e^{-\int_0^{x}\mu_{i,m}(\tau)d\tau} \\ &\quad + P_{i,m}(0,t-x-h)e^{-\int_0^{x}\mu_{i,m}(\tau)d\tau} \\ &\quad - P_{i,m}(0,t-x)e^{-\int_0^{x}\mu_{i,m}(\tau)d\tau}\Big|dx \\ &\quad + \int_0^t \Big|P_{n,l}(0,t-x-h)e^{-\int_0^{x+h}\mu_{n,l}(\tau)d\tau} \\ &\quad - P_{n,l}(0,t-x-h)e^{-\int_0^{x}\mu_{n,l}(\tau)d\tau} \\ &\quad + P_{n,l}(0,t-x-h)e^{-\int_0^{x}\mu_{n,l}(\tau)d\tau} \\ &\quad - P_{n,l}(0,t-x)e^{-\int_0^{x}\mu_{n,l}(\tau)d\tau}\Big|dx \\ &\le \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^t \left|P_{i,m}(0,t-x-h)\right| \\ &\quad \times \left|e^{-\int_0^{x+h}\mu_{i,m}(\tau)d\tau} - e^{-\int_0^{x}\mu_{i,m}(\tau)d\tau}\right|dx \\ &\quad + \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^t \left|P_{i,m}(0,t-x-h) - P_{i,m}(0,t-x)\right| \\ &\quad \times \left|e^{-\int_0^{x}\mu_{i,m}(\tau)d\tau}\right|dx \\ &\quad + \int_0^t \left|P_{n,l}(0,t-x-h)\right| \\ &\quad \times \left|e^{-\int_0^{x+h}\mu_{n,l}(\tau)d\tau} - e^{-\int_0^{x}\mu_{n,l}(\tau)d\tau}\right|dx \\ &\quad + \int_0^t \left|P_{n,l}(0,t-x-h) - P_{n,l}(0,t-x)\right| \\ &\quad \times \left|e^{-\int_0^{x}\mu_{n,l}(\tau)d\tau}\right|dx \end{aligned} \tag{45}$$

Now, we will estimate each of the terms in Eq. (45). Using Eqs. (34)-(35) and the properties of semigroups, we can conclude that

$$\begin{aligned} &\left|P_{i,m}(0,t-x-h)\right| = \left|\lambda_i\eta_m P_i(t-x-h)\right| \\ &= \lambda_i\eta_m\left|P_i(t-x-h)\right| \le \lambda_i\eta_m\left\|P(\cdot,t-x-h)\right\|_X \\ &= \lambda_i\eta_m\left\|S(t-x-h)\psi(\cdot)\right\|_X \le \lambda_i\eta_m\left\|\psi\right\|_X \end{aligned} \tag{46}$$

$$\begin{aligned} &\left|P_{n,l}(0,t-x-h)\right| = \left|\lambda_n P_n(t-x-h)\right| \\ &= \lambda_n\left|P_n(t-x-h)\right| \le \lambda_n\left\|P(\cdot,t-x-h)\right\|_X \end{aligned}$$

$$= \lambda_n \left\| S(t-x-h)\psi(\cdot) \right\|_X \le \lambda_n \left\| \psi \right\|_X \quad (47)$$

By Eqs. (46)-(47) we can estimate the first and third term of Eq. (45) as follows.

$$\begin{aligned} &\int_0^t \left| P_{i,m}(0,t-x-h) \right| \\ &\quad \times \left| e^{-\int_0^{x+h} \mu_{i,m}(\tau)d\tau} - e^{-\int_0^x \mu_{i,m}(\tau)d\tau} \right| dx \\ &\le \lambda_i \eta_m \|\psi\|_X \int_0^t \left| e^{-\int_0^{x+h} \mu_{i,m}(\tau)d\tau} - e^{-\int_0^x \mu_{i,m}(\tau)d\tau} \right| dx \\ &\longrightarrow 0,\ as\ |h| \longrightarrow 0,\ uniformly\ for\ \psi \\ &(1 \le i \le n-1,\ 1 \le m \le M+1) \end{aligned} \quad (48)$$

$$\begin{aligned} &\int_0^t \left| P_{n,l}(0,t-x-h) \right| \\ &\quad \times \left| e^{-\int_0^{x+h} \mu_{n,l}(\tau)d\tau} - e^{-\int_0^x \mu_{n,l}(\tau)d\tau} \right| dx \\ &\le \lambda_n \|\psi\|_X \int_0^t \left| e^{-\int_0^{x+h} \mu_{n,l}(\tau)d\tau} - e^{-\int_0^x \mu_{n,l}(\tau)d\tau} \right| dx \\ &\longrightarrow 0,\ as\ |h| \longrightarrow 0,\ uniformly\ for\ \psi \end{aligned} \quad (49)$$

By applying Eqs. (34)-(35), Proposition 4.1 and the above equations we have

$$\begin{aligned} &\left| P_{i,m}(0,t-x-h) - P_{i,m}(0,t-x) \right| \\ &= \left| \lambda_i \eta_m P_i(t-x-h) - \lambda_i \eta_m P_i(t-x) \right| \\ &= \lambda_i \eta_m \left| \psi_i e^{-\lambda_i (t-x-h)} - \psi_i e^{-\lambda_i (t-x)} \right| \\ &\le \lambda_i \eta_m \left\| \psi \right\|_X \left| e^{-\lambda_i (t-x-h)} - e^{-\lambda_i (t-x)} \right| \\ &\longrightarrow 0,\ as\ \left| h \right| \longrightarrow 0,\ uniformly\ for\ \psi \\ &(1 \le i \le n-1,\ 1 \le m \le M+1) \end{aligned} \quad (50)$$

$$\begin{aligned} &\left| P_{n,l}(0,t-x-h) - P_{n,l}(0,t-x) \right| \\ &= \lambda_n \left\| \psi \right\|_X \left| e^{-\lambda_n (t-x-h)} - e^{-\lambda_i (t-x)} \right| \\ &\longrightarrow 0,\ as\ \left| h \right| \longrightarrow 0,\ uniformly\ for\ \psi \end{aligned} \quad (51)$$

From Eqs. (50)-(51) we estimate the second and last term of Eq. (45) as follows.

$$\begin{aligned} &\int_0^t \left| P_{i,m}(0,t-x-h) - P_{i,m}(0,t-x) \right| e^{-\int_0^x \mu_{i,m}(\tau)d\tau} \\ &\longrightarrow 0, \quad as\ |h| \longrightarrow 0,\ uniformly\ for\ \psi \\ &(1 \le i \le n-1,\ 1 \le m \le M+1) \end{aligned} \quad (52)$$

$$\begin{aligned} &\int_0^t \left| P_{n,l}(0,t-x-h) - P_{n,l}(0,t-x) \right| e^{-\int_0^x \mu_{n,l}(\tau)d\tau} \\ &\longrightarrow 0,\ as\ |h| \longrightarrow 0,\ uniformly\ for\ \psi \end{aligned} \quad (53)$$

Combining Eqs. (48)-(49), Eqs. (52)-(53) with Eq. (45) we deduce, for $x+h \in [0,t)$,

$$\begin{aligned} &\sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^t \left| P_{i,m}(x+h,t) - P_{i,m}(x,t) \right| dx \\ &\quad + \int_0^t \left| P_{n,l}(x+h,t) - P_{n,l}(x,t) \right| dx \longrightarrow 0 \\ &\quad as\ |h| \longrightarrow 0,\ uniformly\ for\ \psi \end{aligned} \quad (54)$$

If $h \in [-t,0)$, $x \in [0,t)$, then $P_{i,m}(x+h,t) = P_{n,l}(x+h,t) = 0$ when $x+h<0$. Thus

$$\begin{aligned} &\sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^t \left| P_{i,m}(x+h,t) - P_{i,m}(x,t) \right| dx \\ &\quad + \int_0^t \left| P_{n,l}(x+h,t) - P_{n,l}(x,t) \right| dx \\ &= \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \left\{ \int_{-h}^t \left| P_{i,m}(x+h,t) - P_{i,m}(x,t) \right| dx \right. \\ &\qquad \left. + \int_0^{-h} \left| P_{i,m}(x,t) \right| dx \right\} \\ &\quad + \int_{-h}^t \left| P_{n,l}(x+h,t) - P_{n,l}(x,t) \right| dx \\ &\quad + \int_0^{-h} \left| P_{n,l}(x,t) \right| dx \end{aligned} \quad (55)$$

For $x+h \in [0,t)$, $x \in [0,t)$, $h \in [-t,0)$, the same way as in Eq. (54) is obtained for the first and third terms in Eq. (55):

$$\begin{aligned} &\int_{-h}^t \left| P_{i,m}(x+h,t) - P_{i,m}(x,t) \right| dx \longrightarrow 0 \quad as\ |h| \longrightarrow 0, \\ &uniformly\ for\ \psi\ \ (1 \le i \le n-1,\ 1 \le m \le M+1) \end{aligned} \quad (56)$$

$$\begin{aligned} &\int_{-h}^t \left| P_{n,l}(x+h,t) - P_{n,l}(x,t) \right| dx \longrightarrow 0 \quad as\ |h| \longrightarrow 0, \\ &\quad uniformly\ for\ \psi \end{aligned} \quad (57)$$

In the following, by using Proposition 4.1 and Eqs. (46)-(47) we estimate the second and last term in Eq. (55):

$$\begin{aligned} &\int_0^{-h} \left| P_{i,m}(x,t) \right| dx \\ &= \int_0^{-h} \left| P_{i,m}(0,t-x) \right| e^{-\int_0^x \mu_{i,m}(\tau)d\tau} dx \\ &= \int_0^{-h} \left| \lambda_i \eta_m P_i(t-x) \right| e^{-\int_0^x \mu_{i,m}(\tau)d\tau} dx \\ &\le \lambda_i \eta_m \left\| \psi \right\|_X \int_0^{-h} e^{-\int_0^x \mu_{i,m}(\tau)d\tau} dx \longrightarrow 0 \\ &\quad as\ |h| \longrightarrow 0,\ uniformly\ for\ \psi \\ &\quad (1 \le i \le n-1,\ 1 \le m \le M+1) \end{aligned} \quad (58)$$

$$\begin{aligned} &\int_0^{-h} \left| P_{n,l}(x,t) \right| dx \le \lambda_n \left\| \psi \right\|_X \int_0^{-h} e^{-\int_0^x \mu_{n,l}(\tau)d\tau} dx \\ &\quad \longrightarrow 0 \quad as\ |h| \longrightarrow 0,\ uniformly\ for\ \psi \end{aligned} \quad (59)$$

Combining Eqs. (55)–(59) we have, for $h \in (-t,0)$,

$$\begin{aligned} &\sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^t \left| P_{i,m}(x+h,t) - P_{i,m}(x,t) \right| dx \\ &\quad + \int_0^t \left| P_{n,l}(x+h,t) - P_{n,l}(x,t) \right| dx \longrightarrow 0 \\ &\quad as\ |h| \longrightarrow 0,\ uniformly\ for\ \psi \end{aligned} \quad (60)$$

Eq. (60) and Eq. (54) show that $\mathcal{W}(t)$ is a compact operator.

Theorem 4.2. Under the conditions of Theorem 4.1, then $\mathcal{V}(t)$ fulfills

$$\|\mathcal{V}(t)\psi\|_X \le e^{-\min\{\lambda_1,\ \underline{\mu_{i,m}},\ \underline{\mu_{n,l}}\}t}\|\psi\|_X, \quad \forall \psi \in X.$$

Proof. For any $\psi \in X$, according to the definition of $\mathcal{V}(t)$ and Proposition 4.1, we obtain

$$\begin{aligned}
\|\mathcal{V}(t)\psi(\cdot)\| &= \sum_{i=1}^{n}\left|\psi_i e^{-\lambda_i t}\right| \\
&\quad + \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_t^\infty \left|\psi_{i,m}(x-t)e^{-\int_{x-t}^{x}\mu_{i,m}(\tau)d\tau}\right|dx \\
&\quad + \int_t^\infty \left|\psi_{n,l}(x-t)e^{-\int_{x-t}^{x}\mu_{n,l}(\tau)d\tau}\right|dx \\
&\le e^{-\lambda_i t}\sum_{i=1}^{n}\left|\psi_i\right| \\
&\quad + \sum_{i=1}^{n-1}\sum_{m=1}^{M+1} e^{-\underline{\mu_{i,m}}}\int_t^\infty \left|\psi_{i,m}(x-t)\right|dx \\
&\quad + e^{-\underline{\mu_{n,l}}}\int_t^\infty \left|\psi_{n,l}(x-t)\right|dx \\
&\overset{y=x-t}{=\!=\!=} e^{-\lambda_i t}\sum_{i=1}^{n}\left|\psi_i\right| \\
&\quad + \sum_{i=1}^{n-1}\sum_{m=1}^{M+1} e^{-\underline{\mu_{i,m}}}\int_t^\infty \left|\psi_{i,m}(y)\right|dy \\
&\quad + e^{-\underline{\mu_{n,l}}}\int_t^\infty \left|\psi_{n,l}(y)\right|dy \\
&\le e^{-\min\left\{\lambda_i,\ \underline{\mu_{i,m}},\ \underline{\mu_{n,l}}\right\}t}\Bigg\{\sum_{i=1}^{n}\left|\psi_i\right| \\
&\quad + \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\|\psi_{i,m}\|_{L^1[0,\infty)} + \|\psi_{n,l}\|_{L^1[0,\infty)}\Bigg\} \\
&= e^{-\min\left\{\lambda_i,\ \underline{\mu_{i,m}},\ \underline{\mu_{n,l}}\right\}t}\|\psi\|_X
\end{aligned} \tag{61}$$

This shows that the result of the theorem is right.

From Theorem 4.1 and Theorem 4.2 we deduce

$$\begin{aligned}
\|S(t)-\mathcal{W}(t)\| = \|\mathcal{V}(t)\psi\|_X &\le e^{-\min\{\lambda_1,\ \underline{\mu_{i,m}},\ \underline{\mu_{n,l}}\}t} \\
&\longrightarrow 0 \quad as\ t \longrightarrow 0
\end{aligned}$$

From which together with Definition 1.85 in [29], we derive that $S(t)$ is a quasi-compact C_0 semigroup in X. Since, $\Omega : X \to \mathbb{R}^{n+1+(n-1)\times(M+1)}$ is a bounded linear operator, it follows that Ω is a compact operator on X. Thus, based on this result and Proposition 2.9 in Nagel [30], we obtain the following result.

Corollary 4.1. Under the conditions of Theorem 4.1, then $\mathrm{T}(t)$ is a quasi-compact C_0-semigroup in X.

Lemma 4.2. The eigenvalue 0 of the operator $(\Phi+\Omega)$ have geometric multiplicity 1.

Proof. Consider the equation $(\Phi+\Omega)P = 0$, i.e,

$$\begin{aligned}
\lambda_1 P_1 &= \sum_{i=1}^{n-1}\int_0^\infty \mu_{i,M+1}(x)P_{i,M+1}(x)dx \\
&\quad + \int_0^\infty \mu_{n,l}(x)P_{n,l}(x)dx
\end{aligned} \tag{62}$$

$$\frac{dP_{i,m}(x)}{dx} = -\mu_{i,m}(x)P_{i,m}(x),$$

$$1 \le i \le n-1,\ 1 \le m \le M+1 \tag{63}$$

$$\lambda_i P_i = \sum_{m=1}^{M}\int_0^\infty \mu_{i-1,m}(x)P_{i-1,m}(x)dx,\ 2 \le i \le n \tag{64}$$

$$\frac{dP_{n,l}(x)}{dx} = -\mu_{n,l}(x)P_{n,l}(x) \tag{65}$$

$$P_{i,m}(0) = \lambda_i\eta_m P_i,\ 1 \le i \le n-1, 1 \le m \le M+1 \tag{66}$$

$$P_{n,l}(0) = \lambda_n P_n \tag{67}$$

By solving Eq. (63), Eq. (65) and using Eqs. (66)-(67) we have

$$P_{i,m}(x) = \lambda_i\eta_m P_i e^{-\int_0^x \mu_{i,m}(\tau)d\tau} \tag{68}$$

$$P_{n,l}(x) = \lambda_n P_n e^{-\int_0^x \mu_{n,l}(\tau)d\tau} \tag{69}$$

Through inserting Eq. (68) into Eq. (64) we obtain

$$\begin{aligned}
P_i &= \sum_{m=1}^{M}\eta_m\frac{\lambda_{i-1}}{\lambda_i}P_{i-1}\int_0^\infty \mu_{i-1,m}(x)e^{-\int_0^x \mu_{i-1,m}(\tau)d\tau}dx \\
&= \sum_{m=1}^{M}\eta_m\frac{\lambda_{i-1}}{\lambda_i}P_{i-1} = \Big(\sum_{m=1}^{M}\eta_m\Big)^{i-1}\frac{\lambda_1}{\lambda_i}P_1,\ 2 \le i \le n
\end{aligned} \tag{70}$$

From Eqs. (68)-(70)we can estimate

$$\begin{aligned}
\|P\| &= \sum_{i=1}^{n}|P_i| + \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty \left|P_{i,m}(x)\right|dx \\
&\quad + \int_0^\infty \left|P_{n,l}(x)\right|dx \\
&= |P_1| + \sum_{i=2}^{n}|P_i| \\
&\quad + \sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty \left|\lambda_i\eta_m P_i e^{-\int_0^x \mu_{i,m}(\tau)d\tau}\right|dx \\
&\quad + \int_0^\infty \left|\lambda_n P_n e^{-\int_0^x \mu_{n,l}(\tau)d\tau}\right|dx \\
&= \left(1 + \lambda_1\sum_{i=2}^{n}\Big(\sum_{m=1}^{M}\eta_m\Big)^{i-1}\frac{1}{\lambda_i}\right)|P_1| \\
&\quad + \lambda_1\sum_{i=1}^{n-1}\Big(\sum_{m=1}^{M}\eta_m\Big)^{i-1}\sum_{m=1}^{M+1}\eta_m \\
&\qquad \times \int_0^\infty e^{-\int_0^x \mu_{i,m}(\tau)d\tau}dx|P_1| \\
&\quad + \lambda_1\Big(\sum_{m=1}^{M}\eta_m\Big)^{n-1}\int_0^\infty e^{-\int_0^x \mu_{n,l}(\tau)d\tau}dx|P_1| \\
&< \infty
\end{aligned}$$

The above equation indicates that 0 is an eigenvalue of the operator $(\Phi+\Omega)$. Furthermore, from Eqs. (4-39)-(4-41), we can deduce that the eigenvector corresponding 0 generates a one-dimensional linear space, which means the geometric multiplicity of 0 is 1 (see Gupur [29], Definition 1.23).

Lemma 4.3. The conjugate operator $(\Phi+\Omega)^*$ of $(\Phi+\Omega)$ is given by

$$(\Phi+\Omega)^* Q^* = (\mathrm{A}+\mathrm{B})Q^*, \quad Q^* \in D(\mathrm{A})$$

where

$$\mathrm{A}Q^*(x) = \begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_{n-1} & \\ & & & A_n \end{pmatrix} \begin{pmatrix} Q_1^* \\ Q_{1,1}^*(x) \\ \vdots \\ Q_{1,M+1}^*(x) \\ \vdots \\ Q_{n-1}^* \\ Q_{n-1,1}^*(x) \\ \vdots \\ Q_{n-1,M+1}^*(x) \\ Q_n^* \\ Q_{n,l}^*(x) \end{pmatrix}$$

$$D(\mathrm{A}) = \left\{ Q^* \in X^* \left| \begin{array}{c} \frac{dQ_{i,m}^*(x)}{dx}, \frac{dQ_{n,l}^*(x)}{dx} \text{ exists and} \\ Q_{i,m}^*(\infty) = Q_{n,l}^*(\infty) = \alpha, \\ 1 \le i \le n-1, \ 1 \le m \le M+1 \end{array} \right. \right\}$$

$(1 \le i \le n-1)$

$$A_i = diag\Big(-\lambda_i, -\frac{d}{dx} - \mu_{i,1}(x), \cdots, -\frac{d}{dx} - \mu_{i,M+1}(x) \Big)$$
$$A_n = diag\Big(-\lambda_n, \ -\frac{d}{dx} - \mu_{n,l}(x) \Big)$$

$$\mathrm{B}Q^*(x) = \begin{pmatrix} \overline{B}_1 & B_1 & & & \\ \vdots & & \ddots & & \\ \overline{B}_{n-1} & & & B_{n-1} & \\ \overline{B}_n & & & & B_n \end{pmatrix} \begin{pmatrix} Q_1^* \\ Q_{1,1}^*(x) \\ \vdots \\ Q_{1,M+1}^*(x) \\ \vdots \\ Q_{n-1}^* \\ Q_{n-1,1}^*(x) \\ \vdots \\ Q_{n-1,M+1}^*(x) \\ Q_n^* \\ Q_{n,l}^*(x) \end{pmatrix}$$

$$+ \begin{pmatrix} \widetilde{B}_1 & & & \\ & \ddots & & \\ & & \widetilde{B}_{n-1} & \\ & & & \widetilde{B}_n \end{pmatrix} \begin{pmatrix} Q_1^* \\ Q_{1,1}^*(0) \\ \vdots \\ Q_{1,M+1}^*(0) \\ \vdots \\ Q_{n-1}^* \\ Q_{n-1,1}^*(0) \\ \vdots \\ Q_{n-1,M+1}^*(0) \\ Q_n^* \\ Q_{n,l}^*(0) \end{pmatrix}$$

$$B_i = \begin{pmatrix} & 0 \\ & \mu_{i,1}(x) \\ \mathbf{0} & \vdots \\ & \mu_{i,M}(x) \\ & 0 \end{pmatrix}_{(M+1)\times(M+2)}, B_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\overline{B}_i = \begin{pmatrix} \mathbf{0} \\ \mu_{i,M+1}(x) \end{pmatrix}_{1\times(M+2)}, \overline{B}_n = \begin{pmatrix} 0 \\ \mu_{n,l}(x) \end{pmatrix}_{1\times 2},$$

$$\widetilde{B}_i = \begin{pmatrix} 0 & \lambda_i\eta_1 & \dots & \lambda_i\eta_{M+1} \\ & & \mathbf{0} & \end{pmatrix}_{(M+2)\times(M+2)},$$

$$\widetilde{B}_n = \begin{pmatrix} 0 & \lambda_n \\ 0 & 0 \end{pmatrix}$$

Proof. By applying integration by parts and the boundary conditions on $P \in D(\Phi)$ and $Q^* \in D(\mathrm{A})$, we have

$$\begin{aligned}
&\big\langle (\Phi + \Omega)P, Q^* \big\rangle \\
&= \Big\{ -\lambda_1 P_1 + \sum_{i=1}^{n-1} \int_0^\infty \mu_{i,M+1}(x) P_{i,M+1}(x)dx \\
&\quad + \int_0^\infty \mu_{n,l}(x) P_{n,l}(x)dx \Big\} Q_1^* \\
&\quad + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty \Big\{ -\frac{dP_{i,m}(x)}{dx} \\
&\qquad - \mu_{i,m}(x) P_{i,m}(x) \Big\} Q_{i,m}^*(x)dx \\
&\quad + \sum_{i=2}^{n} \Big\{ -\lambda_i P_i + \sum_{m=1}^{M} \int_0^\infty \mu_{i-1,m}(x) P_{i-1,m}(x)dx \Big\} Q_i^* \\
&\quad + \int_0^\infty \Big\{ -\frac{dP_{n,l}(x)}{dx} - \mu_{n,l}(x) P_{n,l}(x) \Big\} Q_{n,l}^*(x)dx \\
&= -\sum_{i=1}^{n} \lambda_i Q_1^* P_i + \sum_{i=1}^{n-1} \int_0^\infty \mu_{i,M+1}(x) Q_1^* P_{i,M+1}(x)dx \\
&\quad + \int_0^\infty \mu_{n,l}(x) Q_1^* P_{n,l}(x)dx \\
&\quad - \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty \frac{dP_{i,m}(x)}{dx} Q_{i,m}^*(x)dx \\
&\quad - \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty \mu_{i,m}(x) Q_{i,m}^*(x) P_{i,m}(x)dx \\
&\quad + \sum_{i=2}^{n} \sum_{m=1}^{M} \int_0^\infty \mu_{i-1,m}(x) Q_i^* P_{i-1,m}(x)dx \\
&\quad - \int_0^\infty \frac{dP_{n,l}(x)}{dx} Q_{n,l}^*(x)dx \\
&\quad - \int_0^\infty \mu_{n,l}(x) Q_{n,l}^*(x) P_{n,l}(x)dx \\
&= -\sum_{i=1}^{n} \lambda_i Q_1^* P_i + \sum_{i=1}^{n-1} \int_0^\infty \mu_{i,M+1}(x) Q_1^* P_{i,M+1}(x)dx \\
&\quad + \int_0^\infty \mu_{n,l}(x) Q_1^* P_{n,l}(x)dx + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} Q_{i,m}^*(0) P_{i,m}(0) \\
&\quad + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty \frac{dQ_{i,m}^*(x)}{dx} P_{i,m}(x)dx \\
&\quad - \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty \mu_{i,m}(x) Q_{i,m}^*(x) P_{i,m}(x)dx \\
&\quad + \sum_{i=2}^{n} \sum_{m=1}^{M} \int_0^\infty \mu_{i-1,m}(x) Q_i^* P_{i-1,m}(x)dx \\
&\quad + Q_{n,l}^*(0) P_{n,l}(0) + \int_0^\infty \frac{dQ_{n,l}^*(x)}{dx} P_{n,l}(x)dx \\
&\quad - \int_0^\infty \mu_{n,l}(x) Q_{n,l}^*(x) P_{n,l}(x)dx
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n} \lambda_i Q_1^* P_i + \sum_{i=1}^{n-1} \int_0^\infty \mu_{i,M+1}(x) Q_1^* P_{i,M+1}(x) dx \\
&\quad + \int_0^\infty \mu_{n,l}(x) Q_1^* P_{n,l}(x) dx + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} Q_{i,m}^*(0) \lambda_i \eta_m P_i \\
&\quad + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty \frac{dQ_{i,m}^*(x)}{dx} P_{i,m}(x) dx \\
&\quad - \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \int_0^\infty \mu_{i,m}(x) Q_{i,m}^*(x) P_{i,m}(x) dx \\
&\quad + \sum_{i=2}^{n} \sum_{m=1}^{M} \int_0^\infty \mu_{i-1,m}(x) Q_i^* P_{i-1,m}(x) dx \\
&\quad + Q_{n,l}^*(0) \lambda_n P_n + \int_0^\infty \frac{dQ_{n,l}^*(x)}{dx} P_{n,l}(x) dx \\
&\quad - \int_0^\infty \mu_{n,l}(x) Q_{n,l}^*(x) P_{n,l}(x) dx \\
&= -\sum_{i=1}^{n} \lambda_i Q_1^* P_i + \sum_{i=1}^{n-1} \sum_{m=1}^{M} \int_0^\infty \left\{ \frac{dQ_{i,m}^*(x)}{dx} \right. \\
&\qquad \left. - \mu_{i,m}(x) Q_{i,m}^*(x) + \mu_{i,m}(x) Q_{i+1}^* \right\} P_{i,m}(x) dx \\
&\quad + \sum_{i=1}^{n-1} \int_0^\infty \left\{ \frac{dQ_{i,M+1}^*(x)}{dx} - \mu_{i,M+1}(x) Q_{i,M+1}^*(x) \right. \\
&\qquad \left. + \mu_{i,M+1}(x) Q_1^* \right\} P_{i,M+1}(x) dx \\
&\quad + \int_0^\infty \left\{ \frac{dQ_{n,l}^*(x)}{dx} - \mu_{n,l}(x) Q_{n,l}^*(x) \right. \\
&\qquad \left. + \mu_{n,l}(x) Q_1^* \right\} P_{n,l}(x) dx \\
&\quad + \sum_{i=1}^{n-1} \sum_{m=1}^{M+1} \lambda_i \eta_m Q_{i,m}^*(0) P_i + \lambda_n Q_{n,l}^*(0) P_n \\
&= \left\langle P, (\mathrm{A} + \mathrm{B}) Q^* \right\rangle \qquad (4\text{-}45)
\end{aligned}
$$

From which together with Definition 1.21 [29] we know that the result of this lemma is right.

Lemma 4.4. The eigenvalue 0 of the $(\Phi + \Omega)^*$ have geometric multiplicity 1.

Proof. Consider the equation $(\Phi + \Omega)^* Q^* = 0$, i.e,

$$-\lambda_i Q_i^* + \lambda_i \sum_{m=1}^{M+1} \eta_m Q_{i,m}^*(0) = 0, \ 1 \le i \le n-1 \qquad (71)$$

$$\frac{dQ_{i,m}^*(x)}{dx} = \mu_{i,m}(x) Q_{i,m}^*(x) - \mu_{i,m}(x) Q_{i+1}^*, \quad 1 \le i \le n-1, \ 1 \le m \le M \qquad (72)$$

$$\frac{dQ_{i,M+1}^*(x)}{dx} = \mu_{i,M+1}(x) Q_{i,M+1}^*(x) - \mu_{i,M+1}(x) Q_1^*, \quad 1 \le i \le n-1 \qquad (73)$$

$$-\lambda_n Q_n^* + \lambda_n Q_{n,l}^*(0) = 0 \qquad (74)$$

$$\frac{dQ_{n,l}^*(x)}{dx} = \mu_{n,l}(x) Q_{n,l}^*(x) - \mu_{n,l}(x) Q_1^* \qquad (75)$$

$$Q_{i,m}^*(\infty) = Q_{n,l}^*(\infty) = \alpha, \ 1 \le i \le n-1, \ 1 \le m \le M \qquad (76)$$

By solving Eq. (72), Eq. (73) and Eq. (75) we have

$$
\begin{aligned}
Q_{i,m}^*(x) &= Q_{i,m}^*(0) e^{\int_0^x \mu_{i,m}(\tau) d\tau} \\
&\quad - e^{\int_0^x \mu_{i,m}(\tau) d\tau} \int_0^x \mu_{i,m}(\xi) Q_{i+1}^* e^{-\int_0^\xi \mu_{i,m}(\tau) d\tau} d\xi
\end{aligned} \qquad (77)
$$

$$
\begin{aligned}
Q_{i,M+1}^*(x) &= Q_{i,M+1}^*(0) e^{\int_0^x \mu_{i,M+1}(\tau) d\tau} \\
&\quad - e^{\int_0^x \mu_{i,M+1}(\tau) d\tau} \int_0^x \mu_{i,M+1}(\xi) Q_1^* e^{-\int_0^\xi \mu_{i,M+1}(\tau) d\tau} d\xi
\end{aligned} \qquad (78)
$$

$$
\begin{aligned}
Q_{n,l}^*(x) &= Q_{n,l}^*(0) e^{\int_0^x \mu_{n,l}(\tau) d\tau} \\
&\quad - e^{\int_0^x \mu_{n,l}(\tau) d\tau} \int_0^x \mu_{n,l}(\xi) Q_1^* e^{-\int_0^\xi \mu_{n,l}(\tau) d\tau} d\xi
\end{aligned} \qquad (79)
$$

Multiplying both sides of Eqs. (77)-(79) by $e^{-\int_0^x \mu_{i,m}(\tau) d\tau}$, $e^{-\int_0^x \mu_{i,M+1}(\tau) d\tau}$ and $e^{-\int_0^x \mu_{n,l}(\tau) d\tau}$, and taking the limit as $x \to \infty$, and using Eq. (76), we obtain

$$Q_{i,m}^*(0) = Q_{i+1}^* \int_0^\infty \mu_{i,m}(\xi) e^{-\int_0^\xi \mu_{i,m}(\tau) d\tau} d\xi \qquad (80)$$

$$Q_{i,M+1}^*(0) = Q_1^* \int_0^\infty \mu_{i,M+1}(\xi) e^{-\int_0^\xi \mu_{i,M+1}(\tau) d\tau} d\xi \qquad (81)$$

$$Q_{n,l}^*(0) = Q_1^* \int_0^\infty \mu_{n,l}(\xi) e^{-\int_0^\xi \mu_{n,l}(\tau) d\tau} d\xi \qquad (82)$$

Substituting Eqs. (80)-(82) into Eq. (71), Eq. (74) and using $\sum_{m=1}^{M+1} \eta_m = 1$, we have

$$
\begin{aligned}
Q_1^* &= \sum_{m=1}^{M} \eta_m Q_2^* + \eta_{M+1} Q_1^* \\
Q_2^* &= \sum_{m=1}^{M} \eta_m Q_3^* + \eta_{M+1} Q_1^* \\
&\cdots \\
Q_{n-1}^* &= \sum_{m=1}^{M} \eta_m Q_n^* + \eta_{M+1} Q_1^* \\
Q_n^* &= Q_1^* \\
\Rightarrow Q_i^* &= Q_1^*, \quad 2 \le i \le n
\end{aligned} \qquad (83)
$$

By inserting Eqs. (80)-(82) into Eqs. (77)-(79) and using Eq. (83) we derive

$$
\begin{aligned}
Q_{i,m}^*(x) &= e^{\int_0^x \mu_{i,m}(\tau) d\tau} \int_0^\infty \mu_{i,m}(\xi) Q_{i+1}^* e^{-\int_0^\xi \mu_{i,m}(\tau) d\tau} d\xi \\
&\quad - Q_{i+1}^* e^{\int_0^x \mu_{i,m}(\tau) d\tau} \int_0^x \mu_{i,m}(\xi) e^{-\int_0^\xi \mu_{i,m}(\tau) d\tau} d\xi \\
&= Q_{i+1}^* e^{\int_0^x \mu_{i,m}(\tau) d\tau} \int_x^\infty \mu_{i,m}(\xi) e^{-\int_0^\xi \mu_{i,m}(\tau) d\tau} d\xi \\
&= Q_{i+1}^* e^{\int_0^x \mu_{i,m}(\tau) d\tau} \left(-e^{-\int_0^\xi \mu_{i,m}(\tau) d\tau} \Big|_x^\infty \right) \\
&= Q_{i+1}^* = Q_1^*, \\
&\quad 1 \le i \le n-1, \ 1 \le m \le M+1
\end{aligned} \qquad (84)
$$

$$
\begin{aligned}
Q_{i,M+1}^*(x) &= e^{\int_0^x \mu_{i,M+1}(\tau) d\tau} \\
&\quad \times \int_x^\infty \mu_{i,M+1}(\xi) Q_1^* e^{-\int_0^\xi \mu_{i,M+1}(\tau) d\tau} d\xi \\
&= Q_1^*, \quad 1 \le i \le n-1
\end{aligned} \qquad (85)
$$

$$Q_{n,l}^*(x) = e^{\int_0^x \mu_{n,l}(\tau)d\tau} \int_x^\infty \mu_{n,l}(\xi) Q_1^* e^{-\int_0^\xi \mu_{n,l}(\tau)d\tau} d\xi = Q_1^* \tag{86}$$

Summarizing Eqs. (83)-(86) yields

$$\begin{aligned}|||Q^*||| &= \max\Big\{ \max_{1\le i\le n} |Q_i^*|, \max_{\substack{1\le i\le n\\ 1\le m\le M+1}} \|Q_{i,m}^*\|_{L^\infty[0,\infty)}, \\ &\qquad \|Q_{n,l}^*\|_{L^\infty[0,\infty)} \Big\} \\ &= |Q_1^*| < \infty\end{aligned}$$

From the above, 0 is an eigenvalue of $(\Phi+\Omega)^*$. Furthermore, from Eqs. (83)-(86), it is easy to see that the eigenvector corresponding to 0 generates a one-dimensional linear space, i.e., the geometric multiplicity of 0 is 1.

Combining Lemmas 4.1 and 4.3 with Theorem 3.1, we have that the algebraic multiplicity of 0 is also 1 and that the spectral boundary of $\Phi+\Omega$ is 0, i.e., $s(\Phi+\Omega) = 0$. Thus, by using Theorem 3.1, Corollary 4.1, Lemma 4.1 and Theorem 1.90 [29] the following result can be derived:

Theorem 4.3. If conditions of Theorem 4.1 hold, then there is a positive projection operator $\mathbb{P}$ and suitable constants $\delta > 0$ and $\overline{\mathcal{M}} > 0$ such that

$$\|\mathrm{T}(t) - \mathbb{P}\| \le \overline{\mathcal{M}} e^{-\delta t}$$

where $\mathbb{P} = \frac{1}{2\pi i}\int_{\overline{\Gamma}} (zI - \Phi - \Omega)^{-1} dz$, $\overline{\Gamma}$ is a circle of a sufficiently small radius with its center at 0.

By the proof of Theorem 1.90 [29], Theorem 3.1, Corollary 4.1 and Theorem 4.1, we have that

$$\{\gamma \in \sigma(\Phi+\Omega) \mid \Re\gamma = 0\} = \{0\}$$

In other words, all points on the imaginary axis except zero belong to the resolvent set of $\Phi+\Omega$. This means that only 0 is the spectral point of $\Phi+\Omega$ on the imaginary axis. Thus by using Theorem 1.96 [29] we obtain the following result:

Remark 4.1. If $\mu_{i,m}(x)(1 \le i \le n-1;\ 1 \le m \le M+1)$ and $\mu_{i,m}(x)$ are Lipschitz continuous and there exists $\underline{\mu_{i,m}}$, $\overline{\mu_{i,m}}$, $\underline{\mu_{n,l}}$ and $\overline{\mu_{n,l}}$ such that

$$\begin{aligned}0 < \underline{\mu_{i,m}} \le \mu_{i,m}(x) \le \overline{\mu_{i,m}} < \infty \\ 0 < \underline{\mu_{n,l}} \le \mu_{n,l}(x) \le \overline{\mu_{n,l}} < \infty,\end{aligned}$$

then the time-dependent solution of the system (2-8) strongly converges to its steady-state solution, i.e.,

$$\lim_{t\to\infty} P(x,t) = \langle P(0), Q^* \rangle P(x)$$

where $P(x)$ is the eigenvector corresponding to 0, $P(0)$ is the initial value of the system (2-8), and $Q^*(x)$ is the eigenvector of $\Phi+\Omega$ corresponding to 0.

Lemma 4.5. For $\gamma \in \rho(\Phi+\Omega)$ and $\forall \psi \in X$, we have

$$(\gamma I - \Phi - \Omega)^{-1} \begin{pmatrix} \psi_1 \\ \psi_{1,1}(x) \\ \vdots \\ \psi_{1,M+1}(x) \\ \vdots \\ \psi_{n-1} \\ \psi_{n-1,1}(x) \\ \vdots \\ \psi_{n-1,M+1}(x) \\ \psi_n \\ \psi_{n,l}(x) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_{1,1}(x) \\ \vdots \\ y_{1,M+1}(x) \\ \vdots \\ y_{n-1} \\ y_{n-1,1}(x) \\ \vdots \\ y_{n-1,M+1}(x) \\ y_n \\ y_{n,l}(x) \end{pmatrix}$$

Proof. For $\forall \psi \in X$, we consider the equation

$$(\gamma I - \Phi - \Omega) y = \psi \quad i.e.,$$

$$(\gamma+\lambda_1) y_1 = \sum_{i=1}^{n-1} \int_0^\infty \mu_{i,M+1}(x) y_{i,M+1}(x) dx + \int_0^\infty \mu_{n,l}(x) y_{n,l}(x) dx + \psi_1 \tag{87}$$

$$\frac{dy_{i,m}(x)}{dx} = -(\gamma + \mu_{i,m}(x)) y_{i,m}(x) + \psi_{i,m}(x), \quad 1 \le i \le n-1,\ 1 \le m \le M+1 \tag{88}$$

$$(\gamma+\lambda_i) y_i = \sum_{m=1}^{M} \int_0^\infty \mu_{i-1,m}(x) y_{i-1,m}(x) dx + \psi_i, \quad 2 \le i \le n \tag{89}$$

$$\frac{dy_{n,l}(x)}{dx} = -(\gamma + \mu_{n,l}(x)) y_{n,l}(x) + \psi_{i,m}(x) \tag{90}$$

$$y_{i,m}(0) = \lambda_i \eta_m y_i,\ 1 \le i \le n-1,\ 1 \le m \le M+1 \tag{91}$$

$$y_{n,l}(0) = \lambda_n y_n \tag{92}$$

By solving Eqs. (88), Eq. (90) and combining it with Eq. (91), Eq. (92), we have

$$\begin{aligned}y_{i,m}(x) &= y_{i,m}(0) e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \\ &\quad + e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \int_0^x \psi_{i,m}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{i,m}(\tau)d\tau} d\xi \\ &= \lambda_i \eta_m y_i e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \\ &\quad + e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \int_0^x \psi_{i,m}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{i,m}(\tau)d\tau} d\xi\end{aligned} \tag{93}$$

$$\begin{aligned}y_{n,l}(x) &= y_{n,l}(0) e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \\ &\quad + e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \int_0^x \psi_{n,l}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{n,l}(\tau)d\tau} d\xi \\ &= \lambda_n y_n e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \\ &\quad + e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \int_0^x \psi_{n,l}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{n,l}(\tau)d\tau} d\xi\end{aligned} \tag{94}$$

Substituting Eq. (93) into Eq. (88) yields

$$\begin{aligned}y_i &= \frac{\lambda_{i-1}}{\gamma+\lambda_i} y_{i-1} \sum_{m=1}^{M} \eta_m \\ &\quad \times \int_0^\infty \mu_{i-1,m}(x) e^{-\gamma x - \int_0^x \mu_{i-1,m}(\tau)d\tau} dx \\ &\quad + \frac{1}{\gamma+\lambda_i} \sum_{m=1}^{M} \int_0^\infty \mu_{i-1,m}(x) e^{-\gamma x - \int_0^x \mu_{i-1,m}(\tau)d\tau} \\ &\quad \times \int_0^x \psi_{i-1,m}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{i-1,m}(\tau)d\tau} d\xi dx \\ &\quad + \frac{1}{\gamma+\lambda_i} \psi_i, \quad 2 \le i \le n\end{aligned}$$

Let

$$\begin{aligned}G_i(\gamma) &= \frac{\lambda_i}{\gamma+\lambda_{i+1}} \sum_{m=1}^{M} \eta_m \\ &\quad \times \int_0^\infty \mu_{i,m}(x) e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} dx \\ F_i(\gamma) &= \frac{1}{\gamma+\lambda_{i+1}} \sum_{m=1}^{M} \int_0^\infty \mu_{i,m}(x) e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \\ &\quad \times \int_0^x \psi_{i,m}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{i,m}(\tau)d\tau} d\xi dx \\ \widetilde{\psi}_i &= \frac{1}{\gamma+\lambda_{i+1}} \psi_{i+1}\end{aligned}$$

then

$$\begin{aligned}y_2 &= G_1(\gamma) y_1 + F_1(\gamma) + \widetilde{\psi}_1 = \prod_{j=1}^{2-1} G_{2-j}(\gamma) y_1 \\ &\quad + \sum_{k=1}^{2-1} \prod_{j=0}^{2-2-k} G_{2-j-1}(\gamma) \Big[F_k(\gamma) + \widetilde{\psi}_k \Big] \\ y_3 &= G_2 G_1(\gamma) y_1 + G_2 F_1(\gamma) + G_2 \widetilde{\psi}_1 + F_2(\gamma) + \widetilde{\psi}_2 \\ &= \prod_{j=1}^{3-1} G_{3-j}(\gamma) y_1 + \sum_{k=1}^{3-1} \prod_{j=0}^{3-2-k} G_{3-j-1}(\gamma) \Big[F_k(\gamma) + \widetilde{\psi}_k \Big] \\ y_4 &= G_3 G_2 G_1(\gamma) y_1 + G_3 G_2 F_1(\gamma) + G_3 G_2 \widetilde{\psi}_1 + G_3 F_2(\gamma) \\ &\quad + G_3 \widetilde{\psi}_2 + G_3 + \widetilde{\psi}_3 \\ &= \prod_{j=1}^{4-1} G_{3-j}(\gamma) y_1 + \sum_{k=1}^{4-1} \prod_{j=0}^{4-2-k} G_{4-j-1}(\gamma) \Big[F_k(\gamma) + \widetilde{\psi}_k \Big] \\ &\quad \cdots \\ y_i &= \prod_{j=1}^{i-1} G_{i-j}(\gamma) y_1 + \sum_{k=1}^{i-1} \prod_{j=0}^{i-2-k} G_{i-j-1}(\gamma) \Big[F_k(\gamma) + \widetilde{\psi}_k \Big], \\ &\quad 2 \le i \le n \end{aligned} \tag{95}$$

By substituting Eq. (95) into Eqs. (93)-(94), we get

$$\begin{aligned}y_{1,m}(x) &= \lambda_1 \eta_m y_1 e^{-\gamma x - \int_0^x \mu_{1,m}(\tau)d\tau} \\ &\quad + e^{-\gamma x - \int_0^x \mu_{1,m}(\tau)d\tau} \int_0^x \psi_{1,m}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{1,m}(\tau)d\tau} d\xi \\ &\quad 1 \le m \le M+1 \end{aligned} \tag{96}$$

$$\begin{aligned}y_{i,m}(x) &= \lambda_i \eta_m y_1 \prod_{j=1}^{i-1} G_{i-j}(\gamma) e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \\ &\quad + \lambda_i \eta_m \sum_{k=1}^{i-1} \prod_{j=0}^{i-2-k} G_{i-j-1}(\gamma) \Big[F_k(\gamma) + \widetilde{\psi}_k \Big] \\ &\quad \times e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \\ &\quad + e^{-\gamma x - \int_0^x \mu_{i,m}(\tau)d\tau} \int_0^x \psi_{i,m}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{i,m}(\tau)d\tau} d\xi, \\ &\quad 2 \le i \le n-1, \; 1 \le m \le M+1 \end{aligned} \tag{97}$$

$$\begin{aligned}y_{n,l}(x) &= \lambda_n y_1 \prod_{j=1}^{i-1} G_{i-j}(\gamma) e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \\ &\quad + \lambda_n \sum_{k=1}^{i-1} \prod_{j=0}^{i-2-k} G_{i-j-1}(\gamma) \Big[F_k(\gamma) + \widetilde{\psi}_k \Big] \\ &\quad \times e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \\ &\quad + e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \int_0^x \psi_{n,l}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{n,l}(\tau)d\tau} d\xi \end{aligned} \tag{98}$$

By Combining Eqs. (96)-(98) with Eq. (87) yields

$$\begin{aligned}&(\gamma+\lambda_1) y_1 \\ &= \lambda_1 \eta_{M+1} y_1 \int_0^\infty \mu_{1,M+1}(x) e^{-\gamma x - \int_0^x \mu_{1,M+1}(\tau)d\tau} dx \\ &\quad + \int_0^\infty \mu_{1,M+1}(x) e^{-\gamma x - \int_0^x \mu_{1,M+1}(\tau)d\tau} \\ &\quad \times \int_0^x \psi_{1,M+1}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{1,M+1}(\tau)d\tau} d\xi dx \\ &\quad + \sum_{i=2}^{n-1} \lambda_i \eta_{M+1} y_1 \prod_{j=1}^{i-1} G_{i-j}(\gamma) \\ &\quad \times \int_0^\infty \mu_{i,M+1}(x) e^{-\gamma x - \int_0^x \mu_{i,M+1}(\tau)d\tau} dx \\ &\quad + \sum_{i=2}^{n-1} \lambda_i \eta_{M+1} \sum_{k=1}^{i-1} \prod_{j=0}^{i-2-k} G_{i-j-1}(\gamma) \Big[F_k(\gamma) + \widetilde{\psi}_k \Big] \\ &\quad \times \int_0^\infty \mu_{i,M+1}(x) e^{-\gamma x - \int_0^x \mu_{i,M+1}(\tau)d\tau} dx \\ &\quad + \sum_{i=2}^{n-1} \int_0^\infty \mu_{i,M+1}(x) e^{-\gamma x - \int_0^x \mu_{i,M+1}(\tau)d\tau} \\ &\quad \times \int_0^x \psi_{i,M+1}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{i,M+1}(\tau)d\tau} d\xi dx \\ &\quad + \lambda_n y_1 \prod_{j=1}^{n-1} G_{n-j}(\gamma) \\ &\quad \times \int_0^\infty \mu_{n,l}(x) e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} dx \\ &\quad + \lambda_n \sum_{k=1}^{n-1} \prod_{j=0}^{n-2-k} G_{n-j-1}(\gamma) \Big[F_k(\gamma) + \widetilde{\psi}_k \Big] \\ &\quad \times \int_0^\infty \mu_{n,l}(x) e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} dx \\ &\quad + \int_0^\infty \mu_{n,l}(x) e^{-\gamma x - \int_0^x \mu_{n,l}(\tau)d\tau} \\ &\quad \times \int_0^x \psi_{n,l}(\xi) e^{\gamma\xi + \int_0^\xi \mu_{n,l}(\tau)d\tau} d\xi dx + \psi_1 \end{aligned}$$

$$\Rightarrow$$

$$y_1 = \Bigg\{ \sum_{i=2}^{n-1} \lambda_i \eta_{M+1} \sum_{k=1}^{i-1} \prod_{j=0}^{i-2-k} G_{i-j-1}(\gamma) \Big[F_k(\gamma) + \widetilde{\psi}_k \Big]$$

$$\times \int_0^\infty \mu_{i,M+1}(x)e^{-\gamma x-\int_0^x \mu_{i,M+1}(\tau)d\tau}dx$$
$$+\sum_{i=1}^{n-1}\int_0^\infty \mu_{i,M+1}(x)e^{-\gamma x-\int_0^x \mu_{i,M+1}(\tau)d\tau}$$
$$\times \int_0^x \psi_{i,M+1}(\xi)e^{\gamma\xi+\int_0^\xi \mu_{i,M+1}(\tau)d\tau}d\xi dx$$
$$+\lambda_n\sum_{k=1}^{n-1}\prod_{j=0}^{n-2-k}G_{n-j-1}(\gamma)\Big[F_k(\gamma)+\widetilde{\psi}_k\Big]$$
$$\times \int_0^\infty \mu_{n,l}(x)e^{-\gamma x-\int_0^x \mu_{n,l}(\tau)d\tau}dx$$
$$+\int_0^\infty \mu_{n,l}(x)e^{-\gamma x-\int_0^x \mu_{n,l}(\tau)d\tau}$$
$$\times \int_0^x \psi_{n,l}(\xi)e^{\gamma\xi+\int_0^\xi \mu_{n,l}(\tau)d\tau}d\xi dx+\psi_1\Bigg\}\Bigg/$$
$$\Bigg\{\gamma+\lambda_1$$
$$-\lambda_1\eta_{M+1}\int_0^\infty \mu_{1,M+1}(x)e^{-\gamma x-\int_0^x \mu_{1,M+1}(\tau)d\tau}dx$$
$$-\sum_{i=2}^{n-1}\lambda_i\eta_{_{M+1}}\prod_{j=1}^{i-1}G_{i-j}(\gamma)$$
$$\times \int_0^\infty \mu_{i,M+1}(x)e^{-\gamma x-\int_0^x \mu_{i,M+1}(\tau)d\tau}dx$$
$$-\lambda_n\prod_{j=1}^{n-1}G_{n-j}(\gamma)\int_0^\infty \mu_{n,l}(x)e^{-\gamma x-\int_0^x \mu_{n,l}(\tau)d\tau}dx\Bigg\} \tag{99}$$

Eqs. (95)-(99) show that the conclusions of this lemma hold.

Theorem 4.4. Assume that $\mu_{i,m}(x)(1\le i\le n-1;\ 1\le m\le M+1)$ and $\mu_{n,l}(x)$ are Lipschitz continuous and there exist $\underline{\mu_{i,m}},\ \overline{\mu_{i,m}},\ \underline{\mu_{n,l}}$ and $\overline{\mu_{n,l}}$ such that

$$0<\underline{\mu_{i,m}}\le\mu_{i,m}(x)\le\overline{\mu_{i,m}}<\infty$$
$$0<\underline{\mu_{n,l}}\le\mu_{n,l}(x)\le\overline{\mu_{n,l}}<\infty,$$

then the TDS $P(x,t)$ of system (2-8) converges exponentially to its SSS $P(x)$, i.e.,

$$\|P(\cdot,t)-P(\cdot)\|\le\overline{\mathcal{M}}e^{-\delta t},\ \forall t\ge 0$$

where, $\overline{\mathcal{M}}$ and δ are the positive constants in Theorem 4.3 and P(x) is the characteristic negotiation in Lemma 4.1.

Proof. Theorems 4.1, Theorems 4.2, Eq. (43) and Eq. (44) imply

$$\ln\|S(t)-\mathcal{W}(t)\|=\ln\|\mathcal{V}(t)\|\le-\min\big\{\lambda_i,\ \underline{\mu_{i,m}},\ \underline{\mu_{n,l}}\big\}t$$
$$\Rightarrow\frac{\ln\|S(t)-\mathcal{W}(t)\|}{t}\le-\min\big\{\lambda_i,\ \underline{\mu_{i,m}},\ \underline{\mu_{n,l}}\big\}$$

From this, together with Proposition 2.10 of Nagel and Engel [32], we have $\omega_{ess}\big(S(t)\big)$ the essential growth bound of $S(t)$ (equivalently, $\omega_{ess}\big(\Phi\big)$ the essential growth bound of Φ), satisfying

$$\omega_{ess}\big(S(t)\big)\le-\min\big\{\lambda_i,\ \underline{\mu_{i,m}},\ \underline{\mu_{n,l}}\big\}$$

Since $\Omega: X\to\mathbb{R}^{n+1+(n-1)\times(M+1)}$ is compact operators, by Proposition 2.12 in Nagel and Engel [32], we have

$$\omega_{ess}(\Phi+\Omega)=\omega_{ess}\big(\mathrm{T}(t)\big)=\omega_{ess}\big(S(t)\big)$$
$$\le-\min\big\{\lambda_i,\ \underline{\mu_{i,m}},\ \underline{\mu_{n,l}}\big\} \tag{100}$$

Theorem 3.1, Corollary 2.11 of Engel and Nagel [32] , Lemma 4.1, and Corollary 4.1 imply that $\omega(\mathrm{T})=0,\ s(\Phi+\Omega)=0$,and thus, from Eq. (4-62), Theorem 1.87 [29],

$$0\in\sigma(\Phi+\Omega)\cap\big\{\gamma\in\mathbb{C}|\Re\gamma>\omega_{ess}(\Phi+\Omega)\big\}$$

we Know that 0 is an isolated eigenvalue of $\Phi+\Omega$ and has algebraic multiplicity 1. In other words, 0 is a 1st order pole of $(\gamma I-\Phi-\Omega)^{-1}$. Therefore, from Theorem 4.3, Lemma 4.5 and the residue theorem, we derive

$$\mathbb{P}\psi(x)=\frac{1}{2\pi i}\int_{\Gamma}(zI-\Phi-\Omega)^{-1}\psi(x)dz$$
$$=\lim_{\gamma\to 0}\gamma(\gamma I-\Phi-\Omega)^{-1}\psi(x) \tag{101}$$

Now, we find the projection operator by evaluating the aforementioned limit. Considering that

$$\int_0^\infty \mu_{i,m}(x)e^{-\int_0^x \mu_{i,m}(\tau)d\tau}dx=-e^{-\int_0^x \mu_{i,m}(\tau)d\tau}\Big|_0^\infty=1,$$
$$1\le i\le n-1,\ 1\le m\le M+1$$
$$\int_0^\infty \mu_{n,l}(x)e^{-\int_0^x \mu_{n,l}(\tau)d\tau}dx=-e^{-\int_0^x \mu_{n,l}(\tau)d\tau}\Big|_0^\infty=1$$

and using L'Hospital's rule, we get

$$\lim_{\gamma\to 0}\gamma\Bigg/\Bigg\{\gamma+\lambda_1-\lambda_1\eta_{M+1}$$
$$\times\int_0^\infty \mu_{1,M+1}(x)e^{-\gamma x-\int_0^x \mu_{1,M+1}(\tau)d\tau}dx$$
$$-\sum_{i=2}^{n-1}\lambda_i\eta_{_{M+1}}\prod_{j=1}^{i-1}G_{i-j}(\gamma)$$
$$\times\int_0^\infty \mu_{i,M+1}(x)e^{-\gamma x-\int_0^x \mu_{i,M+1}(\tau)d\tau}dx$$
$$-\lambda_n\prod_{j=1}^{n-1}G_{n-j}(\gamma)$$
$$\times\int_0^\infty \mu_{n,l}(x)e^{-\gamma x-\int_0^x \mu_{n,l}(\tau)d\tau}dx\Bigg\}$$
$$=\lim_{\gamma\to 0}1\Bigg/\Bigg\{1+\lambda_1\eta_{M+1}$$
$$\times\int_0^\infty x\mu_{1,M+1}(x)e^{-\gamma x-\int_0^x \mu_{1,M+1}(\tau)d\tau}dx$$
$$+\sum_{i=2}^{n-1}\lambda_i\eta_{_{M+1}}\prod_{j=1}^{i-1}G_{i-j}(\gamma)$$
$$\times\int_0^\infty x\mu_{i,M+1}(x)e^{-\gamma x-\int_0^x \mu_{i,M+1}(\tau)d\tau}dx$$
$$+\sum_{i=2}^{n-1}\lambda_i\eta_{_{M+1}}\sum_{j=1}^{i-1}G'_{i-j+1}(\gamma)\prod_{\substack{k=1\\k\ne j}}^{i-1}G_{i-k+1}(\gamma)$$
$$\times\int_0^\infty \mu_{i,M+1}(x)e^{-\gamma x-\int_0^x \mu_{i,M+1}(\tau)d\tau}dx$$

$$
\begin{aligned}
&+\lambda_n \prod_{j=1}^{n-1} G_{n-j}(\gamma) \\
&\quad \times \int_0^\infty x\mu_{n,l}(x)e^{-\gamma x-\int_0^x \mu_{n,l}(\tau)d\tau}dx \\
&+\lambda_n \sum_{j=1}^{i-1} G'_{n-j+1}(\gamma) \prod_{\substack{k=1\\k\neq j}}^{i-1} G_{n-k+1}(\gamma) \\
&\quad \times \int_0^\infty \mu_{n,l}(x)e^{-\gamma x-\int_0^x \mu_{n,l}(\tau)d\tau}dx \Bigg\} \\
= 1 \Bigg/ &\Bigg\{1+\lambda_1\eta_{M+1}\int_0^\infty x\mu_{1,M+1}(x)e^{-\int_0^x \mu_{1,M+1}(\tau)d\tau}dx \\
&+\lambda_1\eta_{M+1}\sum_{i=2}^{n-1}\left(\sum_{m=1}^{M}\eta_m\right)^{i-1} \\
&\quad \times \int_0^\infty x\mu_{i,M+1}(x)e^{-\int_0^x \mu_{i,M+1}(\tau)d\tau}dx \\
&+\lambda_1\eta_{M+1}\sum_{i=2}^{n-1}\left(\sum_{m=1}^{M}\eta_m\right)^{i-1}\sum_{j=0}^{i-2}\frac{1}{\lambda_{i-j}} \\
&+\lambda_1\eta_{M+1}\sum_{i=2}^{n-1}\left(\sum_{m=1}^{M}\eta_m\right)^{i-2}\sum_{j=0}^{i-2}\sum_{m=1}^{M}\eta_m \\
&\quad \times \int_0^\infty x\mu_{i-j-1,m}(x)e^{-\int_0^x \mu_{i-j-1,m}(\tau)d\tau}dx \\
&+\lambda_1\int_0^\infty x\mu_{n,l}(x)e^{-\int_0^x \mu_{n,l}(\tau)d\tau}dx \\
&+\lambda_1\left(\sum_{m=1}^{M}\eta_m\right)^{n-1}\sum_{j=0}^{n-2}\frac{1}{\lambda_{n-j}} \\
&+\lambda_1\left(\sum_{m=1}^{M}\eta_m\right)^{n-2}\sum_{j=0}^{n-2}\sum_{m=1}^{M}\eta_m \\
&\quad \times \int_0^\infty x\mu_{n-j-1,m}(x)e^{-\int_0^x \mu_{n-j-1,m}(\tau)d\tau}dx\Bigg\} \\
= \frac{1}{H} &
\end{aligned}
\tag{102}
$$

From Eq. (102), Lemma 4.5, Fubini's theorem and

$$
\begin{aligned}
&\int_0^\infty \mu_{i,m}(x)e^{-\int_0^x \mu_{i,m}(\tau)d\tau}dx = -e^{-\int_0^x \mu_{i,m}(\tau)d\tau}\Big|_0^\infty \\
&\quad = 1,\ 1\le i\le n-1,\ 1\le m\le M+1
\end{aligned}
\tag{103}
$$

$$
\int_0^\infty \mu_{n,l}(x)e^{-\int_0^x \mu_{n,l}(\tau)d\tau}dx = -e^{-\int_0^x \mu_{n,l}(\tau)d\tau}\Big|_0^\infty = 1
\tag{104}
$$

$$
\sum_{i=1}^{n}\psi_i+\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty \psi_{i,m}(x)dx+\int_0^\infty \psi_{n,l}(x)dx=1
\tag{105}
$$

$$
\begin{aligned}
&\int_0^\infty \mu_{i,m}(x)e^{-\int_0^x \mu_{i,m}(\tau)d\tau}\int_0^x \psi_{i,m}(\xi)e^{\int_0^\xi \mu_{i,m}(\tau)d\tau}d\xi dx \\
&= \int_0^\infty \psi_{i,m}(\xi)e^{\int_0^\xi \mu_{i,m}(\tau)d\tau} \\
&\qquad \int_\xi^\infty \mu_{i,m}(x)e^{-\int_0^x \mu_{i,m}(\tau)d\tau}dxd\xi \\
&= \int_0^\infty \psi_{i,m}(\xi)d\xi = \int_0^\infty \psi_{i,m}(x)dx, \\
&\qquad 1\le i\le n-1,\ 1\le m\le M+1
\end{aligned}
\tag{106}
$$

$$
\begin{aligned}
&\int_0^\infty \mu_{n,l}(x)e^{-\int_0^x \mu_{n,l}(\tau)d\tau}\int_0^x \psi_{n,l}(\xi)e^{\int_0^\xi \mu_{n,l}(\tau)d\tau}d\xi dx \\
&\qquad = \int_0^\infty \psi_{n,l}(\xi)d\xi = \int_0^\infty \psi_{n,l}(x)dx
\end{aligned}
\tag{107}
$$

we derive

$$
\begin{aligned}
&\lim_{\gamma\to 0}\gamma y_1 \\
&= \Bigg\{\eta_{M+1}\sum_{m=1}^{M}\int_0^\infty \psi_{1,m}(x)dx+\eta_{M+1}\psi_2 \\
&\quad +\eta_{M+1}\sum_{m=1}^{M}\eta_m\sum_{m=1}^{M}\int_0^\infty \psi_{1,m}(x)dx \\
&\quad +\eta_{M+1}\sum_{m=1}^{M}\eta_m\psi_2+\eta_{M+1}\sum_{m=1}^{M}\int_0^\infty \psi_{2,m}(x)dx \\
&\quad +\eta_{M+1}\psi_3+\eta_{M+1}\left(\sum_{m=1}^{M}\eta_m\right)^2\sum_{m=1}^{M}\int_0^\infty \psi_{1,m}(x)dx \\
&\quad +\eta_{M+1}\left(\sum_{m=1}^{M}\eta_m\right)^2\psi_2 \\
&\quad +\eta_{M+1}\sum_{m=1}^{M}\eta_m\sum_{m=1}^{M}\int_0^\infty \psi_{2,m}(x)dx \\
&\quad +\eta_{M+1}\sum_{m=1}^{M}\eta_m\psi_3 \\
&\quad +\eta_{M+1}\sum_{m=1}^{M}\int_0^\infty \psi_{3,m}(x)dx+\eta_{M+1}\psi_4 \\
&\quad +\dots \\
&\quad +\left(\sum_{m=1}^{M}\eta_m\right)^{n-2}\sum_{m=1}^{M}\int_0^\infty \psi_{1,m}(x)dx \\
&\quad +\left(\sum_{m=1}^{M}\eta_m\right)^{n-2}\psi_2 \\
&\quad +\left(\sum_{m=1}^{M}\eta_m\right)^{n-3}\sum_{m=1}^{M}\int_0^\infty \psi_{2,m}(x)dx \\
&\quad +\left(\sum_{m=1}^{M}\eta_m\right)^{n-3}\psi_3 \\
&\quad +\dots \\
&\quad +\sum_{m=1}^{M}\eta_m\sum_{m=1}^{M}\int_0^\infty \psi_{n-2,m}(x)dx \\
&\quad +\sum_{m=1}^{M}\eta_m\psi_{n-1}+\sum_{m=1}^{M}\int_0^\infty \psi_{n-1,m}(x)dx \\
&\quad +\psi_n+\int_0^\infty \psi_{n,l}(x)dx+\psi_1\Bigg\}\Bigg/H \\
&= \Bigg\{\left[1+\sum_{m=1}^{M}\eta_m+\dots+\left(\sum_{m=1}^{M}\eta_m\right)^{n-3}\right]
\end{aligned}
$$

$$\times\left(1-\sum_{m=1}^{M}\eta_m\right)\sum_{m=1}^{M}\int_0^\infty\psi_{1,m}(x)dx$$
$$+\left(\sum_{m=1}^{M}\eta_m\right)^{n-2}\sum_{m=1}^{M}\int_0^\infty\psi_{1,m}(x)dx$$
$$+\Big[1+\sum_{m=1}^{M}\eta_m+\ldots+\left(\sum_{m=1}^{M}\eta_m\right)^{n-3}\Big]$$
$$\times\left(1-\sum_{m=1}^{M}\eta_m\right)\psi_2+\left(\sum_{m=1}^{M}\eta_m\right)^{n-2}\psi_2$$
$$+\ldots$$
$$+\Big[1+\sum_{m=1}^{M}\eta_m\Big]\left(1-\sum_{m=1}^{M}\eta_m\right)$$
$$\times\sum_{m=1}^{M}\int_0^\infty\psi_{n-3,m}(x)dx$$
$$+\left(\sum_{m=1}^{M}\eta_m\right)^{2}\sum_{m=1}^{M}\int_0^\infty\psi_{n-3,m}(x)dx$$
$$+\Big[1+\sum_{m=1}^{M}\eta_m\Big]\left(1-\sum_{m=1}^{M}\eta_m\right)\psi_{n-2}$$
$$+\left(\sum_{m=1}^{M}\eta_m\right)^{2}\psi_{n-2}$$
$$+\left(1-\sum_{m=1}^{M}\eta_m\right)\sum_{m=1}^{M}\int_0^\infty\psi_{n-2,m}(x)dx$$
$$+\sum_{m=1}^{M}\eta_m\sum_{m=1}^{M}\int_0^\infty\psi_{n-2,m}(x)dx$$
$$+\left(1-\sum_{m=1}^{M}\eta_m\right)\psi_{n-1}+\sum_{m=1}^{M}\eta_m\psi_{n-1}$$
$$+\sum_{i=1}^{n-1}\int_0^\infty\psi_{i,M+1}(x)dx$$
$$+\sum_{m=1}^{M}\int_0^\infty\psi_{n-1,m}(x)dx+\psi_n$$
$$+\int_0^\infty\psi_{n,l}(x)dx+\psi_1\Bigg\}\Bigg/H$$
$$=\Bigg\{\sum_{m=1}^{M}\int_0^\infty\psi_{1,m}(x)dx+\psi_2$$
$$+\sum_{m=1}^{M}\int_0^\infty\psi_{2,m}(x)dx+\psi_3$$
$$+\ldots$$
$$+\sum_{m=1}^{M}\int_0^\infty\psi_{n-3,m}(x)dx+\psi_{n-2}$$
$$+\sum_{m=1}^{M}\int_0^\infty\psi_{n-2,m}(x)dx+\psi_{n-1}$$
$$+\sum_{m=1}^{M}\int_0^\infty\psi_{n-1,m}(x)dx+\psi_n$$
$$+\sum_{i=1}^{n-1}\int_0^\infty\psi_{i,M+1}(x)dx+\int_0^\infty\psi_{n,l}(x)dx+\psi_1\Bigg\}\Bigg/H$$
$$=\Bigg\{\sum_{i=1}^{n}\psi_i+\sum_{i=1}^{n-1}\sum_{m=1}^{M+1}\int_0^\infty\psi_{i,m}(x)dx$$
$$+\int_0^\infty\psi_{n,l}(x)dx\Bigg\}\Bigg/H$$
$$=\frac{1}{H}=P_1 \tag{108}$$

$$\lim_{\gamma\to0}\gamma y_{i,m}(x)=\frac{\left(\sum\limits_{m=1}^{M}\eta_m\right)^{i-1}\lambda_1\eta_m e^{-\int_0^x\mu_{i,m}(\tau)d\tau}}{H}$$
$$=P_{i,m}(x),\quad 1\le i\le n-1,\ 1\le m\le M+1 \tag{109}$$

$$\lim_{\gamma\to0}\gamma y_i=\frac{\left(\sum\limits_{m=1}^{M}\eta_m\right)^{i-1}}{\lambda_i H}=P_i,\quad 2\le i\le n-1 \tag{110}$$

$$\lim_{\gamma\to0}\gamma y_{n,l}(x)=\frac{\left(\sum\limits_{m=1}^{M}\eta_m\right)^{n-1}\lambda_1 e^{-\int_0^x\mu_{n,l}(\tau)d\tau}}{H}$$
$$=P_{n,l}(x) \tag{111}$$

Combining Theorem 4.3 with Eqs. (109)-(111), Eq. (92), we get

$$\mathbb{P}\psi(x)=P(x) \tag{112}$$

Thereby, the desired result is obtained from Theorem 3.3, Eq. (112), Theorem 4.3, Eq. (105):

$$\big\|P(\cdot,t)-P(\cdot)\big\|=\big\|\mathrm{T}(t)\psi(\cdot)-\mathbb{P}\psi(\cdot)\big\|\le\big\|\mathrm{T}(t)-\mathbb{P}\big\|\|\psi\|$$
$$\le\overline{\mathcal{M}}e^{-\delta t}\|\psi\|=\overline{\mathcal{M}}e^{-\delta t},\quad\forall t\ge0$$

V. Numerical Results

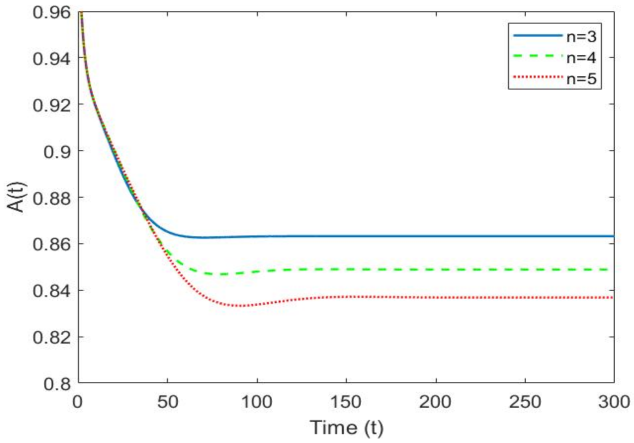
This section, we discuss the effect of system parameters on the transient reliability metrics using an industrial robotic arm for automotive manufacturing as a case study. The robotic system may experience three typical failure modes: servo motor encoder signal drift (Failure Mode 1), reducer gear micro-wear (Failure Mode 2), and control system mainboard burnout (Failure Mode 3). Their occurrence probabilities are denoted as $\eta_m(m=1,2,3)$, satisfying $\sum\limits_{m=1}^{3}\eta_m=1$.

The maintenance strategy is implemented as follows: For Failure Modes 1 and 2, Imperfect repairs are performed with repair rates $\mu_{i,m}$ $(i=1,2;m=1,2)$;for Failure Mode 3, Immediate replacement with a new robotic arm is performed at a repair (replacement) rate of $\mu_{i,3}$ $(i=1,2)$; the system has a maximum failure count of 3, triggering compulsory replacement after the third failure at a repair (replacement) rate of μ_{3l}.

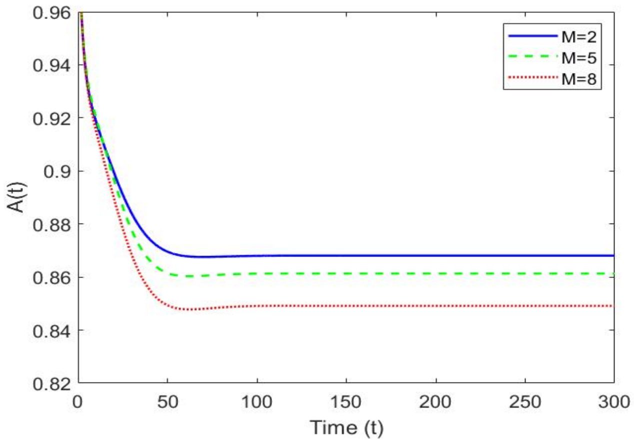
The system failure rate λ_i $(i=1,2,3)$ increases with successive repairs.

We fix the system parameters as follows: $\lambda_1=0.03,\ \lambda_2=0.06,\ \lambda_3=0.08,\ \mu_{1,1}=0.3,\ \mu_{1,2}=0.4,\mu_{1,3}=0.5,\ \mu_{2,1}=0.25,\ \mu_{2,2}=0.35,\ \mu_{2,3}=0.45,\ \mu_{3,l}=0.2,\ \eta_1=0.3,\ \eta_2=0.5,\ \eta_3=0.2$. We then discuss the effects of the variation of each parameter on the transient reliability metrics of the system, such as transient availability $A(t)$, transient failure frequency $m_f(t)$, reliability $R(t)$, and mean time to first failure $(MTTFF)$.

In Fig. 2, the variation of $A(t)$ with time t is depicted for different maximum number of failures n and numbers of failure modes M $(m = 1, \ldots, M + 1)$. From Fig. 2, it can be observed that $A(t)$ decreases rapidly over time and eventually converges to a fixed value, i.e., steady-state availability. Additionally, the $A(t)$ decreases as n and M increase, which aligns with practical scenarios.

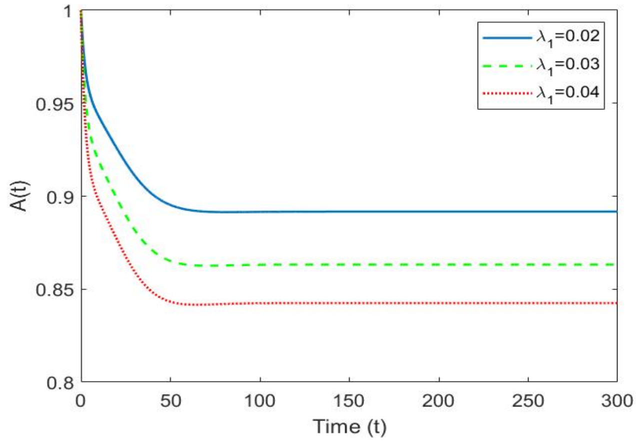


(a). $A(t)$ for various n

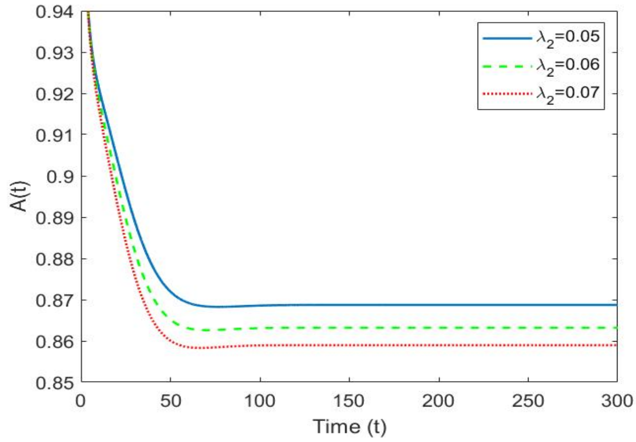


(b). $A(t)$ for various M

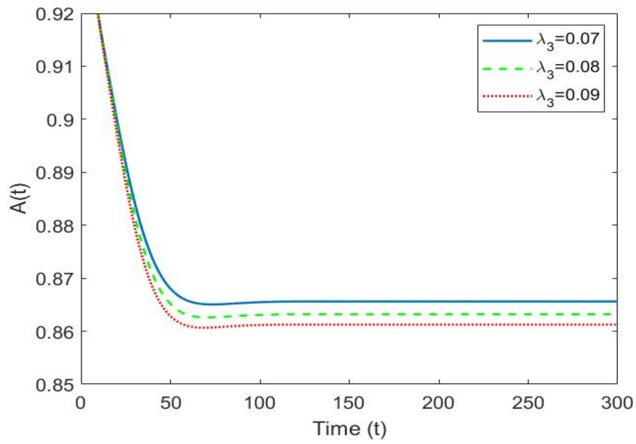
Fig. 2: Variation of $A(t)$ for n and M



(a). $A(t)$ for various λ_1



(b). $A(t)$ for various λ_2

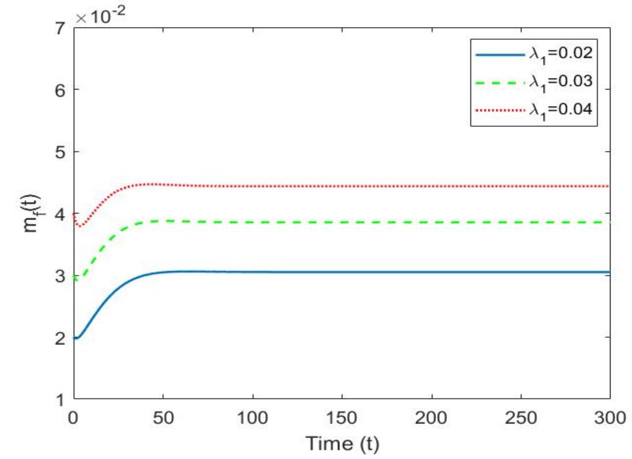


(c). $A(t)$ for various λ_3

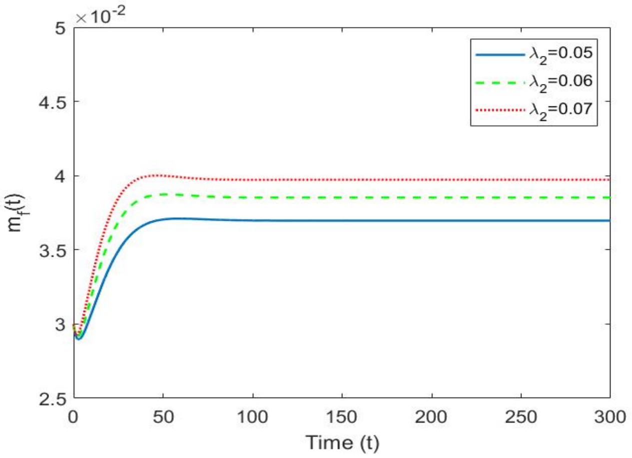
Fig. 3: Effect of λ_i $(i = 1, 2, 3)$ on $A(t)$

As shown in Fig. 3, $A(t)$ decreases rapidly over time and eventually converges to a constant value after a long period of operation. Furthermore, an increase in the failure rate λ_i $(i = 1, 2, 3)$ also leads to a gradual decrease in the $A(t)$.

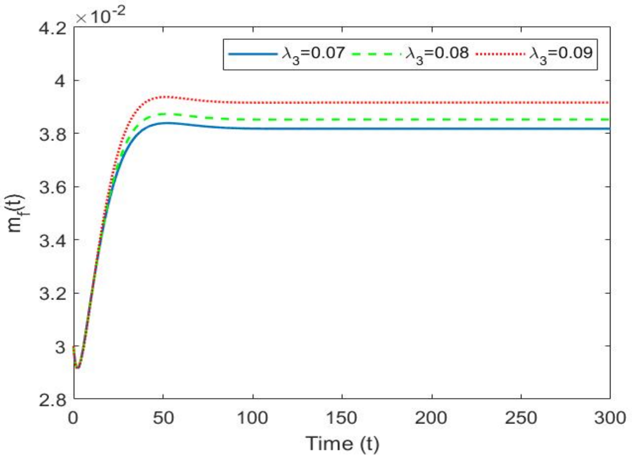
Fig. 4 illustrates that the $m_f(t)$ increases dramatically over time and eventually converges to a constant value after an extended period of operation. In addition, as the failure rate λ_i $(i = 1, 2, 3)$ increases, the $m_f(t)$ also rises correspondingly.



(a). $m_f(t)$ for various λ_1

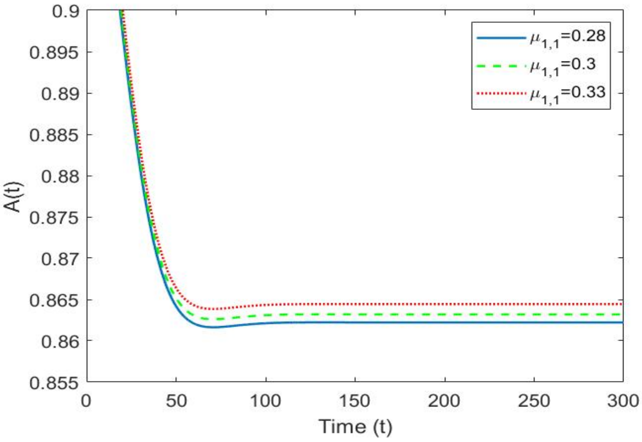


(b). $m_f(t)$ for various λ_2

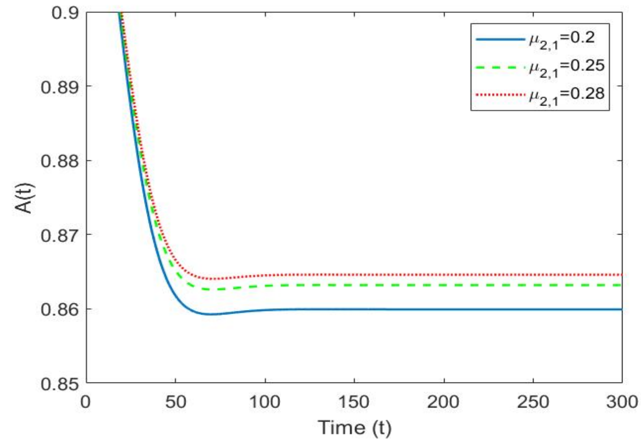


(c). $m_f(t)$ for various λ_3

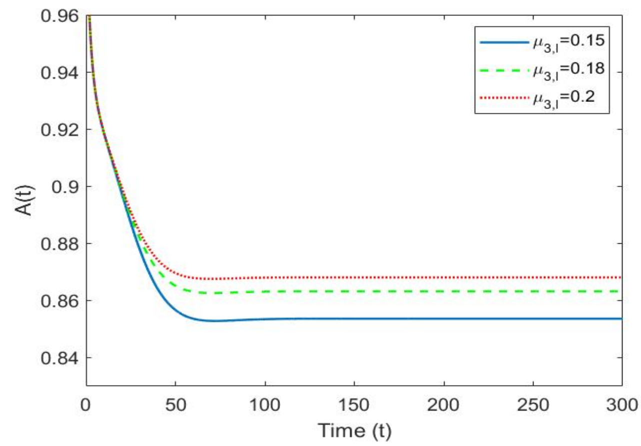
Fig. 4: Effect of λ_i $(i = 1, 2, 3)$ on $m_f(t)$



(a). $A(t)$ for various $\mu_{1,1}$



(b). $A(t)$ for various $\mu_{2,1}$

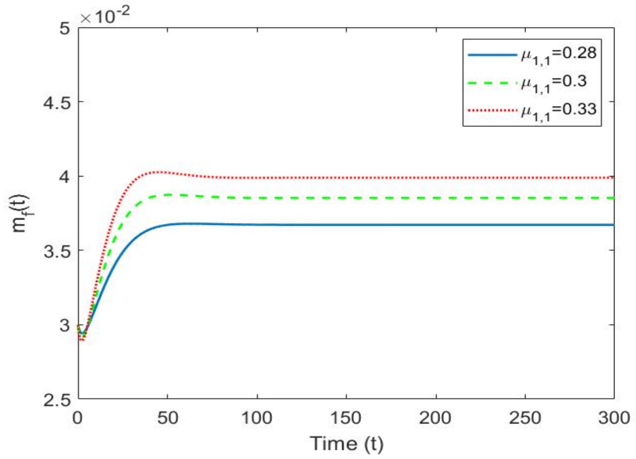


(c). $A(t)$ for various $\mu_{3,l}$

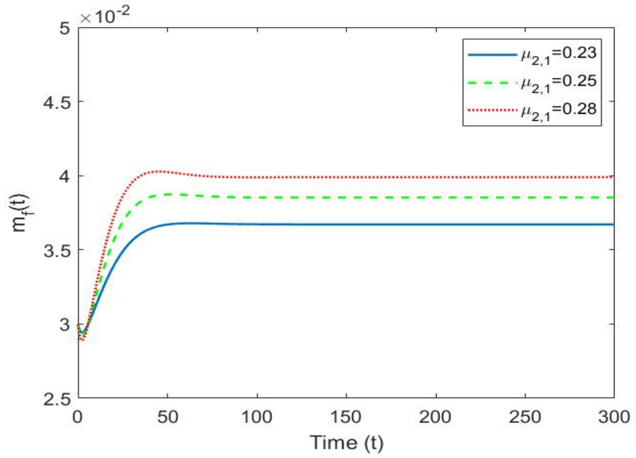
Fig. 5: Effect of $\mu_{i,1}$ $(i = 1, 2)$, $\mu_{3,l}$ on $A(t)$

Fig. 5 shows that the $A(t)$ decreases rapidly over time and eventually converges to a fixed value after a long period of operation. Furthermore, the $A(t)$ gradually increases as the repair rate increases. These trends are consistent for different $\mu_{i,m}$ $(i = 1, 2;\ m = 2, 3)$ and yield similar results.

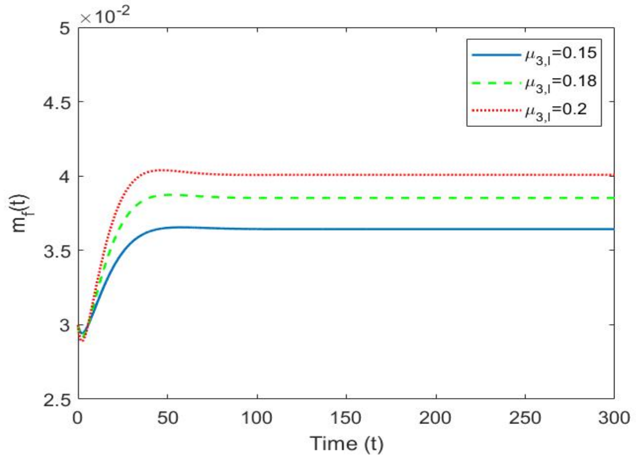
Fig. 6 demonstrates that the $m_f(t)$ increases rapidly over time and eventually converges to a fixed value after a long period of operation. Additionally, $m_f(t)$ gradually increases as the maintenance rate increases. Analogous results can be obtained for different $\mu_{i,m}$ $(i = 1, 2; m = 2, 3)$.



(a).$m_f(t)$ for various $\mu_{1,1}$



(b). $m_f(t)$ for various $\mu_{2,1}$

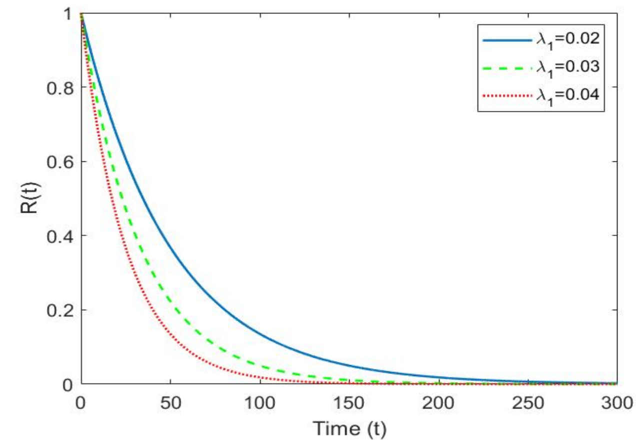


(c). $m_f(t)$ for various $\mu_{3,l}$

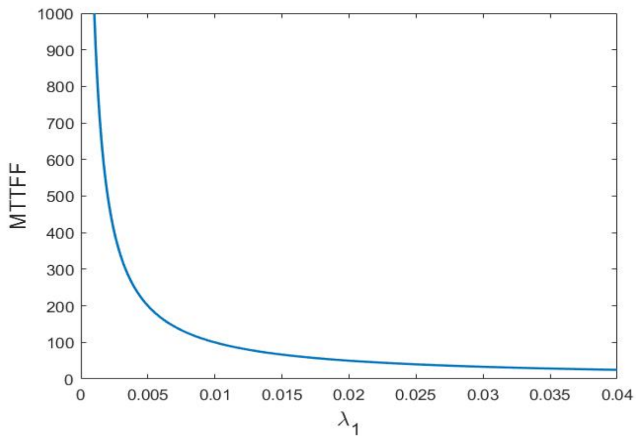
Fig. 6: Effect of $\mu_{i,1}$ $(i = 1, 2)$, $\mu_{3,l}$ on $m_f(t)$

Fig. 7 illustrates the variation of $R(t)$ with time (Fig. 7(a)) and the $MTTFF$ with λ_1 (Fig. 7(b)) for different values of λ_1. It can be observed that $R(t)$ gradually decreases and tends toward zero as time increases, with smaller values of $R(t)$ corresponding to larger values of λ_1. Additionally, $MTTFF$ gradually decreases with an increase in λ_1.

These results show that the system's reliability is primarily influenced by λ_1, with no significant correlation observed with λ_2 and λ_3. Furthermore, the instantaneous reliability indices are found to converge to a fixed value as time approaches infinity. This result further substantiates the rationality of the theoretical results presented in this paper.



(a). The variation of $R(t)$ with time



(b). The variation of $MTTFF$ with λ_1

Fig. 7: The variation of $R(t)$ with time and the variation of $MTTFF$ with λ_1

VI. Conclusion

In this paper, we considered repairable systems with MFM and IR based on the established based by the SVM. First, by applying the C_0-semigroup theory, we demonstrated that the main operator of the model generates a positively contractive strong continuous semigroup. This result ensures the existences of a unique positive TDS that satisfies the probabilistic properties of the model. Next, we showed that the semigroup generated by the main operator is quasi-compact, and that 0 is an eigenvalue of geometric multiplicity 1 for both the main operator and its conjugate operator. We further proved that all points on the imaginary axis except 0 belong to the resolvent set of the main operator. This leads to the strong convergence of the TDS of the model to its SSS, thereby validating the assumptions made in the literature [23]. Subsequently, we applied the residue theorem to derive the expression of the projection operator, which confirms the exponential convergence of the TDS to the SSS.

Additionally, we examined the specific effects of each parameter on system reliability through numerical examples. These examples provide more reliable, safe and cost-effective reliability models. In future research, we plan to further explore the asymptotic expression of the system TDS by determining the spectral distribution of the main operator. This will provide a deeper understanding of the long-term behavior of the system and potentially offer more refined results for reliability analysis.

References

[1] B. S. Dhillon, “Unified availability modeling: a redundant system with mechanical, electrical, software, human and common-cause failures,” *Microelectronics Reliability*, vol. 21, no. 5, pp. 653-659, 1981.

[2] W. K. Chung, “A reliability model for a k-out-of-n:G redundant system with multiple failure modes and common cause failures,” *Microelectronics Reliability*, vol. 27, no. 4, pp. 621-623, 1987.

[3] M. S. Moustafa, “Availability of k-out-of-n:G systems with M failure modes,” *Microelectronics Reliability*, vol. 36, no. 3, pp. 385-388, 1996.

[4] M. S. Moustafa, “Transient analysis of reliability with and without repair for k-out of-n:G systems with M failure modes,” *Reliability Engineering and System Safety*, vol. 59, no. 3, pp. 317-320, 1998.

[5] M. Jain and G. C. Sharma, “Reliability analysis of k-out-of-n:G machining systems with mixed spares and multiple modes of failure,” *International Journal of Engineering Transactions B Applications*, vol. 20, no. 3, pp. 243-250, 2007.

[6] A. V. Nikolov, “K-out-of-n:G system with multiple failure modes and single repair facility,” *Asian Journal of Information Technology*, vol. 10, no. 2, pp. 55-59, 2011.

[7] Q. Qiu and L. Cui, “Availability analysis for periodically inspected systems subject to multiple failure modes,” *International Journal of Systems Science: Operations and Logistics*, vol. 6, no. 3, pp. 258-271, 2019.

[8] R. Subramanian and R. Natarajan, “A 2-unit system, with “imperfect” repair,” *Operations*, vol. 11, no. 4, pp. 617-622, 1980.

[9] A. Biswas and J. Sarkar, “Availability of a system maintained through several imperfect repairs before a replacement or a perfect repair,” *Statistics and Probability Letters*, vol. 50, no. 2, pp. 105-114, 2000.

[10] M. Hajeeh, “Maintenance models for a repairable system,” *Economic Quality Control*, vol. 19, no. 1, pp. 107-114, 2004.

[11] M. Muhammad, M. A. A. Majid and A. A. Mokhtar, “Reliability evaluation for a multi-state system subject to imperfect repair and maintenance,” *International Journal of Engineering and Technology*, vol. 10, no. 1, pp. 80-85, 2010.

[12] M. A. Hajeeh, “Availability of a system with different repair options,” *International Journal of Mathematics in Operational Research*, vol. 4, no. 1, pp. 41-55, 2012.

[13] J. Madhu and S. V. Pratap, “Availability analysis of fault tolerant degraded machining system with imperfect repair and multi-type failures,” *International Journal of Applied and Computational Mathematics*, vol. 9, no. 3, pp. 1-20, 2023.

[14] R. Barlow and L. Hunter, “Optimum preventive maintenance policies,” *Operations Research*, vol. 9, no. 3, pp. 1-20, 2023.

[15] M. Brown and F. Proschan, “Imperfect repair,” *Journal of Applied Probability*, vol. 20, no. 4, pp. 851-859, 1983.

[16] Y. Lin, “Geometric processes and replacement problem,” *Acta Mathematicae Applicatae Sinica. English series*, vol. 4, no. 4, pp. 366-377, 1988.

[17] H. Wang and H. Pham, “Availability and maintenance of series systems subject to imperfect repair and correlated failure and repair,” *European Journal of Operational Research*, vol. 174, no. 3, pp. 1706-1722, 2006.

[18] Y. Chen, X. Meng and S. Chen, “Reliability analysis of a cold standby system with imperfect repair and under poisson shocks,” *Mathematical Problems in Engineering*, vol. 2014, no. 1, 2014.

[19] Q. L. Dong, L. R. Cui LR and H. D. Gao, “A bivariate replacement policy for an imperfect repair system based on geometric processes,” *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 233, no. 4, pp. 670-681, 2019.

[20] S. Haugen, A. Barros, C. V. Gulijk, et al. “Two imperfect repair models for a gamma deteriorating system: A comparison,” *Safety and Reliability – Safe Societies in a Changing World*, pp. 519-527, 2018.

[21] D. S. Brito, V. L. D. Tomazella, P. H. Ferreira, et al. “Reliability analysis of multiple repairable systems under imperfect repair and unobserved heterogeneity,” *Quality and Reliability Engineering International*, vol. 40, no. 7, pp. 3888–3912, 2024.

[22] M. Hajeeh, “Analysis of a repairable system subjected to multiple modes of failure,” *Stochastics and Quality Control* , vol. 19, no. 2, pp. 167-173, 2004.

[23] A. V. Nikolov, “Availability of a system with multiple failure modes and imperfect repairs,” *Economic Quality Control*, vol. 26, no. 2, pp. 99-106, 2011.

[24] Q. Qiu, B. Liu, C. Lin, et al. “Availability analysis and maintenance optimization for multiple failure mode systems considering imperfect repair,” *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 235, no. 6, pp. 982-997, 2021.

[25] E. Kasim and G. Gupur, “Dynamic analysis of a complex system under preemptive repeat repair discipline,” *Bound Value Prob*, vol. 2020, no. 1, pp. 1-37, 2020.

[26] A. Yumaier and E. Kasim, “Dynamic analysis of the multi-state reliability system with priority repair discipline,” *Acta Mathematicae Applicatae Sinica, English Series*, vol. 40, no. 3, pp. 665-694, 2024.

[27] A. Mijit, “Existence and asymptotic behavior of the time-dependent solution of a system composed of two identical units,” *IAENG International Journal of Applied Mathematics*, vol. 55, no. 2, pp. 338-347, 2025.

[28] R. A. Adams, “Sobolev spaces,” *Academic Press, New York*, 1975.

[29] G. Gupur, “Functional analysis methods for reliability models,” *Basel: Springer*, 2011.

[30] R. Nagel, W. Arendt, A. Grabosch, et al. “One parameter semigroups of positive operators,” *Berlin: Springer*, 1986.

[31] H. O. Fattorini, “The cauchy problem,” *Addison Wesley, Massachusetts*, 1984

[32] K. J. Engel, R. Nagel and S. Brendle, “One-parameter semigroups for linear evolution equations,” *New York: Springer*, 2000.