Optimal Routh-Hurwitz Stability Criterion for Fractional Reverse Butterfly-Shaped System

D. A. Arhinful, H. Otoo, and J. Acquah

Abstract—**It is a well-established fact that fractional differential equations provide a more accurate description of several real-life phenomena. Unlike the traditional integer derivatives, fractional derivatives have memory and non-local properties that consider the past behavior of the equation or system. Fractional derivatives are widely studied and used for these reasons, especially in mathematical models or systems involving differential equations. An essential component of research in dynamical systems is the theory of stability, which describes how a differential equation responds to significantly small perturbations. The famous Routh-Hurwitz criterion provides the necessary and sufficient conditions for determining the stability of systems described by integer order differential equations. However, for systems described by fractional-order derivatives, this criterion is only a sufficient condition to guarantee that the zeros of a characteristic polynomial lie in the left half of the complex plane. Therefore, devising techniques for the stability analysis of dynamic systems with derivatives in the fractional sense has become imperative. In this paper, we compute an optimal Routh-Hurwitz criterion obtained from a boundary locus technique for the stability analysis of a fractional-order reverse butterfly-shaped chaotic system. This criterion satisfies the necessary and sufficient conditions for the roots of the system's characteristic polynomial to lie inside the Matignon stability sector. The results provide a better understanding of how the stability criterion of the fractional system is affected by the adjustable control parameter c. The Backward Differential Formula is used to validate the numerical results and is supported by graphical illustrations.**

Index Terms—**Reverse butterfly-shaped system, Fractional order derivative, Matignon stability, Optimal Routh-Hurwitz stability criterion.**

I. Introduction

FRACTIONAL calculus first appeared in the correspondence between Leibniz and L'Hopital, Bernoulli, and Wallis in the years 1695-1697. In a letter to Leibniz', who had invented the classical calculus - d^n/dx^n and $\int dx$, L'Hopital asked 'What if $n = 1/2$?' [1]. This question caught the attention of many mathematicians such as Laplace, Lacroix, Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov, Sonin, Laurent, and Caputo, who contributed to the theory [2].

Today, fractional calculus is considered well-posed. It is well-established that fractional differential equations better describe several real-life phenomena due to their non-local and memory properties, which consider the past behavior of the equation. For these reasons, fractional calculus applies in several areas of science, such as control theory [3], viscoelasticity [4], wave propagation [5], [6], signal and image processing [7], [8], [9].

An essential component of research in dynamical systems is the theory of stability, which describes how a differential equation responds to significantly small perturbations. The famous Routh-Hurwitz criterion provides the necessary and sufficient conditions for determining the stability of systems described by integer-order differential equations. However, for systems described by fractional-order derivatives, this criterion is only sufficient to guarantee that the zeros of a characteristic polynomial lie on the left half of the complex plane. In his elegant research titled 'Stability results for fractional differential equations with applications to control processing,' Matignon developed a theorem for the stability of systems of fractional-order differential equations in the year 1996 [10]. The Matignon stability theorem establishes necessary and sufficient conditions which ensure that all the roots of a characteristic polynomial associated with a system of fractional differential equations lie inside the Matignon stability sector. This major leap has sparked active research in the area. Stability analysis of new and existing systems have been studied in their fractional sense in [11], [12], [13], [14]. However, in applying the Matignon stability criterion, some literature employs techniques that do not establish explicit results. For instance, the stability results of the nontrivial symmetric equilibria of the fractional reverse butterfly-shaped chaotic system presented in Theorem 1 of the paper [15] do not permit explicit stability and bifurcation checks of the system. The paper [16] is another example.

Using a boundary locus technique, Čermák and Nechvátal [17] developed optimal Routh-Hurwitz conditions for the stability analysis of fractional dynamic systems. Their result explicitly states the necessary and sufficient conditions for which the roots of a third-order polynomial satisfy the Matignon stability criterion and, thus, lie inside the Matignon sector. They successfully applied their results to the stability and bifurcation analysis of the fractional Lorenz system [17], Rössler system [18], Chen system [19]. Ng and Phang [20] also applied the optimal Routh-Hurwitz conditions to determine the stability criterion for the fractional Shimizu-Morioka System.

This paper presents an optimal Routh-Hurwitz condition for the stability analysis of the fractional reverse butterfly-shaped dynamic system. The results provide a better understanding of how the stability criterion of the fractional system is affected by the adjustable control parameter c. The Backward Differential Formula is used to validate the numerical results and is supported by graphical illustrations.

The rest of the paper is organized as follows: Section two reviews the stability results of the classical reverse butterfly-shaped system. Section three presents optimal Routh-Hurwitz

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conditions for a fractional system. Section four highlights the stability analysis of the fractional reverse butterfly-shaped chaotic system, and finally, in section five we verify results by numerical simulation.

II. Reverse Butterfly-Shaped System

A. Integer Order Reverse Butterfly-Shaped System

According to the paper [21], the classical reverse butterfly-shaped system is given as

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= bx + kxz \\ \dot{z} &= -cz - hxy \end{aligned} \tag{1}$$

where a, b, c, h, k are positive parameters of the system. The system in Equation (1) has three sets of equilibria which are; the origin $O(0,0,0)$ and a pair of symmetric equilibrium points $E^{\pm}\left(\pm\sqrt{bc/kh}, \pm\sqrt{bc/kh}, -b/k\right)$, and demonstrates chaotic properties at $a = 10, b = 40, c = 2.5, k = 16, h = 1$ as shown in Fig. (1). Taking c as an

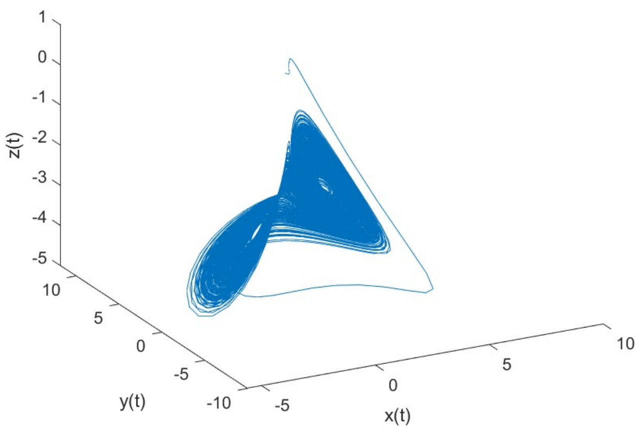

Fig. 1: Chaos at $c = 2.5$

adjustable parameter, the origin is unstable for all possible values of c. The Jacobian Linearization of the system in Equation (1) at the symmetric equilibrium points $E^{\pm}$ gives the same characteristic polynomial as follows;

$$P^{E^{\pm}}_{a,b,c}(\lambda) = \lambda^3 + (a+c)\lambda^2 + (ac+bc)\lambda + 2abc. \tag{2}$$

By the standard Routh-Hurwitz stability criterion, all the roots of the characteristic polynomial (2) lie on the left half of the complex plane if and only if

$$c > c^* = \frac{a(b-a)}{a+b}. \tag{3}$$

Thus, for all possible values of $c > c^*$, the characteristic polynomial (2) has negative roots which implies that the equilibria $E^{\pm}$ are stable. Also, for all positive values of $c < c^*$, the characteristic polynomial (2) has a negative real eigenvalue and a complex conjugate with a positive real part which implies that the equilibria $E^{\pm}$ are unstable. At $c = c^*$, the eigenvalues associated with the equilibria $E^{\pm}$ are nonhyperbolic [22]. At this point, the system (1) experiences a Hopf bifurcation characterized by a center manifold shown in Fig. (2).

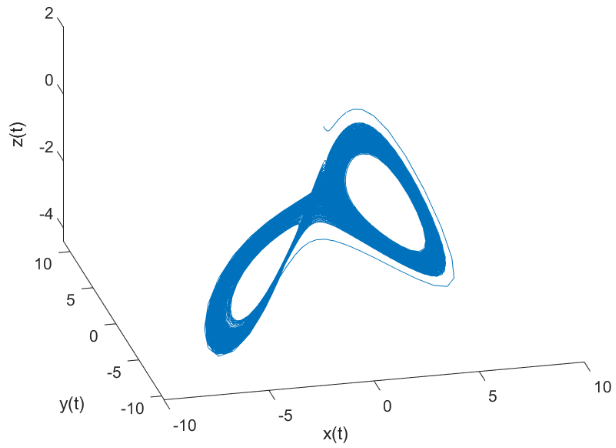

Fig. 2: Center manifold

B. Fractional Order Reverse Butterfly-Shaped System

The fractional form of the reverse butterfly-shaped system (1) is given by

$$\begin{aligned} D^{\alpha}x &= a(y-x) \\ D^{\alpha}y &= bx + kxz \\ D^{\alpha}z &= -cz - hxy \end{aligned} \tag{4}$$

where D^{α} is the Caputo derivative operator defined as follows: For a real function $f(t)$ defined for all $t > 0$, we introduce the fractional integral of the real order is defined as $\beta > 0$ by

$$D_0^{-\beta} f(t) = \int_0^t \frac{(t-\xi)^{\beta-1}}{\Gamma(\xi)} f(\xi) d\xi, t > 0$$

and the Caputo fractional derivative of the real order $0 < \alpha < 1$ by

$$D_0^{\alpha} f(t) = D_0^{-(1-\alpha)} \left(\frac{d}{dt} f(t)\right), \quad t > 0.$$

.

III. Optimal Routh-Hurwitz Stability Conditions for Fractional System

We consider a classical 3-dimensional system (the case $\alpha = 1$) having the characteristic polynomial;

$$P(\lambda; p, q, r) = \lambda^3 + p\lambda^2 + q\lambda + r \tag{5}$$

where p, q, r are real coefficients. The roots of the polynomial (5) λ_i, $i = 1, 2, 3$ have negative real parts if and only if

$$p > 0, q > 0, \text{and} 0 < r < pq$$

according to the standard Routh-Hurwitz conditions [23]. For the fractional case, this condition is only a sufficient condition for all the zeros of λ_i, $i = 1, 2, 3$ of (5) to lie inside the Matignon stability sector

$$|arg(\lambda)| > \frac{\alpha\pi}{2} \tag{6}$$

To guarantee (6) of (5), we use the boundary locus technique. Following [17], we define the boundary locus $BL(\alpha)$ as follows;

$$BL(\alpha) = \{(p,q,r) \in R\} : \exists \lambda \in C, |arg(\lambda)| > \frac{\alpha\pi}{2} \text{ and } P(\lambda; p,q,r) = 0, 0 < \alpha < 1. \tag{7}$$

where

$$\begin{aligned}\lambda =&\omega e^{i\alpha\pi/2}\\ =&\omega\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right)\\ =&\left(\omega\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)^3\\ &+p\left(\omega\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)^2\\ &+\text{q}\left(\omega\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)+r\end{aligned}\tag{8}$$

for a suitable $\omega \geq 0$.
Separating the real and imaginary parts, we obtain

$$\begin{aligned}&\omega^3\cos\left(\frac{3\alpha\pi}{2}\right)+p\omega^2\cos(\alpha\pi)+r\omega\cos\left(\frac{\alpha\pi}{2}\right)+l=0\\ &\omega\left(\omega^2\sin\left(\frac{3\alpha\pi}{2}\right)+p\omega\sin(\alpha\pi)+r\sin\left(\frac{\alpha\pi}{2}\right)\right)=0\end{aligned}\tag{9}$$

with the two solutions:
Solution 1: $l = 0, \omega = 0$
Solution 2:

$$\begin{aligned}&q=-\omega(4\cos(\frac{\alpha\pi}{2})^2\omega+2\cos(\frac{\alpha\pi}{2})p-\omega),\\ &r=2\cos(\frac{\alpha\pi}{2})\omega^3+\omega^2p.\end{aligned}\tag{10}$$

From Equation $(10)_1$, we obtain

$$\omega^{\pm}=\frac{-\cos\left(\frac{\alpha\pi}{2}\right)p\pm\sqrt{\cos\left(\frac{\alpha\pi}{2}\right)^2p^2-4\cos\left(\frac{\alpha\pi}{2}\right)^2q+q}}{4\cos\left(\frac{\alpha\pi}{2}\right)^2-1}.\tag{11}$$

To determine the dependence of the constant r on p and q, we substitute Equation (11) into Equation $(10)_2$ to get;

$$r^{\pm}=\frac{-pq\pm 2k_\alpha\left(p^2-4qk_\alpha^2+q\right)\sqrt{k_\alpha^2p^2-4k_\alpha^2q+q}+\Gamma}{\left(4k_\alpha^2-1\right)^3}\tag{12}$$

where

$$k_\alpha = cos(\frac{\alpha\pi}{2}),$$

and

$$\Gamma = 2pk_\alpha^2\left(-p^2+4qk_\alpha^2+q\right)$$

Theorem 1: Let $2/3 < \alpha < 1$. All the zeros of the polynomial (5) satisfy Equation (6) if and only if the following conditions holds:

(i) $p > 0, q > 0, 0 < r < r^-(p, q; \alpha)$;
(ii) $p \leq 0, \overline{q} > p^2/4cos^2(\alpha\pi/2), 0 < r < r^-(p, q; \alpha)$
(iii) $p > 0, \hat{q}(p; \alpha) \leq q \leq 0, r^+(p, q; \alpha) < r < r^-(p, q; \alpha)$.

Theorem 2: Let $1/2 < \alpha < 2/3$, All the zeros λ_i of the polynomial (5) satisfy Equation (6) if and only if any of the following conditions holds:

(i) $p < 0, q \leq \bar{q}(p; \alpha), r > r^+(p, q; \alpha)$;
(ii) $p < 0, \overline{q} < q < \hat{q}(p; \alpha), 0 < r < r^-(p, q; \alpha)$ or $r > r^+(p, q; \alpha)$;
(iii) $p < 0, q > \hat{q}(p; \alpha), r > 0$;
(iv) $p \geq 0, q < 0, r > r^+(p, q; \alpha)$;
(v) $p \geq 0, q \geq 0, r > 0$.

Theorem 3: Let $0 < \alpha \leq 1/2$, All the zeros λ_i of the polynomial (5) satisfy Equation (6) if and only if any of the following conditions holds:

(i) $p < 0, q \leq \bar{q}(p; \alpha), r > r^+(p, q; \alpha)$;
(ii) $p < 0, q > \bar{q}(q; \alpha), r > 0$;
(iii) $p \geq 0, q \geq 0, r > 0$.

Proof: The proof of theorems (1)-(3) can be seen in [17]. ■

IV. Stability Analysis of the Fractional Reverse Butterfly Shaped System

In this section, we compute an optimal Routh-Hurwitz stability criterion to determine the stability of the equilibria $E^{\pm}$ of the fractional reverse butterfly-shaped system in Equation (4). From the characteristic polynomial in Equation (2), we have $p = a + c > 0$, $q = c(a + b) > 0$, and $r = 2abc > 0$. Applying Theorem (3), it is easy to establish that the equilibria $E^{\pm}$ are asymptotically stable for $0 < \alpha < 1/2$. Also, according to Theorem (2), the system is asymptotically stable for $0 < \alpha < 2/3$. Therefore, we conclude that $E^{\pm}$ has all roots located in the Matignon stability sector in Equation (6), which guarantees that the fractional system in Equation (4) is locally asymptotically stable for $0 < \alpha < 2/3$.
For $2/3 < \alpha < 1$, we apply Theorem (1) to show that the inequality $r < r^-(p, q, r)$ is satisfied. And so, substituting $p = a + c$, $q = c(a + b)$, and $r = 2abc$ into $r < r^-(p, q, r)$ we obtain;

$$\begin{aligned}2abc <& \{-c(a+c)(a+b)-2k_\alpha\left((a+c)^2\right.\\ &-4c(a+b)k_\alpha^2+c(a+b)\\ &\times\sqrt{k_\alpha^2(a+c)^2-4c(a+b)k_\alpha^2+c(a+b)}\\ &+2(a+c)\left(-(a+c)^2+4c(a+b)k_\alpha^2\right.\\ &+c(a+b)k_\alpha^2\}/\left(4k_\alpha^2-1\right)^3\end{aligned}\tag{13}$$

where $k_\alpha = \cos\left(\frac{\alpha\pi}{2}\right)$.
Further simplification of Equation (13) yields

$$Ac^3+Bc^2+Dc+E>(Fc^2+Gc+H)\sqrt{Ic^2+Jc+K}\tag{14}$$

where

$A = 2\cos^2\left(\frac{\alpha\pi}{2}\right)$

$B = -2(a+b)\cos^2\left(\frac{\alpha\pi}{2}\right) - 8(a+b)\cos^4\left(\frac{\alpha\pi}{2}\right) + 6a\cos^2\left(\frac{\alpha\pi}{2}\right) + (a+b)$

$D = -8a(a+b)\cos^4\left(\frac{\alpha\pi}{2}\right) + \left(2ab-4a^2\right)\cos^2\left(\frac{\alpha\pi}{2}\right) + a^2 + ab\left(1+2\left(4\cos^4\left(\frac{\alpha\pi}{2}\right)-1\right)^3\right)$

$E = 2a^3\cos^2\left(\frac{\alpha\pi}{2}\right)$

$F = -2\cos\left(\frac{\alpha\pi}{2}\right)$

$G = -4a\cos\left(\frac{\alpha\pi}{2}\right) + 8\cos^3\left(\frac{\alpha\pi}{2}\right) - 2\cos\left(\frac{\alpha\pi}{2}\right)(a+b)$

$H = -2a^2\cos\left(\frac{\alpha\pi}{2}\right)$

$I = \cos^2\left(\frac{\alpha\pi}{2}\right)$

$J = (-2a-4b)\cos^2\left(\frac{\alpha\pi}{2}\right) + (a+b)$

$K = a^2\cos^2\left(\frac{\alpha\pi}{2}\right)$.

Taking the limit as $\alpha \to 1$, one obtains $F = G = H =$

0. This forces the right side of the inequality (14) to zero. However, on the left side, we get;

$$A = 0, B = (a + b), D = a(a - b), E = 0. \quad (15)$$

Substituting these into the inequality (14) gives

$$c > \frac{a(b - a)}{a + b}. \quad (16)$$

The inequality in Equation (16) is the same as Equation (3), the classical case ($\alpha = 1$) where we used the standard Routh-Hurwitz stability criterion. Therefore, the stability criterion in Equation (14) is the corresponding fractional extension of the Routh-Hurwitz criterion. Also, by squaring both sides of the inequality in Equation (14), we obtain the polynomial in Equation (17) which could equally be used for our analysis in place of the inequality (14).

$$\begin{aligned}Q(c) =&(F^2I - A^2)c^6 + (F^2J + 2FGI - 2AB)c^5\\ &+ (F^2K + 2FGJ + 2FHI + G^2I - 2AD - B^2)c^4\\ &+ (2FGK + 2FHJ + G^2J + 2GHI - 2AE-\\ &2AD)c^3 + (2FHK + G^2K + 2GHJ + H^2I\\ &- 2BE - D^2)c^2 + (2GIK + H^2J - 2DE)c\\ &+ H^2K - E^2\end{aligned} \quad (17)$$

Fixing the original values of the parameters $a = 10, b = 40, k = 16, h = 1$, the dependence of the adjustable control parameter c on the derivative order α can be analyzed. The fractional order reverse butterfly-shaped system thus becomes;

$$\begin{aligned}D^\alpha x &= 10(y - x)\\ D^\alpha y &= 40x + 16xz\\ D^\alpha z &= -cz - xy.\end{aligned} \quad (18)$$

With c as the adjustable control parameter, from the inequality (14) the following deductions and phase analysis can be arrived at;

$$\begin{aligned}&A \equiv A(\alpha) = 2\cos^2\left(\frac{\alpha\pi}{2}\right)\\ &B \equiv B(\alpha) = -400\cos^4\left(\frac{\alpha\pi}{2}\right) - 40\cos^2\left(\frac{\alpha\pi}{2}\right) + 50\\ &D \equiv D(\alpha) = -4000\cos^4\left(\frac{\alpha\pi}{2}\right) - 400\cos^2\left(\frac{\alpha\pi}{2}\right) +\\ &800\left(4\cos^2\left(\frac{\alpha\pi}{2}\right) - 1\right)^3 + 500\\ &E \equiv E(\alpha) = 2000\cos^2\left(\frac{\alpha\pi}{2}\right)\\ &F \equiv F(\alpha) = -2\cos^2\left(\frac{\alpha\pi}{2}\right)\\ &G \equiv G(\alpha) = -140\cos\left(\frac{\alpha\pi}{2}\right) - 400\cos^3\left(\frac{\alpha\pi}{2}\right)\\ &H \equiv H(\alpha) = -200\cos^2\left(\frac{\alpha\pi}{2}\right)\\ &I \equiv I(\alpha) = \cos^2\left(\frac{\alpha\pi}{2}\right)\\ &J \equiv J(\alpha) = -180\cos^2\left(\frac{\alpha\pi}{2}\right) + 50\\ &K \equiv K(\alpha) = 100\cos^2\left(\frac{\alpha\pi}{2}\right).\end{aligned} \quad (19)$$

Let

$$\begin{aligned}&Ac^3 + Bc^2 + Dc + E = f(c;\alpha),\\ &(Fc^2 + Gc + H)\sqrt{Ic^2 + Jc + K} = g(c;\alpha)\end{aligned} \quad (20)$$

and write inequality (14) as

$$f(c;\alpha) > g(c;\alpha) \quad (21)$$

where $2/3 < \alpha < 1$, and $c > 0$. Observing the graphs of (21) in the interval $2/3 < \alpha < 1$, it is seen that

$$F(\alpha) < 0, G(\alpha) < 0, H(\alpha) < 0.$$

Thus, $g(c;\alpha) < 0$ for $c > 0$ and $2/3 < \alpha < 1$. Graphs of the components of the left side of (21) reveals that

$$A(\alpha) > 0, B(\alpha) > 0, E(\alpha) > 0$$

in the interval $2/3 < \alpha < 1$. However, $D(\alpha) \geq 0$ in the interval

$$2/3 < \alpha \leq \alpha_o \approx 0.8727337622.$$

Therefore, the criterion in Equation (14) is satisfied for $c > 0$, and $2/3 < \alpha < 0.8727337622$. Also, $f(c;\alpha) > 0$ for all $0 < c < c^* = 6.0$.

For our next analysis, we assume $\alpha_0 < \alpha < 1$, and $c > 6.0$. Using elementary computations, the signs of the first and second derivatives of $g(c;\alpha)$ are determined. It is observed from analysis that $g(c;\alpha)$ decreases whereas $f(c;\alpha)$ increases in $(0,\infty)$ for any fixed value of $\alpha_0 < \alpha < 1$.

Following [17],

$$\frac{dg}{d\gamma}(c;\gamma) = \frac{U(c;\gamma)}{V(c;\gamma)}$$

where;

$$U(c;\gamma) = 1600c^3k^4 - 4c^4k^2 - 288000c^2k^4 + 440c^3k^2 + 160000ck^4 + 109600c^2k^2 - 100c^3 + 44000ck^2 - 7000c^2 - 40000k^2 - 10000c$$

and

$$V(c;\gamma) = \sqrt{c^2k^2 - 180ck^2 + 100k^2 + 50c}$$

The interval $(\alpha_0, 1)$ is bijectively mapped by $\gamma = cos(\alpha\pi/2)$ onto $(0, \gamma_0)$ where

$$\gamma_0 = cos(\frac{\alpha_0\pi}{2}) = 0.1985804760.$$

Next, we determine the point α_{cr} where the two curves $f(c;\alpha)$ and $g(c;\alpha)$ intersect by formulating the following system of equations;

$$\begin{aligned}&A(\alpha)c^3 + B(\alpha)c^2 + D(\alpha)c + E(\alpha)\\ &= \big(F(\alpha)c^2 + G(\alpha)c + H(\alpha)\big)\sqrt{I(\alpha)c^2 + J(\alpha)c + K(\alpha)}\end{aligned} \quad (22)$$

$$\begin{aligned}&6F(\alpha)I(\alpha)c^3 + (5F(\alpha)J(\alpha) + 4G(\alpha)I(\alpha))c^2 +\\ &(4F(\alpha)K(\alpha) + 3G(\alpha)J(\alpha) + 2HI)c + 2G(\alpha)K(\alpha) +\\ &H(\alpha)J(\alpha)\\ &= \big(6A(\alpha)c^2 + 4B(\alpha)c + 2D(\alpha)\big)\\ &\times\sqrt{I(\alpha)c^2 + J(\alpha)c + K(\alpha)}\end{aligned} \quad (23)$$

with c and α unknown. Solving the system in Equations (22) and (23) numerically, gives the critical value of α,

$$\alpha_{cr} \approx 0.9504050658$$

in the interval $[2/3, 1)$.

V. Numerical Results

In this section, we test the stability results of the fractional system in Equation (18) via numerical simulations. Garrappa's code flmm2 is used in MATLAB to obtain numerical solutions of the fractional system. This code is available on MathWorks and discussed in [24]. Three optional techniques are included in this code; we have opted for the fractional backward differential formula (BDF) method with a step size of 10^{-4} for the construction of time series and phase portrait.

Following previous discussions in section (IV), if we let $a = 10, b = 40, k = 16, h = 1$ and $\alpha_{cr} = 0.9504050658$, we get $c = 0.8242472099$. At these points, the fractional system in Equation (18) experiences a Hopf bifurcation, which results in a centre manifold. This is analogous to the Hopf bifurcation of the classical system at c = 6.0 from Equation (2). The center manifold is displayed in Fig. (3) where we have truncated $c = 0.824$ and the critical value $\alpha_{cr} = 0.95$ for convenience.

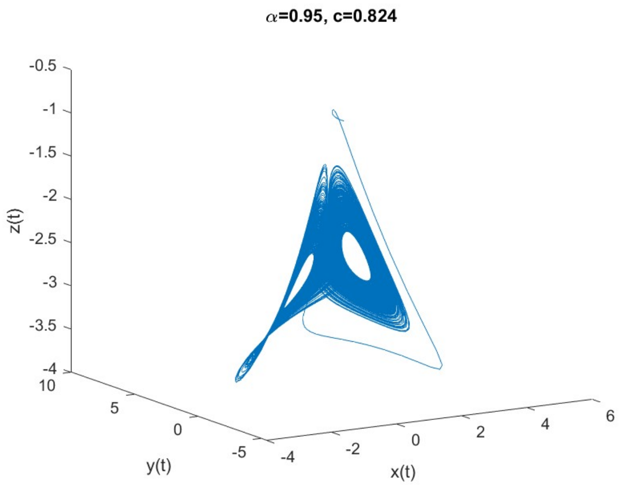

Fig. 3: Centre manifold of the system (18)

In terms of α, the stability criterion in Equation (14) can be expressed as

$$A(\alpha)c^3 + B(\alpha)c^2 + D(\alpha)c + E(\alpha) > \left(F(\alpha)c^2 + G(\alpha)c + H(\alpha)\right)\sqrt{I(\alpha)c^2 + J(\alpha)c + K(\alpha)}. \tag{24}$$

The stability results of the fractional system (18) based on the computed criterion in Equation (14) are summarized in Table (I) below. From the Table (I), if we let $\alpha = 0.94$ (assumption) which is less than the critical value α_{cr}, then for any choice of the bifurcation parameter c, the equilibria $E^{\pm}$ of the fractional system (18) are asymptotically stable as shown in the time series and phase portrait in Fig. (4) and Fig. (5) respectively for the case $c = 1.5$.

α	Range of c	Stability Condition
$\alpha < \alpha_{cr}$	$0 < c < \infty$	Stable
$\alpha > \alpha_{cr}$	Depends on (24)	Stable

TABLE I: Stability conditions for fractional Reverse Butterfly Shaped System (18)

For the condition $\alpha > \alpha_{cr}$, the stability of the fractional system (18) depends on the choice of the value of c. The range of c for which the fractional system (18) is stable at the equilibria $E^{\pm}$ can be obtained from the inequality (24). For instance, if we let $a = 10, b = 40, k = 16, h = 1$ and $\alpha = 0.96$ which is greater than α_{cr}, then the range of c for which the equilibria $E^{\pm}$ are stable can be determined by substituting $\alpha = 0.96$ into the inequality (24) which yields the range $[2.164979049, \infty)$. This is shown in Fig. (6) and Fig. (7).

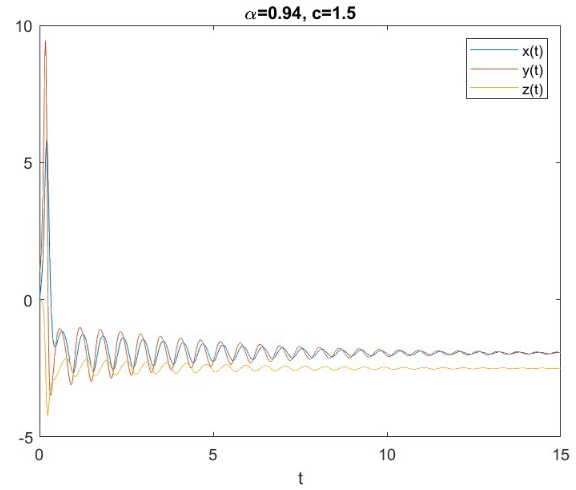

Fig. 4: Stable solution of the system (18) at $\alpha = 0.94 < \alpha_{cr}$

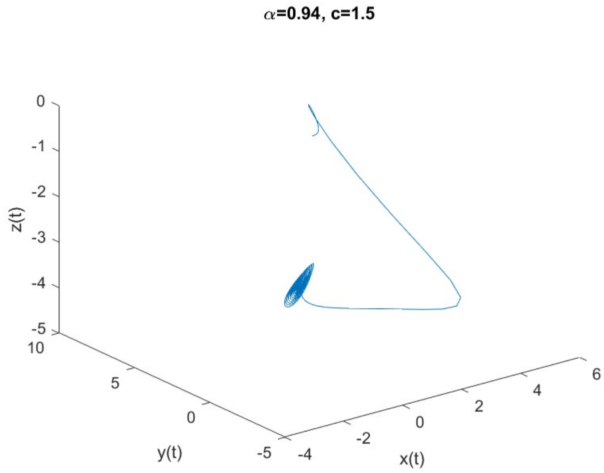

Fig. 5: Phase portrait of stable solution of system (18) at $\alpha = 0.94 < \alpha_{cr}$

The fractional system is unstable for any choice of c outside this range regardless of the value of α as shown in the time series solution and phase portrait in Fig. (8) and (9) respectively for the case $\alpha = 0.96$ and $c = 1.0$.

Table (II) gives a summary of the values of α and the corresponding ranges of c values where the equilibria $E^{\pm}$ of the fractional reverse butterfly-shaped system in Equation 18 are stable.

α	Range of c	Stability Condition
$0.951 > \alpha_{cr}$	$(2.164979049,\infty)$	Stable
$0.96 > \alpha_{cr}$	$(1.06894812, \infty)$	Stable
$0.94 < \alpha_{cr}$	$(0,\infty)$	stable
$0.93 < \alpha_{cr}$	$(0,\infty)$	Stable

TABLE II: Ranges of c for stability of the system (18)

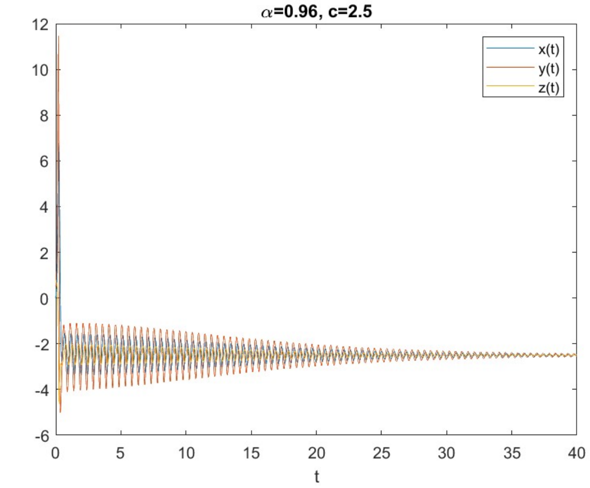

Fig. 6: Stable solution of the system (18) at $\alpha = 0.94 > \alpha_{cr}$, and $c = 2.5 \in [2.164979049, \infty)$

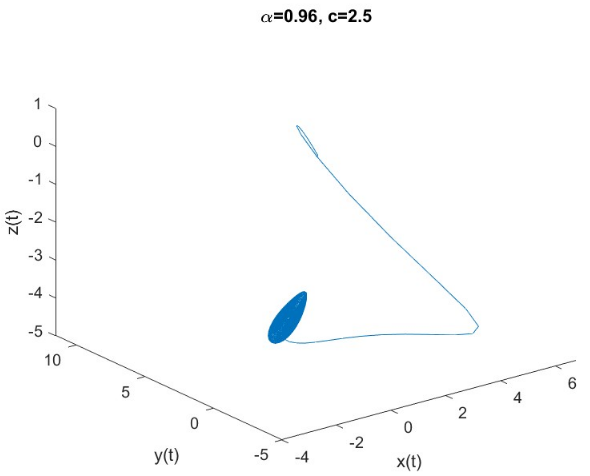

Fig. 7: Phase portrait of system (18) at $\alpha = 0.94 > \alpha_{cr}$, and $c = 2.5 \in [2.164979049, \infty)$

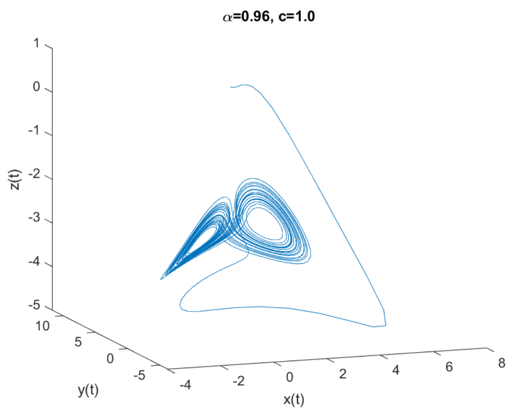

Fig. 8: Phase portrait of unstable solution of the system (18) at $\alpha = 0.96 > \alpha_{cr}$

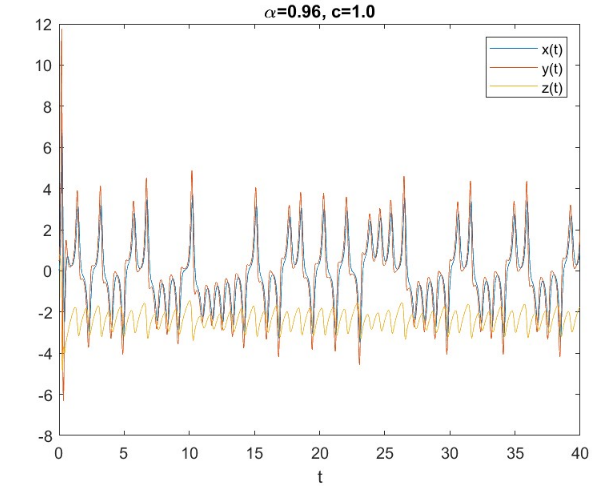

Fig. 9: Unstable solutions of the system (18) at $\alpha = 0.96 > \alpha_{cr}$.

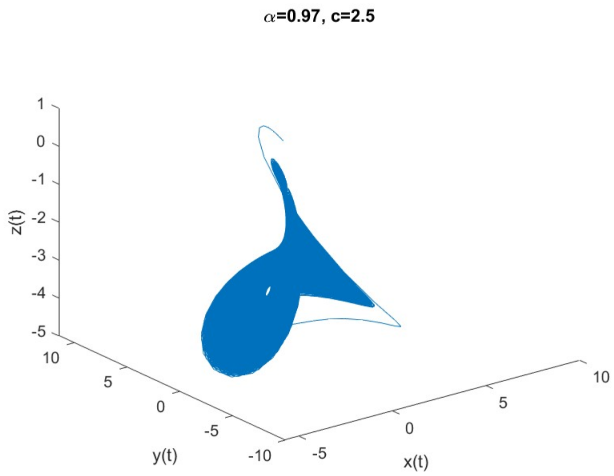

Fig. 10: Chaotic attractor.

Also, the fractional system (18) is chaotic at $\alpha = 0.97$ as shown in the figure (10).

VI. Conclusion

In conclusion, the boundary locus technique is efficient and provides explicit conditions that guarantee that the fractional system's eigenvalues lie inside the Matignon stability sector. In addition, unlike other methods, the results from this technique allow for bifurcation analysis of the fractional system. An optimal Routh-Hurwitz criterion using the boundary locus technique has been computed to perform a stability analysis of the fractional order reverse butterfly-shaped system. The derived optimal Routh-Hurwitz criterion can determine values of the fractional order α for which the fractional reverse butterfly-shaped system is stable and unstable. Also, we have explored how the fractional reverse butterfly-shaped system's stability depends on the adjustable parameter c where α is above its stability level. The Backward Differential Formula (BDF), a fractional linear multi-step method for numerically solving fractional differential equations, verified the numerical results. This research is limited to the case of α within the interval $(0, 1)$. Also, the boundary locus technique is still open to applying to fractional systems of dimension beyond 3. We recommend it for future study.

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