Square-mean Almost Periodic Solutions in Shifts Delta(+/-) of Nonautonomous Semilinear Stochastic Dynamic Equations on Time Scales

Meng Hu, Pingli Xie, Lili Wang

***Abstract*—In this paper, we first define the square-mean almost periodic ($\Delta-$ almost periodic) stochastic process in shifts $\delta_\pm$ on time scales, then the existence of square-mean almost periodic solution in shifts $\delta_\pm$ to a class of nonautonomous stochastic dynamic equations is studied. Using the theory of calculus on time scales and the Acquistapace-Terreni conditions, sufficient conditions for the existence and uniqueness of square-mean almost periodic mild solution in shifts $\delta_\pm$ to those equations on a real separable Hilbert space are established. Finally, two examples are given to illustrate the feasibility and effective of the results.**

***Index Terms*—Stochastic dynamic equation; Square-mean almost periodic solution; Shift operator; Mild solution; Time scale.**

I. Introduction

A Time scale is a nonempty closed subset of $\mathbb{R}$. In recent years, with the development of the theory of time scales (see [1,2]), the existence of almost periodic solutions of dynamic equations on time scales received many researchers' special attention; see, for example, [3-6]. In [7], with the aid of the shift operators $\delta_\pm$, we studied almost periodic dynamic equations in shifts $\delta_\pm$ on time scales, and established the existence and uniqueness theorem of almost periodic solution in shifts $\delta_\pm$ on time scales. However, almost periodicity in shifts $\delta_\pm$ of stochastic dynamic equations on time scales has not been studied so far.

In this paper, we first define the square-mean almost periodic stochastic process in shifts $\delta_\pm$ on time scales, and then we shall study the existence and uniqueness of square-mean almost periodic mild solution in shifts $\delta_\pm$ of the following nonautonomous semilinear stochastic dynamic equations on time scales:

$$\begin{aligned}\Delta x(t) &= A(t)x(t)\Delta t + f(t, x(t))\Delta t \\ &\quad + g(t, x(t))\Delta w(t), \qquad (1)\end{aligned}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an almost periodic time scale in shifts $\delta_\pm$; w is a Wiener process.

Manuscript received February 10, 2025; revised May 4, 2025.

This work was supported in part by the Scientific and Technological Breakthrough Foundation of Henan Province under Grant 252102320118, Key Scientific Research Projects of Universities in Henan Province under Grant 25B110007, and Natural Science Foundation of Henan Province under Grant 252300421484.

M. Hu is an associate professor of School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China (corresponding author to provide phone: +86-0372-2900038; fax: +86-0372-2900038; e-mail: humeng2001@126.com).

P. Xie is an associate professor of School of Mathematics and Statistics, Henan University of Technology, Zhengzhou 450001, China (e-mail: xiepl_03@163.com).

L. Wang is a lecturer of School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China (e-mail: ay_wanglili@126.com).

II. Preliminaries

The theory of time scales and the theory of dynamic equations on time scales, see [1].

Assume that $(\mathbb{H}_1, \|\cdot\|_{\mathbb{H}_1})$ and $(\mathbb{H}_2, \|\cdot\|_{\mathbb{H}_2})$ are real separable Hilbert spaces, $(\Omega, \mathcal{F}, P)$ is a probability space, and $L_2(\mathbb{H}_1, \mathbb{H}_2)$ is a space of all Hilbert-Schmidt operators from $\mathbb{H}_1$ to $\mathbb{H}_2$, equipped with the Hilbert-Schmidt norm $\|\cdot\|_2$.

For a symmetric nonnegative operator $Q \in L_2(\mathbb{H}_1, \mathbb{H}_2)$ with finite trace, we assume that $\{w(t), t \in \mathbb{T}\}$ is a Q-Wiener process defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{H}_1$, and $\mathcal{F}_t = \sigma\{w(s), s \leq t\}$.

The collection of all strongly measurable, square-integrable $\mathbb{H}$-valued random variables, denoted by $L^2(P, \mathbb{H})$, and $L^2(P, \mathbb{H})$ is a Banach space equipped with the norm $\|x\|_{L^2(P,\mathbb{H})} = (\mathbf{E}\|x\|^2)^{1/2}$.

Let $\mathbb{H}_0 = Q^{1/2}K$, and $L_2^0 = L_2(\mathbb{H}_0, \mathbb{H})$ with respect to the norm

$$\|z\|_{L_2^0}^2 = \|zQ^{1/2}\|_2^2 = \mathrm{Trace}(zQz^*).$$

Assume that $A(t) : D(A(t)) \subset L^2(P; \mathbb{H}) \to L^2(P; \mathbb{H})$ is a family of densely defined closed linear operators on a common domain $D = D(A(t))$, which is independent of t and dense in $L^2(P; \mathbb{H})$, and $F : \mathbb{T} \times L^2(P; \mathbb{H}) \to L^2(P; \mathbb{H})$ and $G : \mathbb{T} \times L^2(P; \mathbb{H}) \to L^2(P; L_2^0)$ are jointly continuous functions.

Let $\mathbb{T}^*$ is a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ is a fixed number, define operators $\delta_\pm : [t_0, +\infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, +\infty)_\mathbb{T}$ in $\delta_\pm(s, t)$ is called the shift size. The value $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_\pm := \{(s, t) \in [t_0, +\infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\pm(s, t) \in \mathbb{T}^*\}$$

are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0, +\infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist; see [8].

Definition 1. ([7]) *Let $\mathbb{T}$ is a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be almost periodic in shifts $\delta_\pm$ if there exists $p \in (t_0, +\infty)_{\mathbb{T}^*}$ such that $(p, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$, that is,*

$$\{p \in (t_0, +\infty)_{\mathbb{T}^*} : (p, t) \in \mathbb{D}_\pm, \forall t \in \mathbb{T}^*\} \neq \emptyset.$$

Let $(\mathbb{B}, \|\cdot\|)$ is a Banach space.

Definition 2. *A stochastic process $x : \mathbb{T} \to L^2(P;\mathbb{B})$ is said to be continuous if*

$$\lim_{t\to s} \boldsymbol{E}\|x(t) - x(s)\|^2 = 0.$$

Definition 3. *(Square-mean almost periodic stochastic process in shifts $\delta_\pm$) Let $\mathbb{T}$ is an almost periodic time scale in shifts $\delta_\pm$. A continuous stochastic process $x : \mathbb{T}^* \to L^2(P;\mathbb{B})$ is said to be square-mean almost periodic in shifts $\delta_\pm$ if the ε-translation set of x*

$$E\{\varepsilon, x\} = \{(p,t) \in \mathbb{D}_\pm : \sup_{t\in\mathbb{T}} \boldsymbol{E}\|x(\delta_\pm^p(t)) - x(t)\|^2 < \varepsilon, \forall t \in \mathbb{T}^*\}$$

is a relatively dense set in $\mathbb{T}^*$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > t_0$, $(l(\varepsilon), t) \in \mathbb{D}_\pm$, such that in any interval $[t, \delta_+^{l(\varepsilon)}(t)]([\delta_-^{l(\varepsilon)}(t), t])$, there exists at least a $p \in E\{\varepsilon, x\}$ such that

$$\sup_{t\in\mathbb{T}} \boldsymbol{E}\|x(\delta_\pm^p(t)) - x(t)\|^2 < \varepsilon,$$

where $\delta_\pm^p(t) := \delta_\pm(p,t)$, p is called the ε-translation number of x, $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, x\}$.

Definition 4. *(square-mean Δ-almost periodic stochastic process in shifts $\delta_\pm$) Let $\mathbb{T}$ is an almost periodic time scale in shifts $\delta_\pm$. A continuous stochastic process $x : \mathbb{T}^* \to L^2(P;\mathbb{B})$ is said to be square-mean Δ-almost periodic in shifts $\delta_\pm$ if the ε-translation set of x*

$$E\{\varepsilon, x\} = \{(p,t) \in \mathbb{D}_\pm : \sup_{t\in\mathbb{T}} \boldsymbol{E}\|x(\delta_\pm^p(t))\delta_\pm^{\Delta p}(t) - x(t)\|^2 < \varepsilon, \forall t \in \mathbb{T}^*\}$$

is a relatively dense set in $\mathbb{T}^*$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > t_0$,, $(l(\varepsilon), t) \in \mathbb{D}_\pm$, such that in any interval $[t, \delta_+^{l(\varepsilon)}(t)]([\delta_-^{l(\varepsilon)}(t), t])$, there exists at least a $p \in E\{\varepsilon, x\}$ such that

$$\sup_{t\in\mathbb{T}} \boldsymbol{E}\|x(\delta_\pm^p(t))\delta_\pm^{\Delta p}(t) - x(t)\|^2 < \varepsilon,$$

where $\delta_\pm^p(t) := \delta_\pm(p,t)$, p is called the ε-translation number of x, $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, x\}$.

The collection of all stochastic processes $x : \mathbb{T} \to L^2(P;\mathbb{B})$ which are square-mean almost periodic in shifts $\delta_\pm$ is then denoted by $APS(\mathbb{R}; L^2(P;\mathbb{B}))$. The space $APS(\mathbb{R}; L^2(P;\mathbb{B}))$ of square-mean almost periodic processes in shifts $\delta_\pm$ equipped with the norm

$$\|x\|_\infty = \sup_{t\in\mathbb{T}} (\mathbf{E}\|x(t)\|^2)^{1/2}$$

is a Banach space.

Lemma 1. *If x belongs to $APS(\mathbb{R}; L^2(P;\mathbb{B}))$, then*

(i) the mapping $t \to \boldsymbol{E}\|x(t)\|^2$ is uniformly continuous;
(ii) there exists a constant $\widetilde{M} > 0$ such that $\boldsymbol{E}\|x(t)\|^2 \le \widetilde{M}$, for all $t \in \mathbb{T}$.

Let $(\mathbb{B}_1, \|\cdot\|_1)$, $(\mathbb{B}_2, \|\cdot\|_2)$ are Banach spaces, and $L^2(P;\mathbb{B}_1)$, $L^2(P;\mathbb{B}_2)$ are their corresponding L^2-spaces, respectively.

Lemma 2. *Let $f : \mathbb{T} \times L^2(P;\mathbb{B}_1) \to L^2(P;\mathbb{B}_2)$, $(t,x) \to f(t,x)$ is a square-mean almost periodic function in shifts $\delta_\pm$ in $t \in \mathbb{T}$ uniformly in $x \in \mathbb{S}$ ($\mathbb{S} \subset L^2(P;\mathbb{H})$ is a compact subspace). Moreover, there exists a positive constant $\widehat{M} > 0$ such that*

$$\boldsymbol{E}\|f(t,x) - f(t,y)\|^2 \le \widehat{M}\boldsymbol{E}\|x - y\|_1^2,$$

for all $x, y \in L^2(P;\mathbb{B}_1)$, and for each $t \in \mathbb{T}$. Then for any square-mean almost periodic in shifts $\delta_\pm$ process $\Phi : \mathbb{R} \to L^2(P;\mathbb{B}_1)$, the stochastic process $t \to f(t, \Phi(t))$ is square-mean almost periodic in shifts $\delta_\pm$.

Lemma 3. ([1]) *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b g(s)\nu^\Delta(s)\Delta s = \int_{\nu(a)}^{\nu(b)} g(\nu^{-1}(s))\tilde{\Delta} s.$$

III. Main results

In this section, we shall study the existence of mild solutions of equation (1). Firstly, we make the following assumptions:

(H_1) Assume that the equation

$$x^\Delta(t) = A(t)x(t), t \ge s,$$
$$x(s) = \phi \in L^2(P;\mathbb{H}),$$

has an associated evolution family of operators $\{X(t,s) : t \ge s \text{ with } t, s \in \mathbb{T}\}$, which is uniformly exponentially stable, that is, there exist positive constants $M, \alpha > 0$, such that

$$\|X(\delta_\pm^p(t), \delta_\pm^p(s))\| \le Me_{\ominus\alpha}(t, \sigma(s)), \forall t \ge s.$$

(H_2) The functions $f : \mathbb{T} \times L^2(P;\mathbb{H}) \to L^2(P;\mathbb{H})$, $(t,x) \to f(t,x)$ and $g : \mathbb{T} \times L^2(P;\mathbb{H}) \to L^2(P;L_2^0)$, $(t,y) \to g(t,y)$ are square-mean Δ-almost periodic functions in shifts $\delta_\pm$ in $t \in \mathbb{T}$ uniformly in $x \in \mathbb{S}$ ($\mathbb{S} \subset L^2(P;\mathbb{H})$ is a compact subspace), and

$$\mathbf{E}\|f(\delta_\pm^p(t), x(\delta_\pm^p(t)))\delta_\pm^{\Delta p}(t) - f(t, x(t))\|^2 < \eta;$$
$$\mathbf{E}\|g(\delta_\pm^p(t), x(\delta_\pm^p(t)))\delta_\pm^{\Delta p}(t) - g(t, x(t))\|_{L_2^0}^2 < \eta;$$

where $\eta \to 0$ as $\varepsilon \to 0$. Moreover, there exist positive constants $K_1, K_2 > 0$ such that

$$\sup_{t\in\mathbb{T}} \mathbf{E}\|f(t, x(t))\|^2 \le K_1;$$
$$\sup_{t\in\mathbb{T}} \mathbf{E}\|g(t, x(t))\|_{L_2^0}^2 \le K_2.$$

(H_3) The functions f and g which have been defined in (H_2) are Lipschitz in the sense

$$\mathbf{E}\|f(t,x) - f(t,y)\|^2 \le L^f\mathbf{E}\|x - y\|^2,$$
$$\mathbf{E}\|g(t,x) - g(t,y)\|_{L_2^0}^2 \le L^g\mathbf{E}\|x - y\|^2,$$

for all $t \in \mathbb{T}, x, y \in L^2(P;\mathbb{H})$, and $L^f, L^g > 0$ are positive constants.

Lemma 4. *Assume that $A(t)$ satisfies the Acquistapace-Terreni conditions, $X(t,s)$ is exponentially stable, then for any $\varepsilon > 0$, there exists a constant $l(\varepsilon) > t_0$ (t_0 is the initial point), such that in any interval $[t, \delta_+^{l(\varepsilon)}(t)]([\delta_-^{l(\varepsilon)}(t), t])$, there exists at least a $p \in E\{\varepsilon, X\}$ ($E\{\varepsilon, X\}$ is a relatively dense set) such that*

$$\|X(\delta_\pm^p(t), \delta_\pm^p(s)) - X(t,s)\| < \varepsilon e_{\ominus\frac{\alpha}{2}}(t, \sigma(s)),$$

for all $t - s \geq \varepsilon$.

Theorem 1. *Assume that the graininess function* $\mu(t)$ *is bounded on time scale* $\mathbb{T}$, $(H_1) - (H_3)$ *and the conditions of Lemma 4 hold, and*

$$\lambda = \left(\frac{2M^2L^f}{\tilde{\alpha}^2} + \frac{2trQM^2L^g}{\tilde{\alpha}}\right)^{\frac{1}{2}} < 1,$$

where $\tilde{\alpha} = \inf\{-\ominus\alpha | t \in \mathbb{T}\}$, *then equation (1) has a unique square-mean almost periodic mild solution in shifts* $\delta_{\pm}$, *and*

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} X(t,s)f(s,x(s))\Delta s \\ &\quad + \int_{-\infty}^{t} X(t,s)g(s,x(s))\Delta w(s), t \in \mathbb{T}. \end{aligned}$$

Proof: Let $x(t) = \Phi x(t) + \Psi x(t)$, and

$$\Phi x(t) = \int_{-\infty}^{t} X(t,s)f(s,x(s))\Delta s,$$
$$\Psi x(t) = \int_{-\infty}^{t} X(t,s)g(s,x(s))\Delta w(s).$$

Now, we prove that Φx and Ψx are square-mean almost periodic in shifts $\delta_{\pm}$ whenever x is almost periodic in shifts $\delta_{\pm}$. We first consider the case of Φx. In fact,

$$\begin{aligned} & \|\Phi x(\delta_{\pm}^{p}(t)) - \Phi x(t)\| \\ = & \left\| \int_{-\infty}^{\delta_{\pm}^{p}(t)} X(\delta_{\pm}^{p}(t),s)f(s,x(s))\Delta s \right. \\ & \left. - \int_{-\infty}^{t} X(t,s)f(s,x(s))\Delta s \right\|. \end{aligned}$$

Let $g(s) = X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s))f(\delta_{\pm}^{p}(s), x(\delta_{\pm}^{p}(s)))$ and $\nu(t) = \delta_{\pm}^{p}(t)$, by Lemma 3,

$$\begin{aligned} & \int_{-\infty}^{\delta_{\pm}^{p}(t)} X(\delta_{\pm}^{p}(t),s)f(s,x(s))\Delta s \\ = & \int_{-\infty}^{\nu(t)} g(\nu^{-1}(s))\Delta s \\ = & \int_{-\infty}^{t} g(s)\nu^{\Delta}(s)\Delta s \\ = & \int_{-\infty}^{t} X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s))f(\delta_{\pm}^{p}(s), x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s)\Delta s. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|\Phi x(\delta_{\pm}^{p}(t)) - \Phi x(t)\| \\ = & \left\| \int_{-\infty}^{t} X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s))f(\delta_{\pm}^{p}(s), x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s)\Delta s \right. \\ & \left. - \int_{-\infty}^{t} X(t,s)f(s,x(s))\Delta s \right\| \\ = & \left\| \int_{-\infty}^{t} X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s))[f(\delta_{\pm}^{p}(s), x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s) \right. \\ & -f(s,x(s))]\Delta s \\ & \left. + \int_{-\infty}^{t} [X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s)) - X(t,s)]f(s,x(s))\Delta s \right\| \\ \leq & \left\| \int_{-\infty}^{t} X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s))[f(\delta_{\pm}^{p}(s), x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s) \right. \\ & \left. -f(s,x(s))]\Delta s \right\| \\ & + \left\| \left(\int_{-\infty}^{t-\varepsilon} + \int_{t-\varepsilon}^{t} \right) [X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s)) \right. \\ & \left. -X(t,s)]f(s,x(s))\Delta s \right\|. \end{aligned}$$

Since $(a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2$, then

$$\begin{aligned} & \mathbf{E}\|\Phi x(\delta_{\pm}^{p}(t)) - \Phi x(t)\|^2 \\ \leq & 3\mathbf{E}\Big[\int_{-\infty}^{t} \|X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s))\| \\ & \times \|f(\delta_{\pm}^{p}(s), x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s) - f(s,x(s))\|\Delta s \Big]^2 \\ & +3\mathbf{E}\Big[\int_{-\infty}^{t-\varepsilon} \|X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s)) - X(t,s)\| \\ & \times \|f(s,x(s))\|\Delta s \Big]^2 \\ & +3\mathbf{E}\Big[\int_{t-\varepsilon}^{t} \|X(\delta_{\pm}^{p}(t), \delta_{\pm}^{p}(s)) - X(t,s)\| \\ & \times \|f(s,x(s))\|\Delta s \Big]^2 \\ \leq & 3M^2\mathbf{E}\Big[\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s)) \\ & \times \|f(\delta_{\pm}^{p}(s), x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s) - f(s,x(s))\|\Delta s \Big]^2 \\ & +3\varepsilon^2\mathbf{E}\Big[\int_{-\infty}^{t-\varepsilon} e_{\ominus\frac{\alpha}{2}}(t,\sigma(s))\|f(s,x(s))\|\Delta s \Big]^2 \\ & +3M^2\mathbf{E}\Big[\int_{t-\varepsilon}^{t} 2e_{\ominus\alpha}(t,\sigma(s))\|f(s,x(s))\|\Delta s \Big]^2. \end{aligned}$$

Using Cauchy-Schwarz inequality

$$\begin{aligned} & \mathbf{E}\|\Phi x(\delta_{\pm}^{p}(t)) - \Phi x(t)\|^2 \\ \leq & 3M^2\left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s \right) \\ & \times \Big[\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\mathbf{E}\|f(\delta_{\pm}^{p}(s), x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s) \\ & -f(s,x(s))\|^2\Delta s \Big] \\ & +3\varepsilon^2\left(\int_{-\infty}^{t-\varepsilon} e_{\ominus\frac{\alpha}{2}}(t,\sigma(s))\Delta s \right) \\ & \times \Big[\int_{-\infty}^{t-\varepsilon} e_{\ominus\frac{\alpha}{2}}(t,\sigma(s))\mathbf{E}\|f(s,x(s))\|^2\Delta s \Big] \\ & +12M^2\left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s \right) \\ & \times \Big[\int_{t-\varepsilon}^{t} e_{\ominus\alpha}(t,\sigma(s)) \\ & \times \mathbf{E}\|f(s,x(s))\|^2\Delta s \Big] \\ \leq & 3M^2\left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s \right)^2 \\ & \times \sup_{s\in\mathbb{T}} \mathbf{E}\|f(\delta_{\pm}^{p}(s), x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s) - f(s,x(s))\|^2 \\ & +3\varepsilon^2\left(\int_{-\infty}^{t-\varepsilon} e_{\ominus\frac{\alpha}{2}}(t,\sigma(s))\Delta s \right)^2 \sup_{s\in\mathbb{T}} \mathbf{E}\|f(s,x(s))\|^2 \end{aligned}$$

$$
\begin{aligned}
&+12M^2\left(\int_{t-\varepsilon}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right)^2\\
&\times \sup_{s\in\mathbb{T}}\mathbf{E}\|f(s,x(s))\|^2\\
\leq\;& \frac{3M^2\eta}{\tilde{\alpha}^2}+\frac{12\varepsilon^2K_1}{\tilde{\alpha}^2}+12M^2\varepsilon^2K_1,
\end{aligned}
$$

that is, Φx is square-mean almost periodic in shifts $\delta_{\pm}$.

Next, we consider the case of Ψx. Let $\tilde{w}(s)=w(\delta_{\pm}^{s}(t))-w(s)$, $\tilde{w}$ is also a Wiener process and has the same distribution as w. Then

$$
\begin{aligned}
&\mathbf{E}\|\Psi x(\delta_{\pm}^{p}(t))-\Psi x(t)\|^2\\
=\;& \Big\|\int_{-\infty}^{t} X(\delta_{\pm}^{p}(t),\delta_{\pm}^{p}(s))\\
&\times g(\delta_{\pm}^{p}(s),x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s)\Delta\tilde{w}(s)\\
&-\int_{-\infty}^{t} X(t,s)g(s,x(s))\Delta\tilde{w}(s)\Big\|^2\\
\leq\;& 3\mathbf{E}\Big[\int_{-\infty}^{t}\|X(\delta_{\pm}^{p}(t),\delta_{\pm}^{p}(s))\|\\
&\times\|g(\delta_{\pm}^{p}(s),x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s)\\
&-g(s,x(s))\|\Delta\tilde{w}(s)\Big]^2\\
&+3\mathbf{E}\Big[\int_{-\infty}^{t-\varepsilon}\|X(\delta_{\pm}^{p}(t),\delta_{\pm}^{p}(s))-X(t,s)\|\\
&\times\|g(s,x(s))\|\Delta\tilde{w}(s)\Big]^2\\
&+3\mathbf{E}\Big[\int_{t-\varepsilon}^{t}\|X(\delta_{\pm}^{p}(t),\delta_{\pm}^{p}(s))-X(t,s)\|\\
&\times\|g(s,x(s))\|\Delta\tilde{w}(s)\Big]^2.
\end{aligned}
$$

Using an estimate on the Ito integral,

$$
\begin{aligned}
&\mathbf{E}\|\Psi x(\delta_{\pm}^{p}(t))-\Psi x(t)\|^2\\
\leq\;& 3trQ\Big[\int_{-\infty}^{t}\|X(\delta_{\pm}^{p}(t),\delta_{\pm}^{p}(s))\|^2\\
&\times\mathbf{E}\|g(\delta_{\pm}^{p}(s),x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s)-g(s,x(s))\|_{L_2^0}^2\Delta s\Big]\\
&+3trQ\Big[\int_{-\infty}^{t-\varepsilon}\|X(\delta_{\pm}^{p}(t),\delta_{\pm}^{p}(s))-X(t,s)\|^2\\
&\times\mathbf{E}\|g(s,x(s))\|_{L_2^0}^2\Delta s\Big]\\
&+3trQ\Big[\int_{t-\varepsilon}^{t}\|X(\delta_{\pm}^{p}(t),\delta_{\pm}^{p}(s))-X(t,s)\|^2\\
&\times\mathbf{E}\|g(s,x(s))\|_{L_2^0}^2\Delta s\Big]\\
\leq\;& 3trQM^2\left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right)\\
&\times\sup_{s\in\mathbb{T}}\mathbf{E}\|g(\delta_{\pm}^{p}(s),x(\delta_{\pm}^{p}(s)))\delta_{\pm}^{\Delta p}(s)-g(s,x(s))\|_{L_2^0}^2\\
&+3trQ\varepsilon^2\left(\int_{-\infty}^{t-\varepsilon} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right)\\
&\times\sup_{s\in\mathbb{T}}\mathbf{E}\|g(s,x(s))\|_{L_2^0}^2\\
&+6trQ\varepsilon^2\left(\int_{t-\varepsilon}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right)\\
&\times\sup_{s\in\mathbb{T}}\mathbf{E}\|g(s,x(s))\|_{L_2^0}^2\\
\leq\;& \frac{3trQM^2\eta}{\tilde{\alpha}}+\frac{3trQ\varepsilon^2K_2}{\tilde{\alpha}}+6trQ\varepsilon^3K_2,
\end{aligned}
$$

that is, Ψx is square-mean almost periodic in shifts $\delta_{\pm}$.

Define

$$
\begin{aligned}
\Gamma x(t)\;=\;&\int_{-\infty}^{t} X(t,s)f(s,x(s))\Delta s\\
&-\int_{-\infty}^{t} X(t,s)g(s,x(s))\Delta w(s),
\end{aligned}
$$

then Γ has a unique fixed point. In fact

$$
\begin{aligned}
&\|\Gamma x(t)-\Gamma y(t)\|\\
=\;&\Big\|\int_{-\infty}^{t} X(t,s)(f(s,x(s))-f(s,y(s)))\Delta s\\
&-\int_{-\infty}^{t} X(t,s)(g(s,x(s))-g(s,y(s)))\Delta w(s)\Big\|\\
\leq\;& M\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\|f(s,x(s))-f(s,y(s))\|\Delta s\\
&+\Big\|\int_{-\infty}^{t} X(t,s)(g(s,x(s))-g(s,y(s)))\Delta w(s)\Big\|.
\end{aligned}
$$

Since $(a+b)^2\leq 2a^2+2b^2$, then

$$
\begin{aligned}
&\mathbf{E}\|\Gamma x(t)-\Gamma y(t)\|^2\\
\leq\;& 2M^2\mathbf{E}\Big(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\|f(s,x(s))\\
&-f(s,y(s))\|\Delta s\Big)^2\\
&+2\mathbf{E}\Big(\Big\|\int_{-\infty}^{t} X(t,s)(g(s,x(s))\\
&-g(s,y(s)))\Delta w(s)\Big\|\Big)^2. \qquad (2)
\end{aligned}
$$

Now, we evaluate the right-hand side of (2). Firstly,

$$
\begin{aligned}
&\mathbf{E}\left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\|f(s,x(s))-f(s,y(s))\|\Delta s\right)^2\\
\leq\;& \mathbf{E}\Big[\left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right)\Big(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\\
&\times\|f(s,x(s))-f(s,y(s))\|^2\Delta s\Big)\Big]\\
\leq\;& \left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right)\Big(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\\
&\times\mathbf{E}\|f(s,x(s))-f(s,y(s))\|^2\Delta s\Big)\\
\leq\;& L^f\left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right)\Big(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\\
&\times\mathbf{E}\|x(s)-y(s)\|^2\Delta s\Big)\\
\leq\;& L^f\left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right)^2\\
&\times\sup_{t\in\mathbb{T}}\mathbf{E}\|x(t)-y(t)\|^2\\
\leq\;& L^f\left(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right)^2\|x-y\|_{\infty}^2\\
\leq\;& \frac{L^f}{\tilde{\alpha}^2}\|x-y\|_{\infty}^2.
\end{aligned}
$$

Secondly,

$$\begin{aligned}
&\mathbf{E}\bigg(\bigg\|\int_{-\infty}^{t} X(t,s)(g(s,x(s))-g(s,y(s)))\Delta w(s)\bigg\|\bigg)^2\\
\leq\ & trQ\bigg(\int_{-\infty}^{t}\|X(t,s)\|^2\\
&\times\mathbf{E}\|g(s,x(s))-g(s,y(s))\|^2_{L_2^0}\Delta s\bigg)\\
\leq\ & trQM^2L^g\bigg(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\bigg)\\
&\times\sup_{t\in\mathbb{T}}\mathbf{E}\|x(t)-y(t)\|^2\\
\leq\ & \frac{trQM^2L^g}{\tilde{\alpha}}\|x-y\|^2_{\infty}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\|\Gamma x(t)-\Gamma y(t)\|^2\\
\leq\ & \bigg(\frac{2M^2L^f}{\tilde{\alpha}^2}+\frac{2trQM^2L^g}{\tilde{\alpha}}\bigg)\|x-y\|^2_{\infty},
\end{aligned}$$

and then

$$\begin{aligned}
&\|\Gamma x(t)-\Gamma y(t)\|_{\infty}\\
\leq\ & \bigg(\frac{2M^2L^f}{\tilde{\alpha}^2}+\frac{2trQM^2L^g}{\tilde{\alpha}}\bigg)^{\frac{1}{2}}\|x-y\|_{\infty}\\
=\ & \lambda\|x-y\|_{\infty}.
\end{aligned}$$

Since $\lambda<1$, then Γ has exactly one fixed point, that is, (1) has a unique square-mean almost periodic in shifts $\delta_{\pm}$. The proof is completed. ■

IV. EXAMPLES

Example 1. Consider the following stochastic dynamic system on time scales

$$\begin{aligned}
&\Delta x(t,\eta)\\
=\ & \bigg[\sum_{i,j=1}^{n}\Delta_{x_i}(a_{ij}(t,\eta)\Delta_{x_j})+c(t,\eta)\bigg]x(t,\eta)\Delta t\\
&+F(t,x(t,\eta))\Delta t+G(t,x(t,\eta))\Delta w(t), \qquad (3)\\
&\sum_{i,j=1}^{n}h_i(\eta)a_{ij}(t,\eta)\Delta_{\eta_i}x(t,\eta)=0, \qquad (4)\\
&t\in\mathbb{T},\eta\in\partial\Omega,
\end{aligned}$$

where w is a real valued Brownian motion.

Firstly, we make the following assumptions:

(H_4) The coefficient a_{ij} is symmetric, and $a_{ij}\in C_p^q(\mathbb{T},L^2(P,C(\overline{\Omega})))\cap C_p(\mathbb{T},L^2(P,C^1(\overline{\Omega})))\cap APS(\mathbb{T},L^2(P,L^2(\Omega)))$, $i,j=1,2,\cdots,n$; $c\in C_p^q(\mathbb{T},L^2(P,L^2(\Omega)))\cap C_p(\mathbb{T},L^2(P,C(\overline{\Omega})))\cap APS(\mathbb{T},L^2(P,L^1(\Omega)))$; for some $q\in(\frac{1}{2},1]$.

(H_5) There exists $\varepsilon_0>0$ such that

$$\sum_{i,j=1}^{n}a_{ij}(t,\eta)\zeta_i\zeta_j\geq\varepsilon_0|\zeta|^2,$$

where $(t,\zeta)\in\mathbb{T}\times\overline{\Omega}$, and $\zeta\in\mathbb{T}^n$.

If $(H_4)-(H_5)$ hold, then (H_1) holds, see [9].

Let $H=L^2(\Omega)$. For each $t\in\mathbb{T}$ define an operator $A(t)$ on $L^2(P;H)$ by

$$\begin{aligned}
D(A(t))\ =\ & \{x\in L^2(P;\mathbb{H}^2(\Omega)):\\
&\sum_{i,j=1}^{n}h_i(\eta)a_{ij}(t,\eta)\Delta_{\eta_i}x(t,\eta)=0\},
\end{aligned}$$

and $A(t)x=A(t,\eta)x(\eta)$ for all $x\in D(A(t))$.

Thus under the assumptions $(H_2)-(H_5)$, then the system (3)-(4) has a unique square-mean almost periodic solution in shifts $\delta_{\pm}$, if M is small enough.

Example 2. Consider the following stochastic cellular neural networks on time scales

$$\begin{aligned}
&\begin{pmatrix}\Delta x_1(t)\\ \Delta x_2(t)\end{pmatrix}\\
=\ & \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}\Delta(t)\\
&+\begin{pmatrix}\frac{1}{4}\sin(t)f_1(x_1(t))+\frac{1}{3}\cos(t)f_2(x_2(t))\\ \frac{1}{48}\sin(2t)f_1(x_1(t))+\cos(4t)f_2(x_2(t))\end{pmatrix}\Delta(t)\\
&+\begin{pmatrix}I_1(t)\\ I_2(t)\end{pmatrix}\Delta(t) \qquad (5)\\
&+\begin{pmatrix}\frac{1}{8}\sin(t)f_1(x_1(t))+\frac{1}{6}\cos(t)f_2(x_2(t))\\ \frac{1}{48}\sin(2t)f_1(x_1(t))+\frac{1}{6}\cos(4t)f_2(x_2(t))\end{pmatrix}\\
&\times\Delta w(t),
\end{aligned}$$

where $f_1(x)=f_2(x)=x^3$, $I_1(t)=\cos(t), I_2(t)=\sin(t)$, and w is a real valued Brownian motion.

Choose $\mathbb{T}$ such that $\mu(t)$ is bounded. It is easy to check that $(H_1)-(H_3)$ hold, and if trQ is small enough, then the system (5) has a unique square-mean almost periodic in shifts $\delta_{\pm}$.

V. CONCLUSION

This paper aims to explore the almost periodicity of stochastic dynamic equations on time scales. To this end, this paper firstly defined the square-mean almost periodic s-tochastic process in shifts $\delta_{\pm}$ and the square-mean $\Delta-$almost periodic stochastic process in shifts $\delta_{\pm}$ on time scales. On this basis, the existence and uniqueness theorem of square-mean almost periodic mild solution in shifts $\delta_{\pm}$ of a nonautonomous semilinear stochastic dynamic equations on time scales is established. The research in this paper is a further promotion on the basis of [10]. These theories are the basic theories for studying the almost periodicity of stochastic dynamic equations on time scales.

The results of this paper can be applied to study many other types stochastic dynamics systems on time scales. The relevant literatures can be referred to [11-32].

REFERENCES

[1] M. Bohner, A. Peterson, *Dynamic equations on time scales, An Introduction with Applications*, Boston: Birkhaüser; 2001.

[2] S. Hilger, "Analysis on measure chains-a unified approach to continuous and discrete calculus," *Result. Math.*, vol. 18, pp18-56, 1990.

[3] Y. Li, C. Wang, "Almost periodic functions on time scales and applications," *Discrete Dyn. Nat. Soc.*, vol. 2011, Article ID 727068.

[4] M. Hu, H. Lv, "Almost periodic solutions of a single-species system with feedback control on time scales," *Adv. Difference Equ.*, 2013, 2013:196.

[5] Y. Zhi, Z. Ding, Y. Li, "Permanence and almost periodic solution for an enterprise cluster model based on ecology theory with feedback controls on time scales," *Discrete Dyn. Nat. Soc.*, vol. 2013, Article ID 639138.

[6] H, Zhang, Y. Li, "Almost periodic solutions to dynamic equations on time scales," *J. Egypt. Math. Soc.*, vol. 21, pp3-10, 2013.

[7] M. Hu, L. Wang, "Almost periodicity in shifts delta(+/-) on time scales and its application to Hopfield neural networks," *Engineering Letters*, vol. 29, no.3, pp864-872, 2021.

[8] M. Adıvar, "A new periodicity concept for time scales," *Math. Slovaca*, vol. 63, no. 4, pp817-828, 2013.

[9] L. Maniar, S. Roland, "Almost periodicity of inhomogeneous parabolic evolution equations," *Lecture Notes in Pure and Appl. Math.*, vol. 234, pp. 299-318, 2003.

[10] P. Bezandry, T. Diagana, "Square-mean almost periodic solutions nonautonomous stochastic differdetial equations," *Elect. J. Diff. Equ.*, vol. 2007, no. 117, pp1-10, 2007.

[11] G. Stamov, "On the existence of almost periodic solutions for the impulsive Lasota-Wazewska model," *Appl. Math. Lett.*, vol. 22, pp516-520, 2009.

[12] M. Hu, L. Wang, "Almost periodic solution for a nabla BAM neural networks on time scales," *Engineering Letters*, vol. 25, no. 3, pp290-295, 2017.

[13] J. Liu, H. Li, C. Hu, H. Jiang, J. Cao, "Complete synchronization of discrete-time fractional-order BAM neural networks with leakage and discrete delays," *Neural Networks*, vol. 180, 2024, 106705.

[14] A. Elmwafy, J. J. Oliveira, C. M. Silva, "Existence and exponential stability of a periodic solution of an infinite delay differential system with applications to Cohen-Grossberg neural networks," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 135, 2024, 108053.

[15] X. Meng, Z. Li, J. Cao, "Almost periodic quasi-projective synchronization of delayed fractional-order quaternion-valued neural networks," *Neural Networks*, vol. 169, pp92-107, 2024.

[16] J. Gao, L. Dai, H. Jiang, "Stability analysis of pseudo almost periodic solutions for octonion-valued recurrent neural networks with proportional delay," *Chaos Soliton. Fract.*, vol. 175, Part 2, 2023, 114061.

[17] M. Luo, Z. Jiang, M. Tan, "Mean-square stability of stochastic Clifford-valued Cohen-Grossberg neural networks with variable coefficients and time-varying delays," *Neurocomputing*, vol. 625, No. 7, 2025, 129500.

[18] L. Li and B. Du, "Global asymptotical stability in a stochastic predator-prey system with variable delays," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 2, pp241-246, 2016.

[19] L. Xu, and Y. Liao, "On an Almost Periodic Gilpin-Ayala Competition Model of Phytoplankton Allelopathy," *IAENG International Journal of Applied Mathematics*, vol. 47, no.2, pp182-190, 2017.

[20] L. Wang, "Dynamic analysis on an almost periodic predator-prey model with impulses effects," *Engineering Letters*, vol. 26, no.3, pp333-339, 2018.

[21] Z. Liu, C. Yang, "Permanence and periodic solutions for a two-patch impulsive migration periodic N-species Lotka-Volterra competitive System," *Discrete Dyn. Nat. Soc.*, vol. 2015, Article ID 293050, 2015.

[22] Z. Hou, "On permanence of Lotka-Volterra systems with delays and variable intrinsic growth rates," *Nonlinear Anal.: Real World Appl.*, vol. 14 no. 2, pp960-975, 2013.

[23] J. Zhao, X. Guo, Z. Han, Z. Chen, "Average conditions for competitive exclusion in a nonautonomous two dimensional Lotka-Volterra system," *Math. Comput. Model.*, vol. 57, no. 5-6, pp1131-1138, 2013.

[24] Z. Teng, "On the non-autonomous Lotka-Volterra N-species competing systems," *Appl. Math. Comput.*, vol. 114, pp175-185, 2000.

[25] X. Feng, F. Zhang, K. Wang, X. Li, "Permanence in multispecies nonautonomous Lotka-Volterra competitive systems with delays and impulses," *Discrete Dyn. Nat. Soc.*, vol. 2012, Article ID 125197, 2012.

[26] H. Hu, K. Wang, D. Wu, "Permanence and global stability for nonautonomous N-species Lotka-Volterra competitive system with impulses and infinite delays," *J. Math. Anal. Appl.*, vol. 377, no. 1, pp145-160, 2011.

[27] L. Liu, Y. Shen, "New criteria on persistence in mean and extinction for stochastic competitive Lotka-Volterra systems with regime switching," *J. Math. Anal. Appl.*, vol. 430, no. 1, pp306-323, 2015.

[28] J. Zhao, Z. Zhang, J. Ju, "Necessary and sufficient conditions for permanence and extinction in a three dimensional competitive Lotka-Volterra system," *Appl. Math. Comput.*, vol. 230, pp587-596, 2014.

[29] C. Shi, Z. Li, F. Chen, "Extinction in a nonautonomous Lotka-Volterra competitive system with infinite delay and feedback controls," *Nonlinear Anal.: Real World Appl.*, vol. 13, no. 5, pp2214-2226, 2012.

[30] Z. Hou, "Permanence and extinction in competitive Lotka-Volterra systems with delays," *Nonlinear Anal.: Real World Appl.*, vol. 12, pp2130-2141, 2011.

[31] L. Chen, J. Sun, F. Chen, L. Zhao, "Extinction in a Lotka-Volterra competitive system with impulse and the effect of toxic substances," *Appl. Math. Model.*, vol. 40, no. 3, pp2015-2024, 2016.

[32] Y. Shi, C. Li, "Almost periodic solution, local asymptotical stability and uniform persistence for a harvesting model of plankton allelopathy with time delays," *IAENG International Journal of Applied Mathematics*, vol. 49, no. 3, pp307-312, 2019.