Convergence of Series Solutions for Spatial Distribution Order and Time Fractional Order Diffusion Equations and Accuracy Estimation of Numerical Format

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Abstract—**The difficulty in solving distributed order differential equations lies in the fact that the order of the derivative is distributed within a finite interval. Considering the time derivative Caputo type and spatial derivative Riesz type distributed diffusion equation, we obtain its series solution by spectral methods and prove its convergence, moreover, a specific accuracy estimate of its numerical format is given.**

Index Terms—**Caputo-type derivative; Riesz-type derivative; Distributed order derivative; Spectral method; convergence.**

I. Introduction

In recent decades, fractional calculus theory, as a new mathematical tool, has been widely used in many fields such as physics, viscoelastic mechanics and non-Newtonian fluid forces. After the concept of variable-order integral and variable-order derivative was proposed by Samko [1] in 1993, the variable-order derivative model was applied to the modeling of viscoelastic materials and viscous fluids, and the distribution order derivative whose derivative distributed in a finite interval has also been more and more widely used, for example, the phenomenon of ultra-low velocity diffusion or strong anomalous diffusion in polymer physics is usually described by the distribution-order diffusion equation [2]. The distribution-order diffusion equation can also be used to describe the sub-diffusion stochastic process belonging to the Wiener process, and its diffusion index decreases with time. Many complex diffusion processes with time-varying diffusion indices, such as decelerated hyperdiffusion and accelerated slow diffusion, decelerated slow diffusion and accelerated hyperdiffusion, can be described by the distributional-order diffusion equation [3].

In the past 10 or 20 years, the study of distribution order differential equations has attracted much attention. Zhang Hui [4] studied the two-dimensional Riesz spatially distributed convective diffusion equation, and proposed a Gaussian quadratic formula with higher accuracy than the midpoint quadrature, which can be used to discretize the spatial distribution derivative, and transform the equation into amultinomial spatial fractional order equation. The numerical solution of the equation was obtained by the Crank-Nicolson alternating direction Legendre spectrum method. Wang Mengru [5] constructed the difference format for the time distribution order with nonlinear source terms and the Riesz-type spatial fractional order diffusion equation by approximating the integral term of the equation using the midpoint quadrature, discretizing time fractional derivative using the backward finite difference formula and the central difference method, and proved their stability and convergence. Atanackovic,T.M. et al. [6] obtained the basic solution of the time-distribution order diffusion wave equation by means of Fourier transform and Laplace transform. Sandev, T. et al. [7] studied the distribution-order diffusion equation characterized by a multifractal memory kernel, in contrast to the simple power-law kernel of the ordinary time fractional diffusion equation. Ye, H. and Liu, F. et al. [8] considered the numerical analysis of the time distribution order and the spatial Riesz-type fractional diffusion equation on a bounded domain with Dirichlet boundaries. Zheng, D. and Chen, J. [9] proposed the separation of variables method and spectral method, they are used to solve distributed order diffusion equation in three dimensional space. Here we study the time Caputo-type and spatial Riesz-type mixed distribution order diffusion equation, use the spectral method to find its series solution, prove the convergence of the series solution, and give its accuracy estimate in numerical format.

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II. Preliminary knowledge

In the history of the development of fractional calculus, scholars have obtained several definitions of fractional derivatives from different perspectives. In this article ,the definitions of Caputo type fractional calculus and Riesz type fractional calculus are presented as follow.

Definition 2.1. The Caputo fractional operator for $k-1<\alpha\leq k$ on a infinite interval $a\leq t<+\infty$ is defined as

$$ {}_a^C D_t^{\alpha}\Psi(t)=\frac{1}{\Gamma(k-a)}\int_a^t\frac{\Psi^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}}d\tau, $$

where $\Psi(t)$ is k-order differentiability on a infinite interval $a\leq t<+\infty$.

Definition 2.2. [10] The Riesz fractional operator for $k-1<\alpha\leq k$ on a bounded interval $0\leq t\leq l$ is defined as

$$ \frac{\partial^{\alpha}}{\partial|x|^{\alpha}}\Psi(x,t)=-c_{\alpha}({}_0D_x^{\alpha}+{}_xD_l^{\alpha})\Psi(x,t), $$

where

$$\begin{aligned} c_\alpha &= \frac{1}{2cos(\frac{\pi\alpha}{2})},\ \alpha\neq 1,\\ {}_0D_x^\alpha\Psi(x,t) &= \frac{1}{\Gamma(k-\alpha)}\frac{\partial^k}{\partial x^k}\int_0^x\frac{\Psi(\xi,t)}{(x-\xi)^{\alpha+1-k}}d\xi,\\ {}_xD_l^\alpha\Psi(x,t) &= \frac{(-1)^k}{\Gamma(k-\alpha)}\frac{\partial^k}{\partial x^k}\int_x^l\frac{\Psi(\xi,t)}{(\xi-x)^{\alpha+1-k}}d\xi. \end{aligned}$$

Lemma 2.1[10] For the functions $\Psi(x)$ defined in $(-\infty,+\infty)$, the following equations hold,

$$\begin{aligned} -(-\Delta)^{\frac{\alpha}{2}}\Psi(x) &= -\frac{1}{2\cos(\frac{\pi\alpha}{2})}[{}_{-\infty}D_x^\alpha\Psi(x)\\ &\quad +{}_xD_\infty^\alpha\Psi(x)]\\ &= \frac{\partial^\alpha}{\partial|x|^\alpha}\Psi(x),\ k-1<\alpha<k. \end{aligned}$$

Define

$$\Psi^*(x)=\begin{cases}\Psi(x),\ x\in(0,l),\\ 0,\qquad x\notin(0,l).\end{cases}$$

It follows from Lemma 2.1, one has

Corollary 2.1[10]

$$\begin{aligned} -(-\Delta)^{\frac{\alpha}{2}}\Psi^*(x) &= -\frac{1}{2\cos(\frac{\pi\alpha}{2})}[{}_0D_x^\alpha\Psi(x)+{}_xD_l^\alpha\Psi(x)]\\ &= \frac{\partial^\alpha}{\partial|x|^\alpha}\Psi(x). \end{aligned}$$

Lemma 2.2[10] Suppose the Laplacian $(-\Delta)$ has a complete set of orthonormal eigenfunctions ϕ_k corresponding to eigenvalues λ_k^2 on a bounded region D, i.e., $(-\Delta)\phi_k=\lambda_k^2\phi_k$ on a bounded region D; $B(\phi)=0$ on ∂D, where $B(\phi)$ is one of the standard three homogeneous boundary conditions. Let

$$\Re_\gamma=\{f=\sum_{k=1}^\infty c_k\phi_k, c_k=\langle f,\phi_k\rangle,\sum_{k=1}^\infty|c_k|^2|\lambda|_k^\gamma<\infty,\\ \gamma=\max(\alpha,0)\},$$

then for any $f\in\Re_\gamma$, $(-\Delta)^{\frac{\alpha}{2}}$ is defined by

$$(-\Delta)^{\frac{\alpha}{2}}f=\sum_{k=1}^\infty c_k(\lambda_k^2)^{\frac{\alpha}{2}}\phi_k.$$

Proposition 2.1[11] For $k-1<\mu\leq k$, Laplace transform for Caputo type fractional order derivative

$$L\{{}_0^CD_t^\mu p(t);s\}=s^\mu P(s)-\sum_{n=0}^{k-1}s^{\mu-n-1}p^{(n)}(0),$$

where $P(s)$ is Laplace transform function of $p(t)$.

Proposition 2.2[11] Laplace transform for Mittag-Leffler functions

$$t^{\alpha n+\beta-1}E_{\alpha,\beta}^{(n)}(-at^\alpha)\overset{L}{\leftrightarrow}\frac{n!s^{\alpha-\beta}}{(s^\alpha+a)^{n+1}},$$

where $E_{\alpha,\beta}(-at^\alpha)$ is Mittag-Leffler Dual parameter function.

III. Series solutions of the time Caputo-type and spatial Riesz-type distribution order diffusion equation

Consider the time Caputo-type and spatial Riesz-type mixed distribution order diffusion equation

$$\begin{cases}{}_0^CD_t^\gamma\Psi(x,t)=\int_1^2\frac{\partial^\alpha}{\partial|x|^\alpha}\Psi(x,t)d\alpha,\\ t\geq 0, 0<x<l, 0<\gamma<1,\end{cases}\tag{1}$$

with the boundary and initial conditions given by

$$\begin{cases}\Psi(0,t)=\Psi(l,t)=0,\\ \Psi(x,0)=g(x),\end{cases}\tag{2}$$

where both $\Psi(x,t), g(x)$ are real-valued function and sufficiently smooth, $g(x)$ is bounded in $[0,l]$, and ${}_0^CD_t^\gamma$ is Caputo-type derivative.

Let

$$\Psi(x,t)=\sum_{k=1}^\infty T_k(t)\sin\frac{k\pi x}{l}.\tag{3}$$

By Corollary 2.1 and Lemma 2.2, we have

$$\begin{aligned} \frac{\partial^\alpha}{\partial|x|^\alpha}\Psi(x,t) &= -(-\Delta_x)^{\frac{\alpha}{2}}\Psi(x,t)\\ &= -\sum_{k=1}^\infty T_k(t)(\frac{k^2\pi^2}{l^2})^{\frac{\alpha}{2}}\sin\frac{k\pi x}{l}. \end{aligned}\tag{4}$$

Substitute both (3) and (4) into (1), we obtain

$$\begin{aligned} &\sum_{k=1}^\infty\sin\frac{k\pi x}{l}{}_0^CD_t^\gamma T_k(t)\\ &=-\sum_{k=1}^\infty T_k(t)\sin\frac{k\pi x}{l}\int_1^2(\frac{k^2\pi^2}{l^2})^{\frac{\alpha}{2}}d\alpha. \end{aligned}\tag{5}$$

It follows from above equation

$${}_0^CD_t^\gamma T_k(t)+[\int_1^2(\frac{k^2\pi^2}{l^2})^{\frac{\alpha}{2}}d\alpha]T_k(t)=0.\tag{6}$$

Combining (2) and (3) implies

$$\Psi(x,0)=\sum_{k=1}^\infty T_k(0)\sin\frac{k\pi x}{l}=g(x).$$

Obviously, by the properties of Fourier series, one has

$$T_k(0)=\frac{2}{l}\int_0^l g(x)\sin\frac{k\pi x}{l}dx.$$

Denote

$$\lambda_k=\int_1^2(\frac{k^2\pi^2}{l^2})^{\frac{\alpha}{2}}d\alpha,$$

which come from (6), calculating this integral, we can easily get

$$\lambda_k=\frac{(\frac{\pi}{l})^2k^2-\frac{\pi}{l}k}{\ln k+\ln\frac{\pi}{l}}.$$

By Proposition 2.1, Laplace transform on both sides of (6) leads to

$$s^\gamma\hat{T}_k(s)-s^{\gamma-1}T_k(0)+\lambda_k\hat{T}_k(s)=0,$$

thus

$$\hat{T}_k(s)=\frac{s^{\gamma-1}}{s^\gamma+\lambda_k}T_k(0).\tag{7}$$

Performing an inverse Laplace transform on both sides of equation (7) by Proposition 2.2 result in

$$\begin{aligned} T_k(t) &= E_\gamma(-\lambda_k t^\gamma)T_k(0) \\ &= E_\gamma(-\lambda_k t^\gamma)\frac{2}{l}\int_0^l g(x)\sin\frac{k\pi x}{l}dx. \end{aligned}$$

Consequently, we find the solutions of equation (1) and (2)

$$\begin{aligned} &\Psi(x,t) \\ =\ &\sum_{k=1}^{\infty}\sin\frac{k\pi x}{l}E_\gamma(-\lambda_k t^\gamma)\frac{2}{l}\int_0^l g(x)\sin\frac{k\pi x}{l}dx. \quad (8) \end{aligned}$$

IV. Convergence analysis of the series solutions

Lemma 4.1[11] If $0<\alpha<2$, β is a real number, for any $C>0$ and μ: $\frac{\pi\alpha}{2}<\mu<\min\{\pi,\pi\alpha\}$, then

$$|E_{\alpha,\beta}(Z)| \le \frac{C}{1+|z|}, \mu \le |arg(z)| \le \pi, |z| \ge 0.$$

Theorem 4.1 For any given pair (x,t), the series solutions (8) of equation (1) and (2) is convergent.

Proof. $g(x)$ is bounded on interval $[0,l]$, for $k\in Z^+$, we can easily get

$$\begin{aligned} |T_k(0)| &= \frac{2}{l}|\int_0^l g(x)\sin\frac{k\pi x}{l}dx| \\ &\le \frac{2}{l}|\sup_{x\in[0,l]} g(x)||\int_0^l \sin\frac{k\pi x}{l}dx| \\ &\le m, \end{aligned}$$

where m is a constant $0<m<\infty$.

Furthermore, when $\frac{l}{\pi}\notin Z^+$, for any $k\in Z^+$, we have

$$\begin{aligned} &|\sin\frac{k\pi x}{l}E_\gamma(-\lambda_k t^\gamma)\frac{2}{l}\int_0^l g(x)\sin\frac{k\pi x}{l}dx| \\ &\le m|E_\gamma(-\lambda_k t^\gamma)| \\ &\le m|\frac{c}{1+|\lambda_k t^\gamma|}| \\ &\le \frac{mc}{t^\gamma}|\frac{1}{\lambda_k}| \\ &= \frac{mcl^2}{t^\gamma\pi^2}|\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}|. \end{aligned}$$

Firstly, consider convergence of series of positive terms $\sum_{k=1}^{\infty}|\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}|$, for the fact that

$$\begin{aligned} &\lim_{k\to\infty}\frac{\ln k/k(k-\frac{l}{\pi})}{1/k^{\frac{3}{2}}} \\ =\ &\lim_{k\to\infty}\frac{\sqrt{k}\ln k}{k-\frac{l}{\pi}} \\ =\ &\lim_{k\to\infty}\frac{\ln k+2}{2\sqrt{k}} \\ =\ &\lim_{k\to\infty}\frac{\frac{1}{k}}{\frac{1}{\sqrt{k}}} \\ =\ &\lim_{k\to\infty}\frac{1}{\sqrt{k}} \\ =\ &0. \end{aligned}$$

Additionally, p-series $\sum_{k=1}^{\infty}\frac{1}{k^{\frac{3}{2}}}$ is convergent, thus, $\sum_{k=1}^{\infty}\frac{\ln k}{k(k-\frac{l}{\pi})}$ is absolutely convergent.

Secondly, consider $\sum_{k=1}^{\infty}|\frac{\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}|$ similarly,

$$\begin{aligned} &\lim_{k\to\infty}\frac{\ln\frac{\pi}{l}/k(k-\frac{l}{\pi})}{1/k^{\frac{3}{2}}} \\ =\ &\lim_{k\to\infty}\frac{\sqrt{k}\ln\frac{\pi}{l}}{k-\frac{l}{\pi}} \\ =\ &\lim_{k\to\infty}\frac{\ln\frac{\pi}{l}}{2\sqrt{k}} \\ =\ &0. \end{aligned}$$

Series $\sum_{k=1}^{\infty}\frac{\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}$ is absolutely convergent in the same way. Combining above two cases , we know that $\sum_{k=1}^{\infty}\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}$ is absolutely convergent. Therefore, the solutions of series form

$$\Psi(x,t)=\sum_{k=1}^{\infty}\sin\frac{k\pi x}{l}E_\gamma(-\lambda_k t^\gamma)\frac{2}{l}\int_0^l g(x)\sin\frac{k\pi x}{l}dx$$

are convergent independent of variable x.

When $k=\frac{l}{\pi}\in Z^+$, we have

$$\lambda_k=\int_1^2(\frac{k\pi}{l})^\alpha d\alpha=\int_1^2 d\alpha=1,$$

and

$$\begin{aligned} &|\sin\frac{k\pi x}{l}E_\gamma(-\lambda_k t^\gamma)\frac{2}{l}\int_0^l g(x)\sin\frac{k\pi x}{l}dx| \\ &\le m|E_\gamma(-\lambda_k t^\gamma)| \\ &\le m|\frac{c}{1+|\lambda_k t^\gamma|}| \\ &\le m|\frac{c}{1+t^\gamma}| \\ &\le mc. \end{aligned}$$

It follows that, given that the $\frac{l}{\pi}$-th term of series

$$\Psi(x,t)=\sum_{k=1}^{\infty}\sin\frac{k\pi x}{l}E_\gamma(-\lambda_k t^\gamma)\frac{2}{l}\int_0^l g(x)\sin\frac{k\pi x}{l}dx$$

is a bounded constant under condition $\frac{l}{\pi}\in Z^+$, the series $\Psi(x,t)$ is convergent as well. This ends the proof.

V. Accuracy estimation in numerical format

From chapter 4, we can take

$$\begin{aligned} &\Psi_K(x,t) \\ =\ &\sum_{k=1}^{K}\sin\frac{k\pi x}{l}E_\gamma(-\lambda_k t^\gamma)\frac{2}{l}\int_0^l g(x)\sin\frac{k\pi x}{l}dx, \quad (9) \end{aligned}$$

as numerical format of solution to Problem (1) and (2).

Lemma 5.1

(1) If $0<l<\pi$, then $\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}<\frac{\frac{2\pi}{l}}{k^{\frac{3}{2}}}$, $k\in Z^+$;

(2) If $l>\pi$ and $\frac{l}{\pi}\notin Z^+$, then $\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}<\frac{\frac{2l}{\pi}}{k^{\frac{3}{2}}}$, $k\in Z^+$.

Proof. (1) If $0<l<\pi$, we have

$$\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}-\frac{\frac{2\pi}{l}}{k^{\frac{3}{2}}}=\frac{1}{k}\frac{\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-\frac{2\pi k}{l}+2}{\sqrt{k}(k-\frac{l}{\pi})}.$$

For the fact

$$\frac{d(\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-\frac{2\pi k}{l}+2)}{dk}=\frac{\ln k+2+\ln\frac{\pi}{l}-\frac{4\pi\sqrt{k}}{l}}{2\sqrt{k}},$$

when $k=1$,

$$\begin{aligned}
& \ln k+2+\ln\frac{\pi}{l}-\frac{4\pi\sqrt{k}}{l} \\
< & \ln k+2+\frac{\pi}{l}-\frac{4\pi\sqrt{k}}{l} \\
= & 2-\frac{3\pi}{l} \\
< & 0,
\end{aligned}$$

and when $k\geq 1$,

$$\frac{d(\ln k+2+\ln\frac{\pi}{l}-\frac{4\pi\sqrt{k}}{l})}{dk}=\frac{1}{k}-\frac{2\pi}{l\sqrt{k}}<0,$$

thus,

$$\frac{d(\sqrt{k}\ln k+\sqrt{k}\ln l-\frac{2\pi k}{l}+2)}{dk}<0.$$

Moreover, when $k=1$,

$$\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-\frac{2\pi k}{l}+2=\ln\frac{\pi}{l}-2(\frac{\pi}{l}-1)<0.$$

(By the fact that when $y\geq 1$, $[\ln y-2(y-1)]'=\frac{1}{y}-2<0$, and when $y=1$, $\ln y-2(y-1)=0$. Which imply that when $y>1$, we have $\ln y-2(y-1)<0$.) Therefore, when $0<l<\pi$, then

$$\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}<\frac{\frac{2\pi}{l}}{k^{\frac{3}{2}}},\quad k\geq 1.$$

(2) If $l>\pi$ and $\frac{l}{\pi}\notin Z^+$, when $1\leq k<\frac{l}{\pi}$, by the fact

$$\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}-\frac{\frac{2l}{\pi}}{k^{\frac{3}{2}}}=\frac{1}{k}\frac{\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-\frac{2kl}{\pi}+2(\frac{l}{\pi})^2}{\sqrt{k}(k-\frac{l}{\pi})},$$

and

$$\begin{aligned}
& \frac{d[\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-\frac{2kl}{\pi}+2(\frac{l}{\pi})^2]}{dk} \\
= & \frac{\ln k+2+\ln\frac{\pi}{l}-\frac{4l\sqrt{k}}{\pi}}{2\sqrt{k}}.
\end{aligned}$$

When $k\geq 1$, one has

$$\frac{d(\ln k+2+\ln\frac{\pi}{l}-\frac{4l\sqrt{k}}{\pi})}{dk}=\frac{1}{k}-\frac{2l}{\pi\sqrt{k}}<0,$$

and when $k=1$, it follows

$$\ln k+2+\ln\frac{\pi}{l}-\frac{4l\sqrt{k}}{\pi}<0.$$

So we get

$$\frac{d[\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-\frac{2kl}{\pi}+2(\frac{l}{\pi})^2]}{dk}<0,\quad k\geq 1.$$

Furthermore, when $k=\frac{l}{\pi}$,

$$\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-\frac{2kl}{\pi}+2(\frac{l}{\pi})^2=0,$$

so we have

$$\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-\frac{2kl}{\pi}+2(\frac{l}{\pi})^2\geq 0,\quad 1\leq k<\frac{l}{\pi}.$$

Consequently, when $1\leq k<\frac{l}{\pi}$, we have

$$\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}<\frac{\frac{2l}{\pi}}{k^{\frac{3}{2}}}.$$

When $k>\frac{l}{\pi}$, we can get

$$\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}-\frac{1}{k^{\frac{3}{2}}}=\frac{1}{k}\frac{\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-k+\frac{l}{\pi}}{\sqrt{k}(k-\frac{l}{\pi})}.$$

By the fact

$$\frac{d[\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-k+\frac{l}{\pi}]}{dk}=\frac{\ln k+2+\ln\frac{\pi}{l}-2\sqrt{k}}{2\sqrt{k}},$$

and

$$\frac{d(\ln k+2+\ln\frac{\pi}{l}-2\sqrt{k})}{dk}=\frac{1}{k}-\frac{1}{\sqrt{k}}<0,\quad k>\frac{l}{\pi},$$

together with the fact

$$\ln k+2+\ln\frac{\pi}{l}-2\sqrt{k}<0,$$

when $k=\frac{l}{\pi}$, it follows

$$\frac{d[\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-k+\frac{l}{\pi}]}{dk}<0,\quad k>\frac{l}{\pi}.$$

Moreover, when $k=\frac{l}{\pi}$, we have

$$\sqrt{k}\ln k+\sqrt{k}\ln\frac{\pi}{l}-k+\frac{l}{\pi}=0,$$

thus, when $k>\frac{l}{\pi}$, we get the result

$$\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}<\frac{1}{k^{\frac{3}{2}}}.$$

In short, if $l>\pi$ and $\frac{l}{\pi}\notin Z^+$, then

$$\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}<\frac{\frac{2l}{\pi}}{k^{\frac{3}{2}}},\quad k\geq 1.$$

This ends the proof.

Lemma 5.2[12] Let $p(x)$ is a continuous function which is nonnegative , non-increased and defined in $[1,+\infty)$, then the series $\sum_{n=1}^{\infty}p(n)$ is convergent if and only if generalized integral $\int_1^{\infty}p(x)$ is convergent and

$$\sum_{k=2}^{n}p(k)\leq\int_1^n p(x)dx\leq\sum_{k=1}^{n-1}p(k).$$

Theorem 5.1 For any given pair (x,t), the estimation accuracy of numerical format (9) is expressed as

$$|\Psi(x,t)-\Psi_K(x,t)|\leq\frac{M}{K^{\frac{1}{2}}},$$

where M is a constant independent of K.

Proof. According to the proof of Theorem 4.1, we have

$$\begin{aligned}
|\Psi(x,t)-\Psi_K(x,t)| &= |\sum_{k=K+1}^{\infty}\sin\frac{k\pi x}{l}E_\gamma(-\lambda_k t^\gamma)T_k(0)| \\
&\leq m|\sum_{k=K+1}^{\infty}E_\gamma(-\lambda_k t^\gamma)| \\
&\leq \frac{mcl^2}{t^\gamma\pi^2}\sum_{k=K+1}^{\infty}|\frac{\ln k+\ln\frac{\pi}{l}}{k(k-\frac{l}{\pi})}|,
\end{aligned}$$

when $0 < l < \pi$, it follows

$$\begin{aligned}
|\Psi(x,t) - \Psi_K(x,t)| \quad & \leq \frac{mcl^2}{t^\gamma \pi^2} \sum_{k=K+1}^{\infty} \left| \frac{\ln k + \ln \frac{\pi}{l}}{k(k - \frac{l}{\pi})} \right| \\
& \leq \frac{mcl^2}{t^\gamma \pi^2} \sum_{k=K+1}^{\infty} \frac{\frac{2\pi}{l}}{k^{\frac{3}{2}}} \\
& = \frac{2mcl}{t^\gamma \pi} \sum_{k=K+1}^{\infty} \frac{1}{k^{\frac{3}{2}}} \\
& \leq \frac{2mcl}{t^\gamma \pi} \int_K^\infty \frac{1}{x^{\frac{3}{2}}} dx \\
& = [\frac{mcl^2}{t^\gamma \pi^2}] \frac{2}{\sqrt{K}} \\
& = \frac{M_1}{\sqrt{K}},
\end{aligned}$$

where M_1 is independent of K when t is given. If $l > \pi$ and $\frac{l}{\pi} \notin Z^+$, we can deduce

$$\begin{aligned}
|\Psi(x,t) - \Psi_K(x,t)| \quad & \leq \frac{mcl^2}{t^\gamma \pi^2} \sum_{k=K+1}^{\infty} \left| \frac{\ln k + \ln \frac{\pi}{l}}{k(k - \frac{l}{\pi})} \right| \\
& \leq \frac{mcl^2}{t^\gamma \pi^2} \sum_{k=K+1}^{\infty} \frac{\frac{2l}{\pi}}{k^{\frac{3}{2}}} \\
& = \frac{2mcl^3}{t^\gamma \pi^3} \sum_{k=K+1}^{\infty} \frac{1}{k^{\frac{3}{2}}} \\
& \leq \frac{2mcl^3}{t^\gamma \pi^3} \int_K^\infty \frac{1}{x^{\frac{3}{2}}} dx \\
& = [\frac{2mcl^3}{t^\gamma \pi^3}] \frac{2}{\sqrt{K}} \\
& = \frac{M_2}{\sqrt{K}},
\end{aligned}$$

where M_2 is independent of K when t is given. When $\frac{l}{\pi} \in Z^+$, given $K \geq \frac{l}{\pi}$, similarly, we have

$$|\Psi(x,t) - \Psi_K(x,t)| \leq \frac{M_2}{\sqrt{K}}.$$

This ends the proof.

VI. SUMMARY

For the time Caputo-type spatial Riesz-type distribution order diffusion equation, we have used the spectral method to find its series solution, proved the convergence of the series solution, and given its accuracy estimation of its numerical format.

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