On Twin Edge Colouring of Some Graphs and its Parametrized Graphs

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***Abstract*—In this article, the twin edge coloring of graphs has been analyzed. Graph colouring is one of the popular, extensively researched and studied fields of graph theory having many applications and conjectures, which are still open and studied by various mathematicians along the world. Inspired by edge colouring and total colouring, the edge induced vertex colouring called twin edge colouring is introduced. In this article, the twin chromatic index is investigated for some sunlet, bistar, Mycielskian and pencil graph families. Also it has been analyzed that the twin chromatic index of a subgraph is greater than its graph since $\chi'_t(C_k) > \chi'_t(S_k)$.**

***Index Terms*—Twin Edge colouring, Mycielskian of graph, Sunlet graph, Bistar graph, Pencil graph.**

I. Introduction

LET $G = (V, E)$ be a simple connected undirected graph. Andrews.E et al. introduced Twin Edge Colouring in 2014 [1] and discussed the results of path, cycle, complete graph, star graph, complete bipartite graph and bound for the twin chromatic index for all connected graph of order at least three is found to be $\chi'_t(G) \leq 4\Delta(G) - 3$. Daniel Johnston(2015) in his thesis explored the results of some tree structures such as a broom, double star, regular tree and Petersen graph, cartesian product of cycle and K_2 and the cartesian product of path and K_2. TIAN, S-L et al. discussed the results of finite 2-dimensional grids, d-infinite path, infinite lattice and direct product of paths. in the view of construction of triangle free graph, Mycielski developed the transformation of graph called Mycielskian of a graph. This article analyze the twin chromatic index of Mycielskian of graph, sunlet, bistar and pencil graph families.

II. Preliminaries

Definition 2.1. Let k be a positive integer. An adjacent vertex distinguishing(AVD) total k-colouring ϕ of a graph G is a proper total k-colouring of G such that no pair of adjacent vertices have the same set of colours, where the set of colours at a vertex v is $\{\phi(v)\} \cup \{\phi(e) : e\, is\, incident\, to\, v\}$.

Definition 2.2. The twin edge colouring of a graph is a proper edge colouring $c : E(G) \to Z_k$ that induces a proper vertex colouring $c' : V(G) \to Z_k$ is defined by $c'(v)$ is the sum of colours of edges incident with v where the sum in calculated in Z_k. The minimum value k among the twin edge colourings is called the twin chromatic index of G and is denoted by $\chi'_t(G)$.

Definition 2.3. For a graph G, the Mycielskian of G is the graph $\mu(G)$ with vertex set consisting of the disjoint union $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{x'y : xy\} \cup \{x'u : x''\}$. We call x' the twin of x in $\mu(G)$ and vice versa and u, the root of $\mu(G)$.

Definition 2.4. The *n-sunlet graph* on 2n vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n.

Definition 2.5. The *line graph of a graph G*, denoted by L(G), is a graph whose vertices are the edges of G and if u,v ∈E(G) then uv ∈E(L(G)) if u and v share a vertex in G.

Definition 2.6. Let G be a graph with vertex set V(G) and edge set E(G). *The middle graph of G* denoted by M(G) is defined as follows. The vertex set of M(G) is V(G)∪E(G). Two vertices x,y of M(G) are adjacent in M(G) in case one of the following holds: (i) x,y are in E(G) and x,y are adjacent in G, and (ii) x is in V(G), y is in E(G), and x,y are incident in G.

Definition 2.7. Let G be a graph with vertex set V(G) and edge set E(G). *The total graph of G* denoted by T(G) is defined in the following way. The vertex set of T(G) is V(G)∪E(G) . Two vertices x,y of T(G) are adjacent in T(G) in case one of the following holds: (i) x,y are in V(G) and x is adjacent to y in G, (ii) x,y are in E(G) and x,y are adjacent in G, and (iii)x is in V(G), y is in E(G), and x,y are incident in G.

Definition 2.8. The graph acquired by joining the centre vertices of two copies of $K_{1,n}$ is called bistar graph $B_{n,n}$.

Definition 2.9. The *square graph* of a simple connected graph G is defined by consisering the similar vertex set as of V(G) and edge set is obtained by joining two vertices if they are at a distance 1 or 2 away from each other in G and is represented by G^2.

Definition 2.10. Let G' *and* G'' be two copies of connected graph G. The *shadow graph D_2(G)* is obtained by joining each vertex v′ in G′ to the neighbours of the corresponding vertex v′′ in G′′.

Definition 2.11. *Splitting graph of G* is constructed by including a new vertex equivalent to each vertex of G with the property that the adjacent vertices of vertex in G is the same to the adjacent vertices of newly added vertices and is indicated by $S'(G)$.

Definition 2.12. Let G = (V(G),E(G)) be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \cdots S_i \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree

Manuscript received May 1, 2023; revised March 13, 2025.
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and T = V $\setminus(\cup$ S$_i$). The degree splitting graph of G denoted by DS(G) is obtained from G by adding vertices w$_1$,w$_2$,w$_3$,...,w$_t$ and joining to each vertex of S_i for $1 \leq i \leq t$.

III. TWIN EDGE COLOURING OF SUNLET FAMILIES OF GRAPHS

In this section the twin edge colouring of sunlet families of graphs are explored.

Observation 3.1. If two maximum degree vertices of a connected graph are adjacent then the twin chromatic index is greater than or equal to maximum degree plus one [1].

Lemma 3.1. If P_n is a path of order $n \geq 3$, then $\chi'_t(P_n) = 3$ [1].

Lemma 3.2. If C_n is a cycle of order $n \geq 3$, then [1]

$$\chi'_t(C_n) = \begin{cases} 3 & : n \equiv 0 \pmod 3 \\ 4 & : n \not\equiv 0 \pmod 3 \text{ and } n \neq 5 \\ 5 & : n = 5 \end{cases}$$

Theorem 3.1. For positive integer $k \geq 3$, the twin chromatic index of comb graph $\chi'_t(CB_k) = 4$.

Proof: Let $V(CB_k) = \{u_i, v_i\}$, where $e_i = u_i v_i$ for $1 \leq i \leq k$ and $e_{i+k} = u_i u_{i+1}$ for $1 \leq i \leq k-1$. Since $deg(u_i) = deg(u_{i+1}) = 3 = \Delta(CB_k)$ for $2 \leq i \leq k-2$ and u_i and u_{i+1} are adjacent, by Observation 3.1., $\chi'_t(CB_k) \geq 4$.
The following colouring pattern shows that $\chi'_t(CB_k) \leq 4$.
If k is odd, then the colouring is defined as $c : E(CB_k) \to Z_4$ that induces $c' : V(CB_k) \to Z_4$.
The edges of CB_k receives the following colours $(c(e_1), c(e_2), ..., c(e_k)) = (0, 0, 3, 0, 3, 0, ..., 3, 0)$, $(c(e_{k+1}), c(e_{k+2}), ..., c(e_{2k-1})) = (1, 2, 1, 2, ..., 1, 2)$.
Then the induced vertex colouring is given by $(c'(v_1), c'(v_2), ..., c'(v_k)) = (0, 0, 3, 0, 3, 0, ..., 3, 0)$, $(c'(u_1), c'(u_2), ..., c'(u_k)) = (1, 3, 2, 3, 2, 3, ..., 2, 3)$.
Similarly, when k is even, the colouring is defined.
Therefore $\chi'_t(CB_k) = 4$. ■

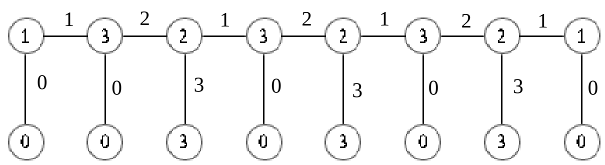

Fig. 1. Twin Edge colouring of Shadow graph of $B_4, 4$.

Theorem 3.2. If S_k is a sunlet graph, where $k \geq 3$, then the twin chromatic index is $\chi'_t(S_k) = 4$.

Proof: Let the vertices of sunlet graph be u_i, v_i for i varies from 1 to k and the edge set is $e_i = u_i v_i$ and $e_{i+k} = u_i u_{i+1}$. Since the two maximum degree vertices u_i and u_{i+1} are adjacent by Observation 3.1., $\chi'_t(S_k) \geq 4$.
When k is odd, the edges e_i are coloured using 2 and 3 for odd and even vertices and the edges e_{i+k} are coloured by 0,1 and 2. The induced colouring produces a proper vertex colouring. This shows that $\chi'_t(S_k) \leq 4$. First considering k as odd we define the colouring
$c : E(S_k) \to Z_4$ that induces $c' : V(S_k) \to Z_4$.

$$c(e_{k+i}) = \begin{cases} 0 & : i \text{ is odd}, i \neq k \\ 1 & : i \text{ is even} \\ 2 & i = k \end{cases}$$

$$c(e_i) = c'(v_i) = \begin{cases} 3 & : i \text{ is odd} \\ 2 & : i \text{ is even} \end{cases}$$

$$c'(u_i) = \begin{cases} 0 & : i \text{ is odd}, i \neq 1, k \\ 1 & i = 1 \\ 2 & i = k \\ 3 & : i \text{ is even} \end{cases}$$

Similarly considering k as even we define the colouring $c : E(S_k) \to Z_4$ that induces $c' : V(S_k) \to Z_4$.

$$c(e_{k+i}) = \begin{cases} 0 & : i \text{ is odd} \\ 1 & : i \text{ is even} \end{cases}$$

$$c(e_i) = c'(v_i) = \begin{cases} 3 & : i \text{ is odd} \\ 2 & : i \text{ is even} \end{cases}$$

$$c'(u_i) = \begin{cases} 0 & : i \text{ is odd} \\ 3 & : i \text{ is even} \end{cases}$$

Thus $\chi'_t(S_k) = 4$. ■

Theorem 3.3. If $L(S_k)$ is the line graph of sunlet graph where $k \geq 3$, then the twin chromatic index is $\chi'_t(L(S_k)) = 5$.

Proof: The line graph of sunlet graph is the edge graph of S_k. As in sunlet graph, the two maximum degree vertices are adjacent, therefore $\chi'_t(L(S_k)) \geq 5$.
When k is odd the edge set e_i, e_{i+k}, e_{i+2k} are coloured using $(1, 0), (0, 1, 2, 3)$ and $(4, 3)$ respectively. The induced colouring produces a proper vertex colouring.

Let $L(S_k) = \{u_i, v_i\}$, where $e_i = u_i v_i$, $e_{i+k} = v_i v_{i+1}$ and $e_{i+2k} = u_{i+1} v_i$ for $1 \leq i \leq k$. In this graph $u_{k+1} = u_1$ and $v_{k+1} = v_1$. The $deg(v_i) = deg(v_{i+1}) = \Delta$ for $1 \leq i \leq k$, v_i and v_{i+1} are adjacent. First considering k as odd, we define the colouring $c : E(L(S_k)) \to Z_5$ that induces $c' : V(L(S_k)) \to Z_5$.

$$c(e_{2k+i}) = \begin{cases} 4 & : i \text{ is odd} \\ 3 & : i \text{ is even} \end{cases}$$

$$c(e_i) = \begin{cases} 1 & i = 1, k \\ 0 & \text{otherwise} \end{cases}$$

$$c(e_{k+i}) = \begin{cases} 0 & i = k \\ 1 & i \text{ is even}, i \neq k-1 \\ 2 & i \text{ is odd}, i \neq k \\ 3 & i = k-1 \end{cases}$$

$$c'(u_i) = \begin{cases} 0 & i = 1, k \\ 3 & i \text{ is even} \\ 4 & i \text{ is odd}, i \neq 1, k \end{cases}$$

$$c'(v_i) = \begin{cases} 1 & i \text{ is odd}, i \neq k \\ 2 & i \text{ is even} \\ 3 & i = k \\ 4 & i = k-1 \end{cases}$$

In the same way when k is even, the coloring is defined. Thus $\chi'_t(L(S_k)) \leq 5$. This shows that $\chi'_t(L(S_k)) = 5$. ■

Theorem 3.4. For $k \geq 3$, the twin chromatic index of middle graph of sunlet graph is $\chi'_t(M(S_k)) = 8$ when $k \equiv 1 \pmod 4$ (or) $3 \pmod 4$ with $k \not\equiv 0 \pmod 3$.

Proof: Let $M(S_k) = \{u_i, v_i, u'_i, v'_i / 1 \leq i \leq k\}$ where $a_i = u_i u'_i$, $b_i = v_i u'_i$, $c_i = v'_i u'_i$, $d_i = v'_i u'_{i+1}$, $f_i = v'_i v$, $e_i = v'_i v'_{i+1}$ and $g_i = v'_i v_{i+1}$ for $1 \leq i \leq k$. Here $u'_{k+1} = u'_1$ and $v'_{k+1} = v'_1$. Since $deg(v'_i) = deg(v'_{i+1}) = \Delta$ for $1 \leq i \leq k$ and v'_i and v'_{i+1} are adjacent, by Observation 3.1, $\chi'_t(M(S_k)) \geq 7$.

Case(i). First suppose that $k \equiv 1 (\bmod\ 4)$, with $k \not\equiv 0 (\bmod\ 3)$ we define the colouring $c : E(M(S_k)) \to Z_8$ that induces $c' : V(M(S_k)) \to Z_8$.
$c(g_i) = 0$, $c(f_i) = 5$, $c(b_i) = 2$ for all $1 \leq i \leq k$. The edges a_i, c_i, d_i, e_i are coloured by $(1, 0), (7, 3), (4, 1), (2, 6, 4, 7)$ respectively. The induced vertex coloring produces a proper vertex colouring.

$$c(d_i) = \begin{cases} 4 & i = k \\ 1 & otherwise \end{cases}$$

$$c(c_i) = \begin{cases} 7 & i = 1 \\ 3 & otherwise \end{cases}$$

$$c(a_i) = \begin{cases} 1 & i = 1 \\ 0 & otherwise \end{cases}$$

$$c(e_i) = \begin{cases} 2 & i \equiv 1 \pmod 4, i = 1, i \neq k \\ 6 & i \equiv 2 \pmod 4, i = 2, k \\ 4 & i \equiv 3 \pmod 4, i = 3 \\ 7 & i \equiv 0 \pmod 4 \end{cases}$$

Then the vertex colouring is induced as follows
$c'(v_i) = 7$ and $c'(u'_i) = 6 :\ 1 \leq i \leq k$

$$c'(u_i) = \begin{cases} 0 & 2 \leq i \leq k \\ 1 & i = 1 \end{cases}$$

$$c'(v'_i) = \begin{cases} 1 & i \equiv 2 \pmod 4, i = 2, k \\ 3 & i \equiv 3 \pmod 4, i = 3 \\ 4 & i \equiv 0 \pmod 4 \\ 2 & i \equiv 1 \pmod 4, i \neq k \\ 5 & i = 1 \end{cases}$$

In a similar manner when $k \equiv 3 \pmod 4$ with// $k \not\equiv 0 \pmod 3$ the colouring is defined. Hence $\chi'_t(M(S_k)) = 8$. This concludes the proof. ■

Theorem 3.5. If $k \geq 3$, then the twin chromatic index of middle graph of sunlet graph is $\chi'_t(M(S_k)) = 7$ when $k \equiv 0 \pmod 3$ and when $k \equiv 0 \pmod 2$ with $k \not\equiv 0 \pmod 3$.

Proof: Let $M(S_k) = \{u_i, v_i, u'_i, v'_i / 1 \leq i \leq k\}$ where $a_i = u_i u'_i$, $b_i = v_i u'_i$, $c_i = v'_i u'_i$, $d_i = v'_i u'_{i+1}$, $f_i = v'_i v$, $e_i = v'_i v'_{i+1}$ and $g_i = v'_i v_{i+1}$ for $1 \leq i \leq k$.
Here $u'_{k+1} = u'_1$ and $v'_{k+1} = v'_1$. Since $deg(v'_i) = deg(v'_{i+1}) = \Delta$ for $1 \leq i \leq k$ and v'_i and v'_{i+1} are adjacent, by Observation 3.1, $\chi'_t(M(S_k)) \geq 7$.
Case(i). First suppose that $k \equiv 0 \pmod 3$ we define the colouring $c : E(M(S_k)) \to Z_7$ that induces $c' : V(M(S_k)) \to Z_7$.
$c(g_i) = c(a_i) = 0$, $c(f_i) = 5$, $c(d_i) = 1$, $c(b_i) = 2$, $c(c_i) = 3$, for all $1 \leq i \leq k$ and

$$c(e_i) = \begin{cases} 4 & i \equiv 1 \pmod 3, i = 1 \\ 6 & i \equiv 2 \pmod 3, i = 2 \\ 2 & i \equiv 0 \pmod 3 \end{cases}$$

Then the vertex colouring induced is

$c'(v_i) = 0$
$c'(u'_i) = 6 :\ 1 \leq i \leq k$
$c'(u_i) = 0$ for all $1 \leq i \leq k$

$$c'(v'_i) = \begin{cases} 1 & i \equiv 1 \pmod 3, i = 1 \\ 5 & i \equiv 2 \pmod 3, i = 2 \\ 3 & i \equiv 0 \pmod 3 \end{cases}$$

In a similar manner when $k \equiv 0 \pmod 2$ with $k \not\equiv 0 \pmod 3$ the coloring is defined. Hence the above colouring pattern shows that $\chi'_t(M(S_k)) \leq 7$. Therefore $\chi'_t(M(S_k)) = 7$. This concludes the proof. ■

Theorem 3.6. For $k \geq 3$, the twin chromatic index of total graph of sunlet graph is $\chi'_t(T(S_k)) = 8$.

Proof: Let $T(S_k) = \{u_i, v_i, u'_i, v'_i / 1 \leq i \leq k\}$ where $a_i = u_i u'_i$, $b_i = v_i u'_i$, $c_i = v'_i u'_i$, $d_i = v'_i u'_{i+1}$, $f_i = v'_i v$, $e_i = v'_i v'_{i+1}$ and $g_i = v'_i v_{i+1}$, $h_i = v_i u_i$, and $j_i = v_i v_{i+1}$ for $1 \leq i \leq k$. Here $u'_{k+1} = u'_1$ and $v'_{k+1} = v'_1$. Since deg(v'_i)=deg(v'_{i+1})=Δ for $1 \leq i \leq k$ and v'_i and v'_{i+1} are adjacent, by Observation 3.1., $\chi'_t(T(S_k)) \geq 7$.
The following colouring shows $\chi'_t(T(S_k)) \leq 7.(\mathrm{S}_k)) \to Z_8$.

Case(i). When k is even
The edges $a_i, b_i, c_i, d_i, e_i, f_i, g_i$ are coloured by $0, 1, 3, 2, 0, 1, 4, 5, 6, 2, 0, 7$ respectively. The induced vertex coloring produces a proper vertex colouring.
$c(h_i) = 2$, $c(f_i) = 4$, $c(d_i) = 2$, $c(b_i) = 1$, $c(c_i) = 3$, $c(a_i) = 0$ for all $1 \leq i \leq k$

$$c(j_i) = \begin{cases} 0 & i \text{ is odd} \\ 7 & i \text{ is even} \end{cases}$$

$$c(e_i) = \begin{cases} 0 & i \text{ is odd} \\ 1 & i \text{ is even} \end{cases}$$

$$c(g_i) = \begin{cases} 6 & i \text{ is odd} \\ 5 & i \text{ is even} \end{cases}$$

Then the induced vertex colouring

$c'(v_i) = 6$, $c'(u_i) = 2 \,:\, 1 \leq i \leq k$,

$$c'(u'_i) = \begin{cases} 3 & i \; is \; odd \\ 4 & i \; is \; even \end{cases}$$

$$c'(v'_i) = \begin{cases} 0 & i \; is \; odd \\ 7 & i \; is \; even \end{cases}$$

In a same way when k is odd the colouring is defined. Hence $\chi'_t(T(S_k)) = 8$. This concludes the proof. ■

Theorem 3.7. If $S(S_k)$ is the subdivision graph of sunlet graph of order $k \geq 3$, then the twin chromatic index is $\chi'_t(S(S_k)) = 3$.

Proof: As twin edge colouring is a proper colouring $\chi'_t(G) \geq max(\chi(G), \chi'(G))$.
Therefore $\chi'_t(S(S_k)) \geq \Delta$. The following colouring pattern shows that $\chi'_t(S(S_k)) \leq \Delta = 3$.
Let $S(S_k) = \{u_i, v_i, a_i, b_i / 1 \leq i \leq k\}$ where $e_i = a_i v_i$, $d_i = b_i v_i$, $c_i = b_i u_i$ for $1 \leq i \leq k$ and $f_i = a_i v_{i+1}$ for $1 \leq i \leq k$. Here $v_{k+1} = v_1$.
We define the colouring
$c : E(S(S_k)) \rightarrow Z_3$ that induces $c' : V(S(S_k)) \rightarrow Z_3$.
For $1 \leq i \leq k$, $c(e_i) = c(c_i) = 0$, $c(d_i) = 1$, $c(f_i) = 2$, $c'(u_i)$=$c'(v_i) = 0$, $c'(b_i) = 1$, $c'(a_i) = 2$.
Therefore $\chi'_t(S(S_k))$=3. Hence the proof. ■

IV. TWIN EDGE COLOURING OF BISTAR FAMILIES OF GRAPHS

In this section the twin edge colouring of bistar families of graphs are attained and is found to be either $\Delta \; or \; \Delta + 1$.

Theorem 4.1. If $B^2_{k,k}$ is the square graph of bistar graph for $k \geq 3$, then the twin chromatic index is $\chi'_t(B^2_{k,k}) = \Delta + 1$.

Proof: Let $V(B^2_{k,k}) = \{u, u_i, v, v_i\}$ where $a_i = uu_i$, $b_i = vv_i$, $c_i = vu_i$, $d_i = uv_i$ for $1 \leq i \leq k$ and $e = uv$. By Observation 3.1 $\chi'_t(B^2_{k,k}) \geq \Delta + 1$. The following colouring pattern shows that $\chi'_t(B^2_{k,k}) \leq \Delta + 1$. we define the colouring $c : E(B^2_{k,k}) \rightarrow Z_{\Delta+1}$ that induces $c' : V(B^2_{k,k}) \rightarrow Z_{\Delta+1}$.
$c(e) = 2k$, $c(a_i) = k - 1 + i$, $c(d_i) = i - 1$ for all $1 \leq i \leq k$

$$c(c_i) = \begin{cases} i - 1 & \text{if } i = 1, 4 \leq i \leq k \\ 1 & \text{if } i = 3 \\ 2 & \text{if } i = 2 \end{cases}$$

When k is odd,

$$c(b_i) = \begin{cases} k - 1 + i & \text{if } i = 1, 4 \leq i \leq k \\ k + 2 & \text{if } i = 2 \\ \Delta & \text{if } i = 3 \end{cases}$$

When k is even,

$$c(b_i) = \begin{cases} k - 1 + i & \text{if } i = 1, 4 \leq i \leq k \\ \Delta & \text{if } i = 2 \\ k + 1 & \text{if } i = 3 \end{cases}$$

then the induced vertex colouring is given below

When k is considered to be odd

$c'(u) = k + 2$, $c'(v) = 0$, $c'(u_3) = k + 3$, $c'(v_3) = 1$

$$c'(u_i) = \begin{cases} k & \text{if } i = 1 \\ k + 3 & \text{if } i = 2 \\ k - 2 + 2i & \text{if } 4 \leq i \leq \lceil k/2 \rceil \\ 2k + 1 & \text{if } i = \lceil k/2 \rceil + 1 \\ 2i - (k + 4) & \text{if } \lceil k/2 \rceil + 2) \leq i \leq k \end{cases}$$

The above colouring is same for $c'(v_i)$

When k is even

$c'(u) = k + 2$, $c'(v) = \Delta$, $c'(u_2) = k + 3$, $c'(v_2) = 0$

$$c'(u_i) = \begin{cases} k & \text{if } i = 1 \\ k + 3 & \text{if } i = 2 \\ k - 2 + 2i & \text{if } 4 \leq i \leq \lceil k/2 \rceil \\ 2k & \text{if } i = \lceil k/2 \rceil + 1 \\ 2i - (k + 4) & \text{if } \lceil k/2 \rceil + 2) \leq i \leq k \end{cases}$$

The above colouring is same for $c'(v_i)$
Hence $\chi'_t(B^2_{k,k}) = \Delta + 1$. This concludes the proof. ■

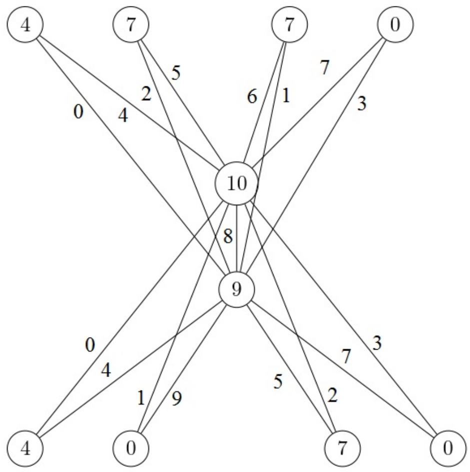

Fig. 2. Twin Edge colouring of Square graph of $B_{4,4}$.

Theorem 4.2. If $D_2(B_{k,k})$ is the shadow graph of bistar graph for $k \geq 3$, then the twin the chromatic index is $\chi'_t(D_2(B_{k,k})) = \Delta + 1$.

Proof: Let $V(D_2(B_{k,k})) = \{u_i, v_i, u'_i, v'_i / 0 \leq i \leq k - 1 \cup u, v, u', v'\}$ and $E(D_2(B_{k,k})) = \{uv, u'v', uv', vu', uu_i, vv_i, u'u'_i, u_iu', uu'_i, v_iv', vv'_i\}$.
By Observation 3.1 $\chi'_t(D_2(B_{k,k})) \geq \Delta + 1$.
Now we show that $\chi'_t(D_2(B_{k,k})) \leq \Delta + 1$.
We define the colouring $c : E(D_2(B_{k,k})) \rightarrow Z_{\Delta+1}$ that induces $c' : V(D_2(B_{k,k})) \rightarrow Z_{\Delta+1}$
$c(uu_i) = c(vv_i) = c(u'u'_i) = c(v'v'_i) = k + i$,
$c(u'u_i) = c(v'v_i) = c(uu'_i) = c(vv'_i) = i :\; 0 \leq i \leq k - 1$
$c(uv) = 2k$, $c(u'v') = 2k + 2$, $c(uv') = 2k + 1$,

$c(u'v) = 2k+3$. Then the induced vertex colouring
$c'(u) = 1$, $c'(v) = 3$, $c'(u') = 5$, $c'(v') = 3$,
$c'(u_i) = c'(v_i) = c'(u'_i) = c'(v'_i) = k+2i \,:\, 0 \le i \le k-1$
Hence $\chi'_t(D_2(B_{k,k})) = \Delta + 1$. Hence the proof. ■

Theorem 4.3. If S'(B$_{k,k}$) is the spliting graph of bistar graph for $k \ge 4$, then the twin chromatic index is $\chi'_t(S'(B_{k,k})) = \Delta + 1$.

Proof: Let $V(S'(B_{k,k})) = \{u_i, v_i, u_i, v_i/1 \le i \le k \cup u, v, u', v'\}$ and $E(S'(B_{k,k})) = \{uv, uu'_i, vv_i, u'u_i, v_iv', vv'_i, uv'_i/1 \le i \le k\}$.
By Observation 3.1 $\chi'_t(S'(B_{k,k})) \ge \Delta + 1$.
The following colouring shows that $\chi'_t(S'(B_{k,k})) \le \Delta + 1$.
We define the colouring $c : E(S'(B_{k,k})) \to Z_{\Delta+1}$ that induces $c' : V(S'(B_{k,k})) \to Z_{\Delta+1}$
$c(uv) = 2k$, $c(uu'_i) = c(vv'_i) = i-1$, $c(uu_i) = k-1+i$ for all $1 \le i \le k$

$$c(vv_i) = \begin{cases} k-1+i & \text{if } 1 \le i \le k-1 \\ 2k+1 & \text{if } i = k \end{cases}$$

$$c(u'u_i) = c(v'v_i) = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{if } i = 3 \\ 2 & \text{if } i = 2 \\ i & \text{if } 4 \le i \le k \end{cases}$$

then the vertex colouring is induced as follows

$c'(u'_i) = c'(v'_i) = i-1 : 1 \le i \le k-1$, $c'(u) = k+2$, $c'(v) = k+4$, $c'(u') = c'(v') = \sum_{i=1}^{k-1} i$ in Z_k,
$c'(u_i) = \sum_{i=1}^{k} [c(uu_i) + c(u_iu')]$ in Z_k,
$c'(v_i) = \sum_{i=1}^{k} [c(vv_i) + c(v_iv')]$ in Z_k
Therefore $\chi'_t(S'(B_{k,k})) = \Delta + 1$. This concludes the proof. ■

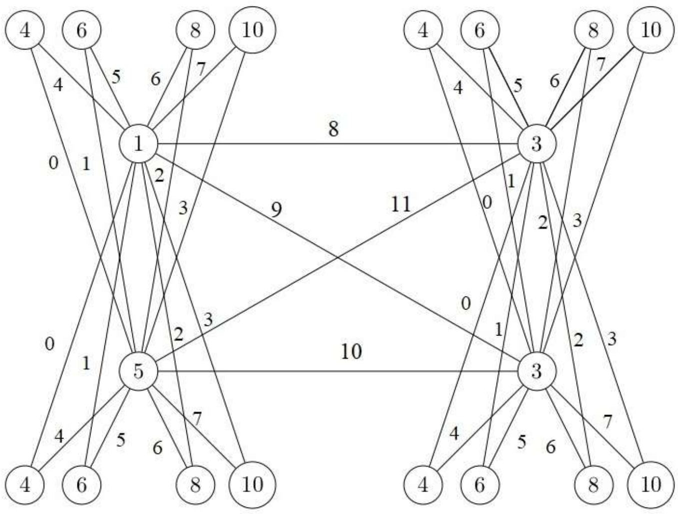

Fig. 3. Twin Edge colouring of Shadow graph of $B_4, 4$.

Theorem 4.4. If $DS(B_{k,k})$ is the degree spliting graph of bistar graph for $k \ge 3$, then the twin chromatic index is $\chi'_t(DS(B_{k,k})) = \Delta$.

Proof: Let $V(DS(B_{k,k}))$=$\{u_i, v_i/1 \le i \le k \cup u, v, w_1, w_2\}$
and $E(DS(B_{k,k})) = \{uv, uw_2, vw_2, w_1u_i, w_1v_i, uu_i, vv_i/1 \le i \le k\}$. As twin edge colouring is a proper colouring $\chi'_t(G) \ge max(\chi(G), \chi'(G))$.
Therefore $\chi'_t(DS(B_{k,k})) \ge \Delta$.
We define the colouring $c : E(DS(B_{k,k})) \to Z_\Delta$ that induces $c' : V(DS(B_{k,k})) \to Z_\Delta$
$c(uv) = 0$, $c(uw_2) = \Delta - 2$, $c(vw_2) = \Delta - 1$,
$c(w_1u_i) = i-1$; $1 \le i \le k$, $c(w_1v_i) = k+i$; $2 \le i \le k$, $c(w_1v_1) = k$, and in a similar way $c(u_iu), c(v_iv)$ are coloured from the set $\{1, 2, ..., k\}$. Then the induced vertex colouring is
$c'(w_1) = k$, $c'(w_2) = \Delta - 3$
and $c'(u') = \sum_{i=1}^{k} i + c(uv) + c(uw_2)$ in Z_k
$c'(v') = \sum_{i=1}^{k} i + c(uv) + c(vw_2)$ in Z_k
$c'(u_i) = c(w_1u_i) + c(uu_i)]$ in Z_k
$c'(v_i) = c(w_1v_i) + c(vv_i)]$ in Z_k
The above colouring shows that $\chi'_t(DS(B_{k,k})) \le \Delta$. Hence $\chi'_t(DS(B_{k,k})) = \Delta$. This concludes the proof. ■

V. TWIN EDGE COLOURING OF PENCIL AND ITS LINE GRAPH

In this section the twin edge colouring of pencil graph and line graph of pencil graph is discussed.

Theorem 5.1. If Pc_k is the pencil graph for $k \ge 3$, then the twin chromatic index is $\chi'_t(Pc_k) = 5$.

Proof: Let $V(Pc_k) = \{u_0, v_0, u_i, v_i/1 \le i \le k\}$ and $E(Pc_k) = \{e, e_i, a_0, a_i, a_k, b_0, b_i, b_k\}$ where $u_0v_0 = e$, $u_iv_i = e_i \,:\, 1 \le i \le k$, $u_iu_{i+1} = a_i \,:\, 1 \le i \le k-1$, $v_iv_{i+1} = b_i \,:\, 1 \le i \le k-1$, $u_0u_1 = a_0$, $u_0v_1 = b_0$, $v_0u_k = a_k$, $v_0v_k = b_k$.

We define the colouring $c : E(Pc_k) \to Z_5$ that induces $c' : V((Pc_k)) \to Z_5$

$c(a_0) = 2$, $c(b_0) = 3$, $c(a_k) = 4$, $c(b_k) = 3$

$$c(a_i) = \begin{cases} 1 & \text{if } i = 1, i \equiv 1 \pmod 3 \\ 3 & \text{if } i = 2, i \equiv 2 \pmod 3 : 1 \le i \le k \\ 2 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c(b_i) = \begin{cases} 1 & \text{if } i = 1, i \equiv 1 \pmod 3 \\ 2 & \text{if } i = 2, i \equiv 2 \pmod 3 : 1 \le i \le k \\ 4 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$c(e_i) = 0$, $c(e) = 1$

then the vertex colouring induced is as follows

$c'(u_0) = 1$, $c'(v_0) = 3$, $c'(u_1) = 3$, $c'(v_1) = 4$

$$c'(u_i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod 3 \\ 4 & \text{if } i = 2, i \equiv 2 \pmod 3 : 2 \le i \le k-1 \\ 0 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c'(v_i) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod 3 \\ 3 & \text{if } i = 2, i \equiv 2 \pmod 3 : 2 \leq i \leq k-1 \\ 1 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c'(u_k) = \begin{cases} 2 & \text{if } i \equiv 0 \pmod 3 \\ 1 & \text{if } i \equiv 1 \pmod 3 : 2 \leq i \leq k-1 \\ 0 & \text{if } i \equiv 2 \pmod 3 \end{cases}$$

$$c'(v_k) = \begin{cases} 0 & \text{if } i \equiv 0 \pmod 3 \\ 2 & \text{if } i \equiv 1 \pmod 3 : 2 \leq i \leq k-1 \\ 4 & \text{if } i \equiv 2 \pmod 3 \end{cases}$$

Therefore $\chi'_t(Pc_k) = 5$. Hence the proof. ∎

Theorem 5.2. If $L(Pc_k)$ is the line graph of pencil graph for $k \geq 3$, then the twin chromatic index is $\chi'_t(L(Pc_k)) = 6$.

Proof: Let $V(L(Pc_k)) = \{u_i, v_i, w_i\}$, $E(L(Pc_k)) = \{u_iu_{i+1}, v_iv_{i+1}, u_iw_{i+1}, v_iw_{i+1} / 0 \leq i \leq k-1 \cup u_iw_i, v_iw_i / 1 \leq i \leq k \cup u_0v_0, u_kv_k, u_0w_0, v_0w_0, w_0u_k, w_0v_k\}$. We define the colouring $c : E(L(Pc_k)) \to Z_6$ that induces $c' : V(L(Pc_k)) \to Z_6$

Case(i). When $k \equiv 0 \pmod 3$ or $1 \pmod 3$

$c(u_0v_0) = c(u_kv_k) = 1$, $c(w_0v_0) = 2$, $c(u_0w_0) = 3$, $c(u_kw_0) = 5$, $c(w_0v_k) = 4$.

$$c(u_iu_{i+1}) = \begin{cases} 0 & \text{if } i = 0, i \equiv 0 \pmod 3 \\ 1 & \text{if } i = 1, i \equiv 1 \pmod 3 \\ 3 & \text{if } i = 2, i \equiv 2 \pmod 3 \end{cases}$$

$$c(v_iv_{i+1}) = \begin{cases} 4 & \text{if } i = 0 \\ 1 & \text{if } i = 1, i \equiv 1 \pmod 3 \\ 0 & \text{if } i = 2, i \equiv 2 \pmod 3 \\ 2 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c(u_iw_{i+1}) = \begin{cases} 3 & \text{if } i = 0 \\ 2 & \text{otherwise} \end{cases}$$

$$c(w_iv_{i-1}) = \begin{cases} 2 & \text{if } i = 1 \\ 3 & \text{otherwise} \end{cases}$$

$c(w_iv_i) = 5 : 1 \leq i \leq k$, $c(u_iw_i) = 4 : 1 \leq i \leq k$

then the vertex colouring induced is as follows $c'(w_i) = 2 : 0 \leq i \leq k$

Subcase (i.1). When $k \equiv 0 \pmod 3$

$$c'(u_i) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1, k, i \equiv 1 \pmod 3 \\ 4 & \text{if } i = 2, i \equiv 2 \pmod 3 : 0 \leq i \leq k \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c'(v_i) = \begin{cases} 4 & \text{if } i = 0, i \equiv 0 \pmod 3 \\ 1 & \text{if } i = 1 \\ 3 & \text{if } i = 2, i \equiv 2 \pmod 3 : 0 \leq i \leq k \\ 5 & \text{if } i \equiv 1 \pmod 3 \end{cases}$$

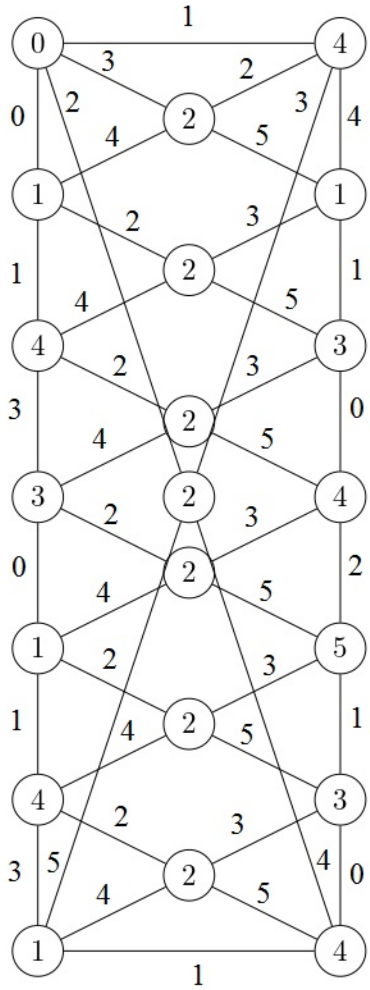

Fig. 4. Twin Edge colouring of Line graph of Pencil graph $L(Pc_6)$.

Subcase (i.2). When $k \equiv 1 \pmod 3$

$$c'(u_i) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1, i \neq k, i \equiv 1 \pmod 3 \\ 4 & \text{if } i = 2, i \equiv 2 \pmod 3 : 0 \leq i \leq k \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$c'(v_i) = \begin{cases} 4 & \text{if } i = 0, i \equiv 0 \pmod 3 \\ 1 & \text{if } i = 1 \\ 3 & \text{if } i = 2, i \equiv 2 \pmod 3 : 0 \leq i \leq k \\ 5 & \text{if } i \neq k, i \equiv 1 \pmod 3 \\ 0 & \text{if } i = k \end{cases}$$

In the same way when $k \equiv 2 \pmod 3$, the colouring is defined. Hence $\chi'_t(L(Pc_k)) = 6$. ∎

VI. TWIN EDGE COLOURING OF MYCIELSKIAN GRAPH

Theorem 6.1. If $\mu(K_{1,k})$ is the Mycielskian of $K_{1,n}$ for $n \geq 3$, then the twin chromatic index is $\chi'_t(\mu(K_{1,k})) = 2n - 1$.

Theorem 6.2. For $\mu(P_k)$, the twin chromatic index is $\chi'_t(\mu(P_k)) = n$.

VII. Conclusion

In this article the twin chromatic index is investigated for some sunlet, bistar and pencil graph families. The upper bound for the bistar families of graph is $\Delta+1$ and sunlet and pencil graph families it is $\Delta+2$. Also it has been analyzed that the twin chromatic index of a subgraph is greater than its graph since $\chi'_t(C_k) > \chi'_t(S_k)$.

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