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On the Phi-Partial Ordering of s-k-Unitary Matrices

Swati and Hemen Bharali

Abstract—**This paper introduces the ϕ-partial order for matrices. It explores this ordering in the context of s-k-unitary matrices, connecting it to other matrix partial orders like Loewner and star. As an application, we study how $P \underset{\phi}{\leq} R$ relates to $P^2 \underset{\phi}{\leq} R^2$. Key findings include the preservation of $\underset{\phi}{\leq}$ partial order under s-k unitary similarity transformations and characterizations related to s-k normal matrices.**

Index Terms—**ϕ-partial order; s-k-Unitary similarity; s-k-Hermitian matrix; s-k-Normal matrix.**

I. Introduction and Preliminaries

The s-k-unitary matrices are an extension of unitary matrices introduced by Krishnamoorthy and Bhuvaneswari [6]. This generalization builds on the structure of secondary diagonal matrices and permutation matrices, earlier described by A. Lee [8] and Hill and Waters [9], respectively. Let $M_n(\mathbb{C})$ denote the set of all complex matrices of order n. For any matrix $A \in M_n(\mathbb{C})$, the notations $\overline{A}$, A^T, A^*, A^S, and A^θ represent the conjugate, transpose, transpose conjugate (primary), secondary transpose, and secondary conjugate transpose respectively. A. Lee has initiated the study of secondary symmetric, skew-symmetric, and orthogonal matrices, and explained usual (primary) transpose $A^T$and secondary transpose (transpose via secondary diagonal) A^S. This is expressed as $A^S = VA^TV$, where the matrix V is a permutation matrix with units at the secondary diagonal, satisfying the properties: $\overline{V} = V^T = V^S = V^* = V^\theta = V$, $V^2 = I$. From this it follows that the secondary conjugate transpose is given by $A^\theta = VA^*V$.

Hill and Waters further extended the concept of k-real and k-Hermitian matrices by generalizing their characterizations through the use of permutation matrices, where K is the associated permutation matrix with a fixed product of disjoint transposition 'k' in S_n. This leads to the definition of the s-k-transpose as $A^{sk} = KA^SK$, with K satisfying:
$\overline{K} = K^T = K^S = K^* = K^\theta = K$, $K^2 = I$.
The s-k-conjugate transpose is then defined as:
$A^\phi = KA^\theta K = KVA^*VK$.
A matrix A is said to be s-k-unitary if $AA^\phi = A^\phi A = I$. Furthermore, both K and V satisfy the property $K^\phi = K$, $V^\phi = V$.
An s-k-eigen value of a matrix is defined as a zero of the polynomial $det[\lambda KV - A]$. A non-zero vector $x \neq 0 \in M_nC$ is said to be the s-k-eigenvector of a complex matrix A with a s-k-eigenvalue λ if it satisfies $Ax = \lambda KVx$.

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In 1978, Drazin [5] introduced the concept of star partial ordering which is defined as,

$$A \underset{*}{\leq} B \Leftrightarrow A^*A = A^*B \;\; and \;\; AA^* = BA^* \tag{1}$$

where A^* denotes the conjugate transpose of A. The unique matrix satisfying the given four conditions of Penrose (i) $AXA = A$ (ii) $XAX = X$ (iii) $(AX)^*= AX$ (iv) $(XA)^* = XA$ is called Penrose inverse of A. The star partial ordering may be characterized by using the concept of Moore-Penrose inverse [5],

$$A \underset{*}{\leq} B \Leftrightarrow A^\dagger A = A^\dagger B \;\; and \;\; AA^\dagger = BA^\dagger \tag{2}$$

where $'\dagger'$ is Moore-Penrose inverse. In equation (2), it can be replaced [or conjugate transpose(primary) in (1)] by a reflexive generalized inverse, satisfying both $AA^\dagger A = A$ and $A^\dagger AA^\dagger = A^\dagger$, which is called plus order [4] and there will be no effect on partial ordering. This leads to the plus order version of star partial ordering:

$$A \underset{*}{\leq} B \Leftrightarrow A^+A = A^+B \;\; and \;\; AA^+ = BA^+ \tag{3}$$

Hartwig [4] introduced a relationship between the rank subtractivity and the plus partial ordering defined as

$$A \underset{rs}{\leq} B \Leftrightarrow rank(B - A) = rank(B) - rank(A) \tag{4}$$

where the rank subtractivity is equal to the plus order. Hartwig and Styan [7] carried further studies on conditions relative to reflexive inverses and adopted the term minus partial ordering to describe the rank subtractivity relation as well.

$$A \underset{-}{\leq} B \Leftrightarrow rank(B - A) = rank(B) - rank(A) \tag{5}$$

In their study, they also identified additional conditions which must be added so that the rank subtractivity become star partial order. In a subsequent development, Bakasalary and Mitra [2] introduced the concept of left star and right star partial ordering. Moreover, several properties of matrix partial ordering was discussed in [1] by Bakasalary, Pukelshein and Styan. The lowener partial ordering is defined as-

$$A \underset{L}{\leq} B \Leftrightarrow (B - A) \geq 0 \tag{6}$$

For further characterizations and in-depth studies of various partial orderings including the core, minus, sharp, star, and diamond partial orderings, particularly in the context of Hermitian matrices, one may refer ([10], [11]).
In this paper, we introduce a new partial order, referred to as the ϕ -partial order, and derive several related results. In addition, its relationship to other partial orders is described, as well as some characterizations for s-k-normal matrices.
The paper is organized as follows: In Section 1, we present

the foundational concepts and established results concerning s-k-unitary matrices and partial orders. In Section 2, we define the ϕ-partial ordering and established key results. Section 3 characterizes the s-k-normal matrices in the context of ϕ- partial ordering and explores its behavior under matrix powers. Also, we present the characterizations of minus partial ordering and demonstrate its equivalence relation with ϕ-partial order. Lastly, the conclusion and suggestions for future research are presented in section 4.

II. ϕ- Partial Ordering

In this section, we formally introduce the concept of ϕ partial ordering and explore its properties through some results. We begin by discussing the fundamental aspect of ϕ- partial ordering, with particular emphasis on its relation to matrix inversion and its preservation under this order. Furthermore, we examine the behavior of powers of matrices that adhere to the ϕ- partial ordering and under what specific conditions this ordering is maintained. Subsequently, we delve into the connections between ϕ- partial ordering and other matrix partial orderings. These relationships provide deeper insights into how matrices interact under various transformation works, like unitary and s-k-unitary similarity. The results presented in this section will highlight the robust nature of ϕ- partial ordering, particularly in the context of products and powers of matrices. We now proceed to define ϕ- partial ordering for matrices.

Definition 2.1: Let $P, R \in M_n(\mathbb{C})$, then the ϕ- partial ordering is defined as:
$P \underset{\phi}{\leq} R \iff P^\phi P = P^\phi R$ and $PP^\phi = RP^\phi$.
A partial order can be related to an equivalence relation when it satisfies certain conditions. Specifically, for two matrices P and R, the ϕ- partial order, denoted as $P \underset{\phi}{\leq} R$, is defined based on the existence of a matrix Q such that $P = RQ^\phi R$. This order can induce an equivalence relation if we further establish that two matrices P and R are equivalent under the ϕ- partial order when both $P \underset{\phi}{\leq} R$ and $R \underset{\phi}{\leq} P$ hold simultaneously. In such cases, the matrices share certain structural properties, particularly concerning their range and null space projections. Thus, the ϕ- partial order serves not only to compare matrices in a partial order but also enables the classification of matrices into equivalence classes based on their mutual comparability under this relation. This concept is useful when analyzing spectral properties. In the following, we present an important theorem that demonstrates how the inversion of matrices also adheres to this partial ordering.

Theorem 2.2: Let P and R be s-k-unitary matrices then $P \underset{\phi}{\leq} R \iff P^{-1} \underset{\phi}{\leq} R^{-1}$.

Proof: Given that
$P \underset{\phi}{\leq} R \implies P^\phi P = P^\phi R$ and $PP^\phi = RP^\phi$.
Consider the relation $P^\phi P = P^\phi R$. Since P is s-k-unitary, it satisfies P^ϕ=P^{-1}. Hence, $P^{-1}P = P^{-1}R$.
Taking s-k-conjugate transpose on both sides yields
$\implies (P^{-1}P)^\phi = (P^{-1}R)^\phi$.
$\implies P^\phi (P^{-1})^\phi = R^\phi (P^{-1})^\phi$

$$P^{-1}(P^{-1})^\phi = R^{-1}(P^{-1})^\phi \tag{7}$$

Now, consider the second condition: $PP^\phi = RP^\phi$. Since P is s-k-unitary, we have

$$PP^{-1} = RP^{-1}.$$

Taking s-k-conjugate transpose:

$$(PP^{-1})^\phi = (RP^{-1})^\phi,$$

which gives,

$$(PP^{-1})^\phi = (RP^{-1})^\phi$$

and, since P and R are s-k-unitary, we have

$$(P^{-1})^\phi P^\phi = (P^{-1})^\phi R^\phi$$
$$(P^{-1})^\phi P^{-1} = (P^{-1})^\phi R^{-1} \tag{8}$$

From equations (7) and (8), it follows that
$P \underset{\phi}{\leq} R \implies P^{-1} \underset{\phi}{\leq} R^{-1}$.
A similar argument, starting from $P^{-1} \underset{\phi}{\leq} R^{-1} \implies P \underset{\phi}{\leq} R$, establishes the reverse implication.
Hence, $P \underset{\phi}{\leq} R \iff P^{-1} \underset{\phi}{\leq} R^{-1}$. ■

With the preservation of inversion under the star partial order established, we now connect this concept to other matrices, particularly s-k-unitary matrices. The following results illustrate how s-k-unitary matrices, known for their stability and structure-preserving properties, interact with ϕ-partial order.

Theorem 2.3: If P is s-k-unitary such that $VKP \underset{\phi}{\leq} PVK$ then P is unitary.

Proof: Suppose VKP is ϕ-partial order. i.e. $VKP \underset{\phi}{\leq} PVK$. Then

$$(VKP)^\phi VKP = (VKP)^\phi (PVK)$$
$$\implies (P^\phi KV)(VKP) = (P^\phi KV)(PVK) \text{ [since } K^\phi = K, V^\phi = V]$$
$$\implies P^\phi KKP = (P^\phi KV)(PVK) \text{ [since } V^2 = I]$$
$$\implies P^\phi P = (P^\phi KV)(PVK) \text{ [since } K^2 = I]$$
$$I = (P^\phi KV)(PVK) \text{ [since } P \text{ is s-k-unitary]}$$

Post-multiplying both sides by KV, we obtain

$$KV = P^\phi KVP(VKKV)$$
$$\implies KV = P^\phi KVP$$
$$\implies KV = (KVP^*VK)KVP$$
$$\implies KV = KVP^*P$$

Pre-multiplying both sides by VK, we have

$$VKKV = (VKKV)P^*P.$$
$$I = P^*P \tag{9}$$

Similarly, from the second condition of the partial order: $VKP \underset{\phi}{\leq} PVK$.

$$\implies (VKP)(VKP)^\phi = (PVK)(VKP)^\phi$$
$$\implies (VKP)(P^\phi KV) = (PVK)(P^\phi KV)$$
$$\implies VKKV = (PVK)(P^\phi KV) \text{[since } P \text{ is s-k-unitary]}$$
$$\implies VV = (PVK)(P^\phi KV)$$
$$\implies I = (PVK)(P^\phi KV).$$

Post-multiplying both sides by VK, we get

$$VK = (PVK)(P^\phi KV)VK$$
$$\implies VK = (PVK)P^\phi$$
$$\implies VK = (PVK)(KVP^*VK)$$
$$\implies VK = PP^*VK$$

Multiplying both sides by KV, we have

$$VKKV = PP^*(VKKV)$$
$$I = PP^*. \tag{10}$$

From equations (9) and (10), we get
$PP^* = P^*P = I. \implies P$ is unitary. ■

Theorem 2.4: If $P\underset{\phi}{\leq}R$, then

(a) $P\underset{\phi}{\leq}R \iff VKP\underset{\phi}{\leq}VKR \iff PVK\underset{\phi}{\leq}RVK$

(b) $P\underset{\phi}{\leq}R \iff KVP\underset{\phi}{\leq}KVR \iff PKV\underset{\phi}{\leq}RKV$

The proof of this theorem directly from the previous theorem. An intriguing aspect of matrix theory is how the partial ordering of two matrices relates to the partial ordering of their powers. Specifically, consider the case where we have $P\underset{\phi}{\leq}R$, and we seek to establish that squares of these matrices also respect the same partial ordering. For this implication to be valid, a commutativity condition between the matrices P and R become important. In particular, if P and R are both s-k-Hermitian and ϕ-partial ordered, then they must be commutative, i.e. $RP = PR$. Firstly, we have the following relation:

$$P\underset{\phi}{\leq}R \implies P^{\phi}P = P^{\phi}R. \text{ and } PP^{\phi} = RP^{\phi}$$

Let us take,

$$P^{\phi}P = P^{\phi}R \implies PP^{\phi} = P^{\phi}R.$$

Since $P^{\phi} = P$, we have

$$\implies RP^{\phi} = P^{\phi}R \text{ [since } PP^{\phi} = RP^{\phi}]$$
$$\implies RP = PR, \text{ where P and R are s-k-Hermitian.}$$

However, to prove the reverse implication
$P^2\underset{\phi}{\leq}R^2 \implies P\underset{\phi}{\leq}R$, an additional condition is necessary, specifically, that the matrices P and R are positive definite, which we discussed in the next result. In this sense, we examine how the ordering of two s-k-Hermitian non-negative definite matrices P and R relates to the ordering of their squares $P^2$and R^2 in terms of ϕ-partial ordering.

Theorem 2.5: Let P and R be s-k-Hermitian matrices and positive definite matrices, then $P\underset{\phi}{\leq}R$ if and only if $P^2\underset{\phi}{\leq}R^2$.

Proof: Assume $P\underset{\phi}{\leq}R$. This implies $P\underset{\phi}{\leq}R \implies P^{\phi}P = P^{\phi}R$ and $PP^{\phi} = RP^{\phi}$.

Using the first condition $P^{\phi}P = P^{\phi}R$, and post -multiplying both sides by P, we have, $P^{\phi}PP = P^{\phi}RP$.

$$P^{\phi}P^2 = P^{\phi}RP,$$
$$\implies P^{\phi}P^2 = P^{\phi}PR \text{ [since PR=RP]}$$
$$\implies P^{\phi}P^2 = P^{\phi}RR \text{ [since } P^{\phi}P = P^{\phi}R]$$
$$\implies P^{\phi}P^2 = P^{\phi}R^2.$$

Pre-multiplying both sides by P^{ϕ} gives
$(P^{\phi})^2P^2 = (P^{\phi})^2R^2$

$$\implies (P^2)^{\phi}P^2 = (P^2)^{\phi}R^2$$

Thus, $P^2\underset{\phi}{\leq}R^2$.

Conversely, assume $P^2\underset{\phi}{\leq}R^2$.

Let, $P^2 = C$, $R^2 = D$

$$C\underset{\phi}{\leq}D \implies C^{\phi}C = C^{\phi}D$$
$$\implies (P^2)^{\phi}P^2 = (P^2)^{\phi}(R^2)$$
$$\implies (P^{\phi})^2P^2 = (P^{\phi})^2R^2$$
$$\implies P^2P^2 = P^2R^2 \text{ [since } P^{\phi} = P \text{ and } R^{\phi} = R]$$
$$\implies P^2 = R^2.$$

$\implies P = R$

Since P and R are s-k-Hermitian and positive definite, it follows that $P^2 = R^2 \implies P = R$.

Multiplying both sides by P^{ϕ}, we have

$P^{\phi}P = P^{\phi}R. \implies P\underset{\phi}{\leq}R$

Hence the result. ∎

The following result can be readily proved by applying Theorem 2.5.

Theorem 2.6: Let P and R be s-k-Hermitian matrices and positive definite such that

(a) $P\underset{\phi}{\leq}R$

(b) $P^2\underset{\phi}{\leq}R^2$

(c) $PR = RP$

Then $(b) \iff (a) \implies (c)$.

Unitary similarity is a well established concept in matrix theory. In this context, s-k-unitary similarity serves as a generalization of unitary similarity, built upon the matrices K and V as discussed earlier.

Two matrices P, $R \in M_n(\mathbb{C})$ are said to be s-k-unitary similar, if there exist a s-k-unitary matrix T such that $T^{\phi}PT = R$. Notably, several matrix partial orderings like lowener, star, minus, and ϕ- partial ordering are preserved under unitary similarity. In this section, we establish the invariance concept related to ϕ- partial ordering under s-k-unitary similarity. For which, our aim to prove that $P\underset{\phi}{\leq}R \iff T^{\phi}PT\underset{\phi}{\leq}T^{\phi}RT$, which is presented in the next theorem. Additionally, we explore the relationship between ϕ- partial ordering and other matrix partial order, providing further insights into their inter-dependencies.

Theorem 2.7: ϕ- partial ordering is preserved under s-k-unitary similarity i.e. $P\underset{\phi}{\leq}R \iff T^{\phi}PT\underset{\phi}{\leq}T^{\phi}RT$, where T is s-k-unitary matrix.

Proof: If $P\underset{\phi}{\leq}R$ then $P\underset{\phi}{\leq}R \implies P^{\phi}P = P^{\phi}R$ and $PP^{\phi} = RP^{\phi}$.

To prove: $T^{\phi}PT\underset{\phi}{\leq}T^{\phi}RT$

Let $J_1 = T^{\phi}PT$ and $J_2 = T^{\phi}RT$, then $J_1\underset{\phi}{\leq}J_2$.

where, $J_1^{\phi}J_1 = J_1^{\phi}J_2$, and $J_1J_1^{\phi} = J_2J_1^{\phi}$.

Now, $J_1^{\phi}J_1 = (T^{\phi}PT)^{\phi}T^{\phi}PT = T^{\phi}P^{\phi}TT^{\phi}PT$

$= T^{\phi}P^{\phi}PT$ [since T is s-k-unitary matrix]

$= T^{\phi}P^{\phi}RT$ [since $P^{\phi}P = P^{\phi}R$]

$= T^{\phi}P^{\phi}TT^{\phi}RT$ [since $TT^{\phi} = I$]

$= (T^{\phi}PT)^{\phi}T^{\phi}RT$

$= J_1^{\phi}J_2$

Similarly,

$J_1J_1^{\phi} = T^{\phi}PT(T^{\phi}PT)^{\phi} = T^{\phi}PTT^{\phi}P^{\phi}T$

$= T^{\phi}PP^{\phi}T$ [since T is s-k-unitary matrix]

$= T^{\phi}RP^{\phi}T$ [since $PP^{\phi} = RP^{\phi}$

$= T^{\phi}RTT^{\phi}P^{\phi}T$

$= (T^{\phi}RT)(T^{\phi}PT)^{\phi}$

$= J_2J_1^{\phi}$

Thus, $J_1^{\phi}J_1 = J_1^{\phi}J_2$ and $J_1J_1^{\phi} = J_2J_1^{\phi}$

$\implies J_1\underset{\phi}{\leq}J_2$. ∎

Relationship of ϕ- partial order with other partial orders.

Theorem 2.8: If $P\underset{L}{\leq}R$ and $P\underset{\phi}{\leq}R$ then $P = R$.

Proof: Suppose $P\underset{\phi}{\leq}R$, and therefore $P^{\phi}P = P^{\phi}R$ and $PP^{\phi} = RP^{\phi}$.

$$\implies P^{\phi}(R - P) = 0$$
$$\implies (R - P)^{\phi}P = 0 \tag{11}$$

Again, $P\underset{L}{\leq}R \implies KV(R - P)^*VK = R - P$.

This follows from the equivalence

$P\underset{L}{\leq}R \iff KVP\underset{L}{\leq}KVR \implies KV(R - P) \geq 0$

which shows that $(R - P)$ is s-k-Hermitian positive definite.

Now, pre-multiplying by VK and post multiplying by KV, we obtain

$$(R - P)^* = VK(R - P)KV \tag{12}$$

Using equation (11), we get $(R-P)^{\phi}P = 0$.
$\implies (KV(R-P)^{*}VK)P = 0$.
Now, by using (12)

$$\begin{aligned}&(KV(VK(R-P)KV)VK)P = 0\\ &I(R-P)IP = 0\\ &(R-P)P = 0\\ &RP - PP = 0\\ &PP = RP\end{aligned}$$

$\implies P = R$. ■

Theorem 2.9: If P and R are s-k-unitary matrices then $P\underset{\phi}{\leq}R \implies VKP\underset{*}{\leq}VKR$.

Proof: Assume $P\underset{\phi}{\leq}R$, which implies $P^{\phi}P = P^{\phi}R$ and $PP^{\phi} = RP^{\phi}$.
From $P^{\phi}P = P^{\phi}R$,
We have, $(KVP^{*}VK)P = (KVP^{*}VK)R$.
$\implies (PVK)^{*}(VKP) = (PVK)^{*}(VKR)$
Since P is s-k-unitary,
Therefore, $PVK = VKP$, and hence, $(PVK)^{*} = (VKP)^{*}$.
Thus,

$$(VKP)^{*}(VKP) = (VKP)^{*}(VKR). \tag{13}$$

Again, $PP^{\phi} = RP^{\phi}$,
which gives, $P(KVP^{*}VK) = R(KVP^{*}VK)$.
Post- multiplying both sides by KV yields,

$$PKVP^{*} = RKVP^{*}$$

Pre- multiplying by VK, we obtain

$$\begin{aligned}VKPKVP^{*} &= VKRKVP^{*}\\ (VKP)(PVK)^{*} &= (VKR)(PVK)^{*}\end{aligned}$$

This implies,

$$(VKP)(VKP)^{*} = (VKR)(VKP)^{*} \tag{14}$$

From equations (13) and (14), we conclude: $VKP\underset{*}{\leq}VKR$.
Hence, the result. ■

Theorem 2.10: If P and R are s-k-unitary matrices then $P\underset{*}{\leq}R \implies KVP\underset{\phi}{\leq}KVR$.

Proof: Given, $P\underset{*}{\leq}R \implies P^{*}P = P^{*}R$ and $PP^{*} = RP^{*}$.
Using, $P^{*}P = P^{*}R$
Pre- multiplying both sides by KV:

$$KVP^{*}P = KVP^{*}R.$$

$$\begin{aligned}KVP^{*}VKKVP &= KVP^{*}VKKVR\\ P^{\phi}KVP &= P^{\phi}KVR\end{aligned}$$

Again, pre-multiplying both sides by VK,
$VKP^{\phi}KVP = VKP^{\phi}KVR$.
Using the fact that

$$(PKV)^{\phi}KVP = (PKV)^{\phi}KVR.$$

Since, P is s-k-unitary, therefore,
$PKV = KVP \implies (PKV)^{\phi} = (KVP)^{\phi}$

$$(KVP)^{\phi}KVP = (KVP)^{\phi}KVR \tag{15}$$

Now, from $PP^{*} = RP^{*}$, post-multiplying both sides by VK gives, $PP^{*}VK = RP^{*}VK$,

$$\begin{aligned}PVKKVP^{*}VK &= RVKKVP^{*}VK,\\ PVKP^{\phi} &= RVKP^{\phi}.\end{aligned}$$

Pre-multiplying both sides by KV gives.

$$\begin{aligned}KVPVKP^{\phi} &= KVRVKP^{\phi},\\ (KVP)(PKV)^{\phi} &= (KVR)(PKV)^{\phi},\\ (KVP)(KVP)^{\phi} &= (KVR)(KVP)^{\phi}.\end{aligned} \tag{16}$$

Combining equations (15) and (16), we get $KVP\underset{\phi}{\leq}KVR$.
This completes the proof. ■

As discussed earlier, Hartwig and Styan [3] proved $P\underset{*}{\leq}R \iff P\underset{rs}{\leq}R$. Based on this and the previous theorem, it follows that the ϕ- partial ordering is also closely related to the minus partial ordering or rank subtractivity.

III. Characterizations of $P\underset{\phi}{\leq}R$

In this section, we present characterizations of ϕ- partial ordering specifically for s-k-normal matrices. For these matrices spectral decomposition becomes significantly more accessible, allowing for a clear understanding of their underlying structure. We will also establish relationships concerning the squares of two s-k-normal matrices under the ϕ- partial ordering. A matrix $P \in M_n(\mathbb{C})$ is said to be s-k-normal if $PP^{\phi} = P^{\phi}P$.

Theorem 3.1: If P and R are two s-k-normal matrices with $1 \leq rank(P) < rank(R)$, then the following statements are equivalent:

(a) $P\underset{\phi}{\leq}R$

(b) There is a s-k-unitary matrix T such that $T^{\phi}PT = \begin{bmatrix} M & 0\\ 0 & 0\end{bmatrix}$ and $T^{\phi}RT = \begin{bmatrix} M & 0\\ 0 & N\end{bmatrix}$,
where M is a s-k-diagonal matrix and $N \neq 0$ is a s-k-diagonal matrix.

(c) There is a s-k-unitary matrix T such that
$$T^{\phi}PT = \begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix} \text{ and } T^{\phi}RT = \begin{bmatrix} E & 0\\ 0 & F\end{bmatrix},$$
where E is a non-singular square matrix and $F \neq 0$.

(d) There is a s-k-unitary matrix T such that
$$T^{\phi}PT = \begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix} \text{ and } T^{\phi}RT = \begin{bmatrix} E' & 0\\ 0 & F\end{bmatrix},$$
where E is a non-singular square matrix and E' is a square matrix of the same dimension and $F \neq 0$, then $E = E'$.

(e) There is a s-k-unitary matrix T such that
$$T^{\phi}PT = \begin{bmatrix} M & 0\\ 0 & 0\end{bmatrix} \text{ and } T^{\phi}RT = \begin{bmatrix} M' & 0\\ 0 & N\end{bmatrix},$$
where M and M' are s-k-diagonal matrices of the same dimension, and $N \neq 0$ is an s-k-diagonal matrix, then $M = M'$.

(f) If T is a s-k-unitary matrix satisfies $T^{\phi}PT = \begin{bmatrix} M & 0\\ 0 & 0\end{bmatrix}$, where M is a non-singular s-k-diagonal matrix, then $T^{\phi}RT = \begin{bmatrix} M & 0\\ 0 & F\end{bmatrix}$, where $F \neq 0$.

(g) All s-k-eigenvectors corresponding to non-zero s-k-eigenvalues of P are s-k-eigen vector of R corresponding to same eigen values.

The proof of this theorem is structured in four parts:
In part(1), we establish the logical sequence
$(a) \implies (b) \implies (c) \implies (a)$.

Proof: $(a) \implies (b)$
Given that $P \underset{\phi}{\leq} R \implies P^{\phi}P = P^{\phi}R$ and $PP^{\phi} = RP^{\phi}$, where both P and R are s-k-normal matrices. Since P is s-k-normal, it satisfies $\implies PP^{\phi} = P^{\phi}P$, which means that P^{ϕ} and P commute. Therefore, they are simultaneously s-k-diagonalizable or contains same eigen vectors. Furthermore, using s-k- normality property, P^{ϕ} and R also commute. i.e.

$$P^{\phi}P = P^{\phi}R$$
$$PP^{\phi} = P^{\phi}R$$

Since, $PP^{\phi} = RP^{\phi}$,
$\implies RP^{\phi} = P^{\phi}R$,
Therefore they are simmultaneouslay s-k-diagonalizable. Using all these results, we can say that P and R are also simmultaneouslay s-k-diagonalizable, hence, P^{ϕ} and R commutative. Let us suppose that M and N are s-k-diagonal matrices of matrices P and R, respectively, then there exists a s-k-unitary matrix T such that

$$P = T\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}T^{\phi} \text{ and } R = T\begin{bmatrix} M' & 0 \\ 0 & N \end{bmatrix}T^{\phi},$$

where M is a s-k-diagonal matrix and $N \neq 0$.

Now, $P^{\phi} = T\begin{bmatrix} M^{\phi} & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}$.
Therefore,

$$P^{\phi}P = T\begin{bmatrix} M^{\phi} & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}T\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}$$

$$P^{\phi}P = T\begin{bmatrix} M^{\phi}M & 0 \\ 0 & 0 \end{bmatrix}T^{\phi} \tag{17}$$

and, $P^{\phi}R = T\begin{bmatrix} M^{\phi} & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}T\begin{bmatrix} M' & 0 \\ 0 & N \end{bmatrix}T^{\phi}$

$$P^{\phi}R = T\begin{bmatrix} M^{\phi}M' & 0 \\ 0 & 0 \end{bmatrix}T^{\phi} \tag{18}$$

From equations (17) and (18), we have
$P^{\phi}P = P^{\phi}R \implies M^{\phi}M = M^{\phi}M' \implies M = M'$. ■

Proof: $(b) \implies (c)$.
This implication follows trivially. ■

Proof: $(c) \implies (a)$

Given, $T^{\phi}PT = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ and $T^{\phi}RT = \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}$,

where E is a non-singular square matrix and $F \neq 0$.
Therefore,

$$P = T\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}T^{\phi},\ R = T\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}T^{\phi}$$

Then, we can write $P^{\phi} = T\begin{bmatrix} E^{\phi} & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}$
Thus,

$$P^{\phi}P = T\begin{bmatrix} E^{\phi} & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}T\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}$$

$$P^{\phi}P = T\begin{bmatrix} E^{\phi}E & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}. \tag{19}$$

Also,

$$P^{\phi}R = T\begin{bmatrix} E^{\phi} & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}T\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}T^{\phi}$$

$$P^{\phi}R = T\begin{bmatrix} E^{\phi}E & 0 \\ 0 & 0 \end{bmatrix}T^{\phi} \tag{20}$$

From equations (19) and (20), we observed that
$P^{\phi}P = P^{\phi}R \implies P \underset{\phi}{\leq} R$. ■

Part (2):
Proof: $(a) \implies (d) \implies (e) \implies (a)$
By applying a similar approach outlined above, we can easily establish this which is trvial modification of previous results. ■

Part (3): We now prove that $(b) \Leftrightarrow (f)$.
Proof: $(b) \implies (f)$;
Assume that condition (b) holds.
Let T be a s-k-unitary matrix such that $T^{\phi}PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$.
By (b), there exists a s-k-unitary matrix U such that
$U^{\phi}PU = \begin{bmatrix} M' & 0 \\ 0 & 0 \end{bmatrix}$, $U^{\phi}RU = \begin{bmatrix} M' & 0 \\ 0 & N \end{bmatrix}$
where M' is a non-singular s-k-diagonal matrix and $N \neq 0$ is a s-k-diagonal matrix.
By interchanging the columns of U if necessary, we assume $M' = M$.
Let $T = (T_1, T_2)$ be a partition.
Then, we have

$$T^{\phi}PT = \begin{bmatrix} {T_1}^{\phi} \\ {T_2}^{\phi} \end{bmatrix} P \begin{bmatrix} T_1 & T_2 \end{bmatrix}$$

$$= \begin{bmatrix} {T_1}^{\phi}PT_1 & {T_1}^{\phi}PT_2 \\ {T_2}^{\phi}PT_1 & {T_2}^{\phi}PT_2 \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \tag{21}$$

For the corresponding partition $U = (U_1, U_2)$, we have,

$$U^{\phi}PU = \begin{bmatrix} {U_1}^{\phi} \\ {U_2}^{\phi} \end{bmatrix} P \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

$$= \begin{bmatrix} {U_1}^{\phi}PU_1 & {U_1}^{\phi}PU_2 \\ {U_2}^{\phi}PU_1 & {U_2}^{\phi}PU_2 \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \tag{22}$$

and

$$U^{\phi}RU = \begin{bmatrix} {U_1}^{\phi} \\ {U_2}^{\phi} \end{bmatrix} R \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

$$= \begin{bmatrix} {U_1}^{\phi}RU_1 & {U_1}^{\phi}RU_2 \\ {U_2}^{\phi}RU_1 & {U_2}^{\phi}RU_2 \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \tag{23}$$

Observe that, (22) $\implies P = U\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}U^{\phi}$

$$\implies P = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} {U_1}^{\phi} \\ {U_2}^{\phi} \end{bmatrix}$$

$$= \begin{bmatrix} U_1M & 0 \end{bmatrix}\begin{bmatrix} {U_1}^{\phi} \\ {U_2}^{\phi} \end{bmatrix}$$

$\implies P = U_1M{U_1}^{\phi}$.

Equation (23) $\implies R = U\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}U^{\phi}$

$$T^{\phi}RT = T^{\phi}U\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}U^{\phi}T$$

$$= \begin{bmatrix} {T_1}^{\phi} \\ {T_2}^{\phi} \end{bmatrix}\begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}\begin{bmatrix} {U_1}^{\phi} \\ {U_2}^{\phi} \end{bmatrix}\begin{bmatrix} T_1 & T_2 \end{bmatrix}$$

$$=\begin{bmatrix} T_1{}^{\phi}U_1 & T_1{}^{\phi}U_2 \\ T_2{}^{\phi}U_1 & T_2{}^{\phi}U_2 \end{bmatrix}\begin{bmatrix} MU_1{}^{\phi}T_1 & MU_1{}^{\phi}T_2 \\ NU_2{}^{\phi}T_1 & NU_2{}^{\phi}T_2 \end{bmatrix}$$

By applying the normality condition, we have

$$=\begin{bmatrix} T_1{}^{\phi}U_1 & 0 \\ 0 & T_2{}^{\phi}U_2 \end{bmatrix}\begin{bmatrix} MU_1{}^{\phi}T_1 & 0 \\ 0 & NU_2{}^{\phi}T_2 \end{bmatrix}$$

$$=\begin{bmatrix} T_1{}^{\phi}U_1MU_1{}^{\phi}T_1 & 0 \\ 0 & T_2{}^{\phi}U_2NU_2{}^{\phi}T_2 \end{bmatrix}$$

$$T^{\phi}RT=\begin{bmatrix} T_1{}^{\phi}PT_1 & 0 \\ 0 & T_2{}^{\phi}U_2NU_2{}^{\phi}T_2 \end{bmatrix}$$

Since, $T_1{}^{\phi}PT_1 = M$, let us take

$T_2{}^{\phi}U_2NU_2{}^{\phi}T_2 = F$, $F \neq 0$.

Therefore, $T^{\phi}RT = \begin{bmatrix} M & 0 \\ 0 & F \end{bmatrix}$.
Hence, $(b) \implies (f)$.

Next, to prove that $(f) \Rightarrow (b)$.
Suppose (f) is true. Let T be a s-k-unitary matrix such that $T^{\phi}PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, where M is non-singular s-k-diagonal matrix. Then by (f), $T^{\phi}RT = \begin{bmatrix} M & 0 \\ 0 & F \end{bmatrix}$, where $F \neq 0$. Since F is s-k-normal, there exists a s-k-unitary matrix Y such that $N = Y^{\phi}FY$, which is a s-k-diagonal matrix.

Let $W = T\begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix}$
Then,

$$W^{\phi}PW = \begin{bmatrix} I & 0 \\ 0 & Y^{\phi} \end{bmatrix} T^{\phi}PT \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix}$$

$$=\begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$W^{\phi}RW = \begin{bmatrix} I & 0 \\ 0 & Y^{\phi} \end{bmatrix} R^{\phi}RT \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix}$$

$$=\begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

Finally, the proof of part (4), $(a) \Leftrightarrow (g)$, is straightforward, since s-k-normality ensures simulaneous s-k-diagonalizability, which preserves s-k-eigenvalues and eigenvectors, leading to the confirmation of ϕ- partial structure.
Hence the result. ■

The next theorem addresses the converse for s-k-normal matrices: if $P \underset{\phi}{\leq} R \implies . P^2 \underset{\phi}{\leq} R^2$.

Theorem 3.2: Let P and R be s-k-normal matrices with $I \leq rank(P) < rank(R)$. Then, the following two statements are equivalent:

(a) $P \underset{\phi}{\leq} R$.
(b) $P^2 \underset{\phi}{\leq} R^2$ and if P and R has non-zero s-k-eigen values α and β respectively such that α^2 and β^2 are eigenvalues of P^2 and R^2 with a common eigenvector Y, then $\alpha = \beta$ and Y is a common eigenvector of P and R.

Proof: Let us consider $P \underset{\phi}{\leq} R \implies P^{\phi}P = P^{\phi}R$.
Let T be a s-k-unitary matrix such that $T^{\phi}PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$ and $T^{\phi}RT = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$.
By part (b) of theorem (3.1), we also know that

$$T^{\phi}P^2T = \begin{bmatrix} M^2 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } T^{\phi}R^2T = \begin{bmatrix} M^2 & 0 \\ 0 & N^2 \end{bmatrix}.$$

Let α and β be non-zero s-k-eigenvalues of P and R respectively. Therefore, α^2 and β^2 have non-zero s-k eigenvalues of P^2 and R^2 respectively. Suppose Y is the common s-k-eigenvector of P^2 and R^2, in that case, we have $\alpha = \beta$ and Y is a common s-k-eigen vector of P and R.

Conversely, suppose that statement (b) holds. Then $T^{\phi}P^2T = \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix}$ and $T^{\phi}R^2T = \begin{bmatrix} \Delta^2 & 0 \\ 0 & \Gamma^2 \end{bmatrix}$, where T, Δ, and Γ are appropriate matrices obtained by applying part (b) of theorem (3.1) to P^2 and R^2.

Let $T_{sk(1)}$, $T_{sk(2)}$....,$T_{sk(m)}$ be the column vectors of T, and denote $r = rank(P)$.
For $i = 1, 2, 3, ..., r$, we have $P^2 t_{sk(i)} = R^2 t_{sk(i)} = \chi_{sk(i)} t_{sk(i)}$, where $\chi_{sk(i)} = diag\Delta$.
Thus, by the second part of (b), there exist complex numbers $d_{sk(1)}$, $d_{sk(2)}$, $d_{sk(3)}$,...., $d_{sk(r)}$ such that for all i=1,2,3,...,r, we have $d^2_{sk(i)} = \chi_{sk(i)}$ and $Pt_{sk(i)} = Rt_{sk(i)} = \chi_{sk(i)} t_{sk(i)}$.
Let D be the s-k-diagonal matrix with $d_{sk(i)} = diagD$. For $i = r+1, r+2, ..., n$,
we have $R^2 t_{sk(i)} = \mu_{sk(i-r)} t_{sk(i)}$, where $\mu_{sk(i)} = diag\Gamma$.
Now, Take complex numbers $n_{sk(1)}$,$n_{sk(2)}$, $n_{sk(3)}$,...., $n_{sk(m-r)}$ satisfying $n^2_{sk(i)} = \mu_{sk(i)}$ for $i = 1, 2, ..., m-r$.
Let N be the s-k-diagonal with $n_{sk(i)} = diag(N)$.

Then, $T^{\phi}PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, and $T^{\phi}RT = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$

By applying part (b) of theorem (3.1), this equation satisfies condition (a).
Hence, we conclude that $P^2 \underset{\phi}{\leq} R^2 \implies P \underset{\phi}{\leq} R$. ■

Theorem 3.3: Let P and R be s-k-normal matrices. If $P \underset{\phi}{\leq} R$, then $PR = RP$.

Proof: Let P and R be s-k-normal matrices. Then, by definition of s-k-normality, we have $PP^{\phi} = P^{\phi}P$ and $RR^{\phi} = R^{\phi}R$. By part (b) of theorem (3.1), we have

$$T^{\phi}PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \text{ and } T^{\phi}RT = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}.$$

$$\implies P = T\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}T^{\phi} \text{ and } R = T\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}T^{\phi}$$

$$\implies PR = T\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}T\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}T^{\phi}$$
$$= T\begin{bmatrix} M^2 & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}.$$

Similarly, $RP = T\begin{bmatrix} M^2 & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}$.
$\implies PR = RP$. ■

Corollary 3.4: Let P and R be s-k-normal matrices whose eigenvalues all have positive real parts. Then $P^2 \underset{\phi}{\leq} R^2$ if and only if $P \underset{\phi}{\leq} R$.

Theorem 3.5: Let P and R be s-k-normal matrices with $I \leq rank(P) < rank(R)$. Then

(a) $P \underset{\phi}{\leq} R$
is equivalent to the following:

(b) $P^2 \underset{\phi}{\leq} R^2$ and if $T^\phi PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, $T^\phi RT = \begin{bmatrix} MM_1 & 0 \\ 0 & N \end{bmatrix}$, where T is a s-k-unitary matrix, M is a non-singular s-k-diagonal matrix, M_1 is a s-k-unitary diagonal matrix, and $N \neq 0$ is a s-k-diagonal matrix, then $M_1 = I$.

Proof: For the first direction $(a) \Rightarrow (b)$, refer to the proof of theorem (3.2) and, for the second part of $(a) \Rightarrow (b)$, see part (e) of theorem (3.1).
Conversely, assume that (b) holds. then, as established in the proof of theorem (3.2), we have
$T^\phi P^2 T = \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix}$ and $T^\phi R^2 T = \begin{bmatrix} \Delta^2 & 0 \\ 0 & \Gamma^2 \end{bmatrix}$.
Hence, $T^\phi PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$ and $T^\phi RT = \begin{bmatrix} M' & 0 \\ 0 & N \end{bmatrix}$
where M and M' are s-k-diagonal matrices satisfying $M^2 = (M')^2 = \Delta$ and N is s-k-diagonal matrices satisfying $N^2 = \Gamma$. Denote $d_{sk(i)} = diagD$, $d'_{sk(i)} = diagD'$, $r = rank(P)$. Then, for all $i = 1, 2, \ldots, r$, we have $d^2_{sk(i)} = (d'_{sk(i)})^2$. Hence, there exist complex numbers $m_{sk(1)}, m_{sk(2)}, \ldots, m_{sk(r)}$ such that $|m_{sk(1)}| = |m_{sk(2)}| = \ldots = |m_{sk(r)}|$ and $d'_{sk(i)} = d_{sk(i)} m_{sk}(i)$ for all $i = 1, 2, 3, \ldots, r$.
Let M_1 be the s-k-diagonal matrix with $m_{sk}(i) = diagM_1$. Then $M' = MM_1$, and so $M' = M$. Thus, condition (a) follows, since part (b) of theorem (3.1) is satisfied. ■

Next, we turn to the characterizations of minus partial ordering (rank subtractivity) for s-k-normal matrices. In this context, we examine the equivalence relation $P \underset{rs}{\leq} R$ and $P^2 \underset{rs}{\leq} R^2 \Leftrightarrow P \underset{\phi}{\leq} R$. To establish this equivalence, we make use of a fundamental result due to Israel and Greville [[12],p.178], which we present below as a lemma:

Lemma 3.6: Let $P \in M_n(\mathbb{C})$, $1 \leq r \leq n$, $s = n - r$, then the following statements are equivalent:

(a) $rank(P) = r$

(b) If $N \in M_r(\mathbb{C})$ is a non-singular submatrix of P, then there are permutation matrices A and $B \in M_n(\mathbb{R})$ and matrices $Q \in M_{s \times r}(\mathbb{C})$, $S \in M_{r \times s}(\mathbb{C})$ such that
$P = A \begin{bmatrix} QNS & QN \\ NS & N \end{bmatrix} R.$

Theorem 3.7: Let P and R be s-k-normal matrices. If $a = rank(P)$, $b = rank(R)$, $1 \leq a \leq b \leq n$ and $p = b - a$, then the following conditions are equivalent:

(a) $P \underset{rs}{\leq} R$.

(b) There exists a s-k-unitary matrix $T \in M_n(\mathbb{C})$ such that
$T^\phi PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$ and
$T^\phi RT = \begin{bmatrix} M + QNS & QN & 0 \\ NS & N & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
where $M \in M_a(\mathbb{C})$, $N \in M_p(\mathbb{C})$ are non-singular s-k-diagonal matrices, $Q \in M_{a \times p}(\mathbb{C})$ and $S \in M_{p \times a}(\mathbb{C})$.

(c) There exists a s-k-unitary matrix $T \in M_n(\mathbb{C})$ such that
$T^\phi PT = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$ and
$T^\phi RT = \begin{bmatrix} M + QES & QE & 0 \\ ES & E & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
where $F \in M_a(\mathbb{C})$ and $E \in M_p(\mathbb{C})$ are non-singular matrices, $Q \in M_{a \times p}(\mathbb{C})$ and $S \in M_{p \times a}(\mathbb{C})$.

Proof: First, we prove that $(c) \implies (a)$.
Assume condition (c) holds. Then, we have

$$T^\phi RT - T^\phi PT = \begin{bmatrix} QES & QE & 0 \\ ES & E & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^\phi (R - P)T = \begin{bmatrix} QES & QE & 0 \\ ES & E & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} QES & QE & 0 \\ ES & E & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

,
where $(B - A) = (TCT^\phi)$ satisfies
$rank(C) = rank(B - A)$.
On the other hand, by Lemma (3.6), we have
$rank(C) = rank(F) = p = b - a = rank(R) - rank(P)$
$rank(R - P) = rank(R) - rank(P)$,
which proves $(a) \implies (b)$.
Assume that P and R satisfy condition (a). Then, using the notations from [Theorem 1, [1]], we have
$T^\phi PU = \begin{bmatrix} \sum & 0 \\ 0 & 0 \end{bmatrix}$, and
$T^\phi RU = \begin{bmatrix} \sum + QNS & QN & 0 \\ NS & N & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
The singular values of a s-k-normal matrix are the absolute values of its s-k-eigenvalues. Therefore, the s-k-diagonal matrix formed by of s-k-eigenvalues of A is $M_0 = \sum_0 J$, where J is a s-k-diagonal matrix of elements with absolute value one. Furthermore, $U = TJ^{-1}$ and $T^\phi PU = M_0 = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, where M is the s-k-diagonal matrix of the non-zero s-k-eigenvalues of P. Let us denote:
$J = \begin{bmatrix} L & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & N \end{bmatrix}$.
Next, $T^\phi RT = T^\phi RUJ = \begin{bmatrix} \sum + QNS & QN & 0 \\ NS & N & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} L_1 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & N_1 \end{bmatrix}$

$$T^\phi RT = \begin{bmatrix} \sum L_1 + QNSL_1 & QNM_1 & 0 \\ NSL_1 & NM_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^\phi RT = \begin{bmatrix} M_1 + QNSL_1 & QNM_1 & 0 \\ NSL_1 & NM_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By (a), we have
$b - a = rank(R - P) = rank((T^\phi (R - P))T)$
$= rank \begin{bmatrix} QNSL_1 & QNM_1 \\ NSL_1 & NM_1 \end{bmatrix}$.

Denote $N' = NM_1$. Since N and M_1 are non-singular, it follows that $rank(N') = b - a$.
Hence, by lemma (3.6), there exists matrices $Q' \in M_{axp}(\mathbb{C})$ and $S' \in M_{a\times p}(\mathbb{C})$ such that
$\begin{bmatrix} QNSL_1 & QNM_1 \\ NSL_1 & NM_1 \end{bmatrix} = \begin{bmatrix} Q'N'S' & Q'N' \\ N'S' & N' \end{bmatrix}$.
Consequently, $T^{\phi}RT = \begin{bmatrix} M + Q'N'S' & Q'N' & 0 \\ N'S' & N' & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
which confirms part (b).
The implication $(b) \implies (c)$ is trivial. ■

Corollary 3.8: Let P, $R \in M_n(\mathbb{C})$. If P is s-k-normal, R is s-k-Hermitian and $P \underset{rs}{\leq} R$, then P is s-k-Hermitian.

Proof: If $rank(P) = 0$ or $rank(P) = rank(R)$, then the result follows immediately.
Otherwise, using theorem 3.7, we proceed as follows:
$P' = T^{\phi}PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, and
$R' = T^{\phi}RT = \begin{bmatrix} M + QNS & QN & 0 \\ NS & N & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
Since R is s-k-Hermitian, R' is also s-k-Hermitian. Therefore, $N^{\phi} = N$ and $NS = (QN)^{\phi} = NQ^{\phi}$, which implies $S = Q^{\phi}$, since N is non-singular.
Now consider, $P' = R' - \begin{bmatrix} QNQ^{\phi} & QN & 0 \\ NQ^{\phi} & N & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
which is the difference of two s-k-Hermitian matrices. Hence, and is therefore s-k-Hermitian. Hence, P' is also s-k-Hermitian, and therefore P is s-k-Hermitian. ■

Theorem 3.9: Let P and R be two s-k-normal matrices with $1 \leq rank(P) < rank(R)$, then the following conditions are equivalent:

(a) $P \underset{\phi}{\leq} R$
(b) There is a s-k-unitary matrix $T \in M_n(\mathbb{C})$ such that $T^{\phi}PT = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, $T^{\phi}RT = \begin{bmatrix} M & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
where M and N are non-singular s-k-diagonal matrices. (if $b = n$, then omit the third block row and block-column of zeros in the expression of R).

Proof: Refer to theorem (3.1). ■

Theorem 3.10: Let P,R be s-k-normal and suppose that $(R - P)$ is s-k-Hermitian. Then the following conditions are equivalent:

(a) $P \underset{\phi}{\leq} R$
(b) $P \underset{rs}{\leq} R$ and $P^2 \underset{rs}{\leq} R^2$.

Proof: $(a) \implies (b)$
The implication follows directly from theorems (3.1) and (3.9).
$(b) \implies (a)$.
Assume that (b) holds. Using the notations from theorem 3.1, we have $P = T\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}$

$$R = T\begin{bmatrix} M + QNS & QN & 0 \\ NS & N & 0 \\ 0 & 0 & 0 \end{bmatrix}T^{\phi}.$$

Since (R-P) is s-k-Hermitian, it follows that $T^{\phi}(R - P)T$ is also s-k-Hermitian. Therefore, N is s-k-Hermitian, and $S = Q^{\phi}$. We can write R as:

$$R = T\begin{bmatrix} M + QNQ^{\phi} & QN & 0 \\ NQ^{\phi} & N & 0 \\ 0 & 0 & 0 \end{bmatrix}T^{\phi}.$$

Furthermore,
$P^2 = T\begin{bmatrix} M^2 & 0 \\ 0 & 0 \end{bmatrix}T^{\phi}$, and

$$R^2 = T\begin{bmatrix} (M + QNQ^{\phi})^2 + QN^2Q^{\phi} & (M + QNQ^{\phi})QN + QN^2 & 0 \\ NQ^{\phi}(M + QNQ^{\phi}) + N^2Q^{\phi} & NQ^{\phi}QN + N^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}T^{\phi}.$$

Thus, the difference becomes

$$R^2 - P^2 = T\begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix}T^{\phi},$$

where,

$$H_1 = T\begin{bmatrix} (MQNQ^{\phi} + QNQ^{\phi})^2 + QN^2Q^{\phi} & (M + QNQ^{\phi})QN + QN^2 \\ NQ^{\phi}(M + QNQ^{\phi}) + N^2Q^{\phi} & NQ^{\phi}QN + N^2 \end{bmatrix}T^{\phi}.$$

Applying row and column operation, which do not affect the rank, we reduce H_1 to:
$rank(H_1) = rank\begin{bmatrix} 0 & MQN \\ NQ^{\phi}M & NQ^{\phi}QN + N^2 \end{bmatrix}$.
Since, $P^2 \underset{rs}{\leq} R^2$, we have
$rank(H_1) = rank(R^2 - P^2) = rank(R^2) - rank(P^2) = b - a = p$.
Because $NQ^{\phi}QN$ is s-k-Hermitian non-singular definite, and N is s-k-Hermitian positive definite, their sum $N' = NQ^{\phi}QN + N^2$ is s-k-Hermitian positive definite, and therefore non-singular.
Applying lemma 3.6 to H_1, we find that there exists a matrix $S \in M_{p\times a}(\mathbb{C})$ such that $S^{\phi}N' = MQN$ and $S^{\phi}N'S = 0$. Since N' is positive definite, it follows that $S = 0$. Therefore, $S^{\phi}N' = MQN$ reduces to $MQN = 0$, which in turn implies $Q = 0$ due to the non-singularity of M and N.
Hence,

$$R = T\begin{bmatrix} M & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & 0 \end{bmatrix}T^{\phi}$$

and condition (a) follows from theorem 3.9. ■

IV. Conclusion

In this paper, we have delved into the concept of ϕ- partial ordering and its connections to other partial orders, with a special focus on the ϕ- partial ordering. We started by defining a ϕ- partial order and proved several theorems demonstrating how this ordering behaves in different contexts, including matrix inversion and multiplication.
Next, we explored the preservation of ϕ- partial order under various matrix operations, such as squares and s-k-unitary similarity. One of the significant contributions of this study was the characterization of ϕ- partial ordering for s-k-normal matrices, as well as the characterizations of minus partial ordering are described in the context of s-k-normal and s-k-unitary matrices.
Furthermore, we established that the squares of s-k normal matrices retain the ϕ- partial order under specific conditions, adding depth to our understanding of how partial orders operate on matrix squares. Through the exploration of ϕ- partial ordering and its broader implications, this paper contributes to the continued development of matrix theory.

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