Coefficient Bounds of Sakaguchi Type Functions Associated with Gegenbauer Polynomials

P. Lokesh, S. Prema* and R. Ambrose Prabhu

Abstract—**In this investigation, two subclasses of Sakaguchi functions associated with Gegenbauer are introduced. Further, coefficient bounds and Fekete-Szegö inequalities for functions belonging to these subclasses are obtained.**

Index Terms—**Analytic function, Subordination, Coefficient bounds, Fekete-Szegö problem, Sakaguchi function, Gegenbauer polynomials.**

I. Introduction

ORTHOGONAL polynomials were discovered by Legendre in 1784 [19], and they are widely analyzed nowadays. In mathematical treatments of model problems, they help in finding solutions to ordinary differential equations under certain conditions imposed by the model.

Orthogonal polynomials play a vital role in contemporary mathematics. They are frequently applied in physics and engineering and play a dominant role in problems of approximation theory. In general, they appear in the theory of differential and integral equations, as well as in mathematical statistics. They are also applied in quantum mechanics, scattering theory, automatic control, signal analysis, and oscillatory symmetric potential theory [14], [15].

Generally speaking, polynomials ρ_n and ρ_m of orders n and m are orthogonal if

$$\int_a^b w(x)\rho_n(x)\rho_m(x)\,dx = 0, \quad \text{for } n \neq m.$$

where $w(x)$ is a non-negative function in the interval (a, b); therefore, the integral is well-defined for all finite-order polynomials $\rho_n(x)$.

In this context, Gegenbauer polynomials act as a special case of orthogonal polynomials. They are representatively related to the typically real function T_n, as discovered in [18], where the representation of typically real functions and the generating function of Gegenbauer polynomials share a common algebraic expression. Subsequently, this leads to several inequalities that arise from the realm of Gegenbauer polynomials.

Typically, real functions play a vital role in geometric function theory due to the relation $T_n = \overline{c_0}S_R$ and their role in estimating coefficient bounds, where S_R denotes the class of univalent functions in the unit disk with real coefficients. The notation $\overline{c_0}S_R$ denotes the closed convex hull of S_R.

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This paper interrelates certain new classes of Sakaguchi functions with Gegenbauer polynomials and then explores some properties of the class at hand. This section paves the way for mathematical notations and definitions.

Let A indicate the class of analytic functions given below:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Furthermore, S denotes the class of all functions in A that are univalent in Δ.

A subordination between two analytic functions f and h is written as $f \prec h$. Conceptually, the analytic function f is subordinate to h if the image under h contains the image under f. Technically, the analytic function f is subordinate to h if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$, such that:

$$f(z) = h(w(z)),$$

Besides, if the function h is univalent in Δ, then the following equivalence holds: refer [23].

$$f(z) \prec h(z) \Leftrightarrow f(0) = h(0),$$

and

$$f(\Delta) \subset h(\Delta).$$

The Koebe one-quarter theorem (see [16]) states that the image of Δ under such univalent functions $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, each function $f \in S$ has an inverse f^{-1} that satisfies:

$$f^{-1}(f(z)) = z (z \in \Delta),$$

and

$$f(f^{-1}(w)) = w\{|w| < r_0(f) : r_0(f) \geq \frac{1}{4}\}.$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4 + ... \tag{2}$$

If both f and f^{-1} are univalent, then $f \in A$ is said to be bi-univalent in A. The class of bi-univalent functions is represented by $\sum$ in Δ, as given in (1). In the literature [20], [24], [29], [30], various information and different examples can be found.

Frasin [12] investigated the coefficient inequalities for certain classes of Sakaguchi-type functions satisfying geometric conditions as follows:

$$\mathbb{R}\left\{\frac{(s-t)z(f'(z))}{f(sz) - f(tz)}\right\} > \alpha \tag{3}$$

for complex numbers s, t but $s \neq t$ and α $(0 \leq \alpha < 1)$.

For a nonzero real constant λ, the generating function of the Gegenbauer polynomials is defined as:

$$B_\lambda(\tau, z) = \frac{1}{(1-2\tau z+z^2)^\lambda}, \quad (4)$$

where $\tau \in [-1,\ 1]$ and $z \in \Delta$. For fixing τ, the function B_λ is analytic in Δ. Hence it can be expanded in a Taylor series as

$$B_\lambda(\tau, z) = \sum_{n=0}^{\infty} C_n^\lambda(\tau) z^n, \quad (5)$$

where $C_n^\lambda(\tau)$ is Gegenbauer polynomial of degree n.

Obviously, B_λ generates nothing when $\lambda = 0$. Therefore, the generating function of the Gegenbauer polynomial is given by:

$$B_\lambda(\tau, z) = 1 - log(1-2\tau z+z^2) = \sum_{n=0}^{\infty} C_n^\lambda(\tau) z^n, \quad (6)$$

for $\lambda = 0$. Moreover, it is worth mentioning that a normalization of λ greater than $\frac{-1}{2}$ is desirable [15], [21], [26]. Gegenbauer polynomials can also be defined by the following recurrence relation:

$$C_n^\lambda(\tau) = \frac{1}{n}[2\tau(n+\lambda-1)C_{n-1}^\lambda(\tau) - (n+2\lambda-2)C_{n-1}^\lambda(x)], \quad (7)$$

with the conditional values

$$C_0^\lambda(\tau) = 1, C_1^\lambda(\tau) = 2\tau\lambda \text{ and } C_2^\lambda(\tau) = 2\lambda(1+\lambda)\tau^2 - \lambda.$$

First, we present some special cases of the polynomials $C_n^\lambda(\tau)$:

1. For $\lambda = 1$, we get the Chebyshev polynomials.
2. For $\lambda = \frac{1}{2}$, we get the Legendre polynomials.

Recently, many researchers have been exploring bi-univalent functions associated with orthogonal polynomials; a few of them are mentioned in [1]–[6], [8]–[11], [22], [25], [27], [28]. For the Gegenbauer polynomial, so far, no one has attempted to study Sakaguchi functions in the literature. The aim of this paper is to introduce two subclasses of starlike and convex functions of Sakaguchi functions associated with Gegenbauer polynomials. Additionally, the coefficient bounds and Fekete-Szegö inequalities belonging to these classes are obtained. Definition 1.1 defines a class of starlike Sakaguchi functions associated with the Gegenbauer polynomial as follows:

Definition 1.1: A function $f \in A$ given by (1) is said to be in the class $G^*(\lambda, m)$ if the following subordinations holds for all $z, w \in \Delta$:

$$\frac{(1-m)\ zf'(z)}{f(z)-f(mz)} \prec B_\lambda(\tau, z), \quad (8)$$

and

$$\frac{(1-m)\ wh'(w)}{h(w)-h(mw)} \prec B_\lambda(\tau, w). \quad (9)$$

where $z \in (\frac{1}{2}, 1]$ and $|m| \leq 1$ but $m \neq 1$, the function $h(w) = f^{-1}(w)$ is defined by (2) and B_λ is the generating function of the Gegenbauer polynomial is given by (4).

The following definition introduces a class of convex Sakaguchi functions associated with the Gegenbauer polynomial:

Definition 1.2: A function $f \in A$ given by (1) is said to be in the class $G_c(\lambda, m)$ if the following subordinations holds for all $z, w \in \Delta$:

$$\frac{((1-m)\ zf'(z))'}{(f(z)-f(mz))'} \prec B_\lambda(\tau, z), \quad (10)$$

and

$$\frac{((1-m)\ wh'(w))'}{(h(w)-h(mw))'} \prec B_\lambda(\tau, w). \quad (11)$$

where $z \in (\frac{1}{2}, 1]$ and $|m| \leq 1$ but $m \neq 1$, the function $h(w) = f^{-1}(w)$ is defined by (2) and B_λ is the generating function of the Gegenbauer polynomial is given by (4).

Remark 1.1: Taking the parameter $m = 0$ that was studied by Ala Amourah et al [7].

II. Coefficient Bounds for the Class $\mathbf{G^*(\lambda, m)}$

This section is devoted to finding the initial coefficient bounds of the class $G^*(\lambda, m)$ of Sakaguchi functions.

Theorem 2.1: For $|m| \leq 1$ but $m \neq 1$, let the function $f \in A$ given by (1) be in the class $G^*(\lambda, m)$. Then

$$|a_2| \leq \frac{2|\lambda|\tau\sqrt{2|\lambda|\tau}}{\sqrt{(1-m)|(4\lambda^2-(1-m)2\lambda(1+\lambda))\tau^2+(1-m)\tau|}}, \quad (12)$$

and

$$|a_3| \leq \frac{2|\lambda|\tau}{2-m-m^2} + \frac{4\lambda^2\tau^2}{(1-m)^2}. \quad (13)$$

Proof: Let $f \in G^*(\lambda, m)$. From (8) and (9), it is known that

$$\frac{(1-m)\ zf'(z)}{f(z)-f(mz)} = B_\lambda(\tau, w(z)), \quad (14)$$

and

$$\frac{(1-m)\ wh'(w)}{h(w)-h(mw)} = B_\lambda(\tau, \vartheta(w)), \quad (15)$$

for some analytic functions

$$w(z) = r_1 z + r_2 z^2 + r_3 z^3 + ... \quad (z \in \Delta),$$

$$\vartheta(w) = s_1 w + s_2 w^2 + s_3 w^3 + ... \quad (w \in \Delta).$$

such that $w(0) = \vartheta(0) = 0$, $|w(z)| < 1$ $(z \in \Delta)$ and $|\vartheta(w)| < 1$ $(w \in \Delta)$.

It follows from (14) and (15) that

$$\frac{(1-m)\ zf'(z)}{f(z)-f(mz)} = 1 + C_1^\lambda(\tau)r_1 z + [C_1^\lambda(\tau)r_2 + C_2^\lambda(\tau)r_1^2]z^2 + ...$$

and

$$\frac{(1-m)\ wh'(w)}{h(w)-h(mw)} = 1 + C_1^\lambda(\tau)s_1 w + [C_1^\lambda(\tau)s_2 + C_2^\lambda(\tau)s_1^2]w^2 + ...$$

By simple calculation show that

$$(1-m)a_2 = C_1^\lambda(\tau)r_1, \quad (16)$$

$$(2-m-m^2)a_3 - (1-m^2)a_2^2 = C_1^\lambda(\tau)r_2 + C_2^\lambda(\tau)r_1^2, \quad (17)$$

$$-(1-m)a_2 = C_1^\lambda(\tau)s_1, \quad (18)$$

$$(2a_2^2 - a_3)(2-m-m^2) - (1-m^2)a_2^2 = C_1^\lambda(\tau)r_2 + C_2^\lambda(\tau)r_1^2, \quad (19)$$

From (16) and (18), it is obtained that

$$r_1 = -s_1, \quad (20)$$

and

$$2(1-m)^2a_2^2=[C_1^\lambda(\tau)]^2(r_1^2+s_1^2). \quad (21)$$

By summing (17) and (19), it is found that

$$2(1-m)a_2^2=C_1^\lambda(\tau)(r_2+s_2)+C_2^\lambda(\tau)(r_1^2+s_1^2). \quad (22)$$

By using (21) in (22), it is obtained that

$$\left[2(1-m)-\frac{2C_2^\lambda(\tau)(1-m)^2}{[C_1^\lambda(\tau)]^2}\right]a_2^2=C_1^\lambda(\tau)(r_2+s_2). \quad (23)$$

It is well-known that [16], if $|w(z)|<1$ and $|\vartheta(w)|<1$, then

$$|r_j|\le 1 \text{ and } |s_j|\le 1 \text{ for all } j\in\mathbb{N}. \quad (24)$$

By considering (7) and (24), the required inequality (12) is obtained from (23).

Next, by subtracting (19) from (17), then

$$\begin{aligned}&2(2-m-m^2)a_3-2(2-m-m^2)a_2^2\\&=C_1^\lambda(\tau)(r_2-s_2)+C_2^\lambda(\tau)(r_1^2-s_1^2). \quad (25)\end{aligned}$$

Further, in view of (20), it follows from (25) that

$$a_3=\frac{C_1^\lambda(\tau)(r_2-s_2)}{2(2-m-m^2)}+a_2^2. \quad (26)$$

By considering (7) and (24), the desired inequality (13) is obtained from (26).

This completes the proof of Theorem 2.1. ■

By taking $\lambda=1$ in theorem 2.1, the following corollary is obtained.

Corollary 2.1: Let the function $f\in A$ given by (1) be in the class $G^*(1,m)$. Then

$$|a_2|\le\frac{2\tau\sqrt{2\tau}}{\sqrt{(1-m)|(4-(1-m)4)\tau^2+(1-m)|}},$$

and

$$|a_3|=\frac{4\tau^2}{(1-m)^2}+\frac{2\tau}{2-m-m^2}.$$

By putting $m=0$, in corollary 2.1, which was studied by Ala Amourah et al [7]

Remark 2.1: Let the function $f\in A$ given by (1) be in the class $G^*(1)$. Then

$$|a_2|\le 2\tau\sqrt{2\tau},$$

and

$$|a_3|\le 4\tau^2+\tau.$$

III. Coefficient Bounds for the Function Class $\mathbf{G_c(\lambda, m)}$

This section is devoted to finding the initial coefficient bounds of the class $G_c(\lambda,m)$ of Sakaguchi functions.

Theorem 3.1: For $|m|\le 1$ but $m\ne 1$, let the function $f\in A$ given by (1) be in the class $G_c(\lambda,m)$. Then

$$|a_2|\le\frac{2|\lambda|\tau\sqrt{2|\lambda|\tau}}{\sqrt{|(2-3m+m^2)4\lambda^2\tau^2-4(1-m)^2(2\lambda(1+\lambda)\tau^2-\lambda)|}}, \quad (27)$$

and

$$|a_3|\le\frac{\lambda^2\tau^2}{(1-m)^2}+\frac{2|\lambda|\tau}{3(2-m-m^2)}. \quad (28)$$

Proof: By using (10) and (11), it is allows that

$$\frac{((1-m)zf'(z))'}{(f(z)-f(mz))'}=B_\lambda(\tau,w(z)), \quad (29)$$

and

$$\frac{((1-m)wh'(w))'}{(h(w)-h(mw))'}=B_\lambda(\tau,\vartheta(w)). \quad (30)$$

for some analytic functions

$$w(z)=r_1z+r_2z^2+r_3z^3+...$$

and

$$\vartheta(w)=s_1w+s_2w^2+s_3w^3+...$$

on the unit disk Δ with $w(0)=\vartheta(0)=0, |w(z)|<1 (z\in\Delta)$ and $|\vartheta(w)|<1 (w\in\Delta)$. By virtue of the generating function of the Gegenbauer polynomial B_λ defined in (4), the equations (29) and (36), can be written as

$$\frac{((1-m)zf'(z))'}{(f(z)-f(mz))'}=1+C_1^\lambda(\tau)r_1z+[C_1^\lambda(\tau)r_2+C_2^\lambda(\tau)r_1^2]z^2+...$$

and

$$\frac{((1-m)wh'(w))'}{(h(w)-h(mw))'}=1+C_1^\lambda(\tau)s_1w+[C_1^\lambda(\tau)s_2+C_2^\lambda(\tau)s_1^2]w^2+...$$

A simple calculation shows that

$$2(1-m)a_2=C_1^\lambda(\tau)r_1, \quad (31)$$

$$3(2-m-m^2)a_3-4(1-m^2)a_2^2=C_1^\lambda(\tau)r_2+C_2^\lambda(\tau)r_1^2, \quad (32)$$

$$-2(1-m)a_2=C_1^\lambda(\tau)s_1, \quad (33)$$

$$3(2-m-m^2)(2a_2^2-a_3)-4(1-m^2)a_2^2=C_1^\lambda(\tau)s_2+C_2^\lambda(\tau)s_1^2, \quad (34)$$

From (31) and (33). it is clear that

$$r_1=-s_1, \quad (35)$$

and

$$8(1-m)^2a_2^2=[C_1^\lambda(\tau)]^2(r_1^2+s_1^2), \quad (36)$$

By adding (32) and (34), it is obtained that

$$2(2-3m+m^2)a_2^2=C_1^\lambda(\tau)(r_2+s_2)+C_2^\lambda(\tau)(r_1^2+s_1^2), \quad (37)$$

By applying (36) in (37),

$$\left[2(2-3m+m^2)-\frac{8C_1^\lambda(\tau)(1-m)^2}{[C_1^\lambda(\tau)]^2}\right]a_2^2=C_1^\lambda(\tau)(r_2+s_2). \quad (38)$$

It is well known that [16], if $|w(z)|<1$ and $|\vartheta(w)|<1$, then

$$|r_j|\le 1 and |s_j|\le 1 for all j\in\mathbb{N}. \quad (39)$$

By considering (7) and (39), the desired inequality (27) is obtained from (28).

Next, by subtracting (30) from (32), it is obtained that

$$\begin{aligned}&6(2-m-m^2)a_3-6(2-m-m^2)a_2^2\\&=C_1^\lambda(\tau)(r_2+s_2)+C_2^\lambda(\tau)(r_1^2-s_1^2), \quad (40)\end{aligned}$$

Further, in view of (35), it is follows from (40) that

$$a_3=a_2^2+\frac{C_1^\lambda(\tau)}{6(2-m-m^2)}(r_2-s_2). \quad (41)$$

By considering (36) and (39), the desired inequality (28) is derived from (41).

This completes the proof the theorem 3.1 ■

By taking $\lambda = 1$ in theorem 3.1, the following corollary is found.

Corollary 3.1: Let the function $f \in A$ given by (1) be in the class $G_c(1, m)$. Then

$$|a_2| \leq \frac{2\tau\sqrt{2\tau}}{\sqrt{|(2-3m+m^2)4\tau^2 - 4(1-m)^2(4\tau^2-1)|}},$$

$$|a_3| \leq \frac{\tau^2}{(1-m)^2} + \frac{2\tau}{3(2-m-m^2)}.$$

By putting $m = 0$, in corollary 3.1, which was introduced by Ala Amourah et al [7].

Remark 3.1: Let the function $f \in A$ given by (1) be in the class $G_c(1)$. Then

$$|a_2| \leq \frac{\tau\sqrt{2\tau}}{\sqrt{|1-2\tau^2|}},$$

and

$$|a_3| \leq \tau^2 + \frac{\tau}{3}.$$

IV. Fekete-Szegö Inequality for the Class $G^*(\lambda, m)$

The Fekete-Szegö inequality is one of the famous problems related to the coefficients of univalent analytic functions. It was first introduced by [17], who stated that, $f \in S$, then

$$|a_3 - \mu a_2^2| \leq 1 + 2e^{\frac{-2\mu}{1-\eta}}. \tag{42}$$

This bound is sharp when μ is real.

This section is devoted to finding the sharp bounds of the Fekete-Szegö functional $a_3 - \mu a_2^2$ for the class $G^*(\lambda, m)$.

Theorem 4.1: For $|m| \leq 1$ but $m \neq 1$, let the function $f \in A$ given by (1) be in the class $G^*(\lambda, m)$. Then for some $\mu \in \mathbb{R}$.

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\lambda|\tau}{2-m-m^2}, & |\mu - 1| \leq |\frac{(1-m)[(4\lambda^2-(1-m)(2\lambda(1+\lambda)))\tau^2+\lambda]}{2\lambda^2\tau^2}| \\ \frac{8|\lambda|^3\tau^3(1-\mu)}{(1-m)|(4\lambda^2-(1-m)(2\lambda(1+\lambda)))\tau^2+\lambda|}, & |\mu - 1| \geq |\frac{(1-m)[(4\lambda^2-(1-m)(2\lambda(1+\lambda)))\tau^2+\lambda]}{2\lambda^2\tau^2}| \end{cases} \tag{43}$$

Proof: Let $f \in G^*(\lambda, m)$. By using (23) and (26) for some $\mu \in \mathbb{R}$, it is claimed that

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\frac{[C_1^\lambda(\tau)]^3(r_2+s_2)}{2(1-m)([C_1^\lambda(\tau)]^2 - (1-m)C_2^\lambda(\tau)} \\ &\quad + \frac{C_1^\lambda(\tau)(r_2-s_2)}{2(2-m-m^2)} \\ &= C_1^\lambda(\tau)\left[(h(\mu) + \frac{1}{2(2-m-m^2)})r_2 + (h(\mu) \right. \\ &\quad \left. - \frac{1}{2(2-m-m^2)})s_2\right] \end{aligned}$$

where $h(\mu) = \frac{[C_1^\lambda(\tau)]^2(1-\mu)}{2(1-m)([C_1^\lambda(\tau)]^2-(1-m)C_2^\lambda(\tau)}$. Then, it is easily concluded that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\lambda|\tau}{2-m-m^2}, & |h(\mu)| \leq \frac{1}{2(2-m-m^2)} \\ 4|h(\mu)||\lambda|\tau, & |h(\mu)| \geq \frac{1}{2(2-m-m^2)} \end{cases}$$

This proves Theorem 4.1 ■

By choosing $m = 0$ in Theorem 4.1, which was investigated by Ala Amourah et al. [7], we derive the following:

Remark 4.1: Let the function $f \in A$ given by (1) be in the class $G^*(\lambda)$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} |\lambda|\tau, & |\mu - 1| \leq \left|\frac{2\lambda\tau^2-2\tau^2+1}{2\lambda\tau^2}\right| \\ \frac{8|\lambda|^3\tau^3(1-\mu)}{|2\lambda(\lambda-1)\tau^2+\lambda|}, & |\mu - 1| \geq \left|\frac{2\lambda\tau^2-2\tau^2+1}{2\lambda\tau^2}\right| \end{cases}$$

Taking $\mu = 1$ in theorem 4.1, the following corollary found.

Corollary 4.1: Let the function $f \in A$ given by (1) be in the class $G^*(\lambda)$. Then

$$|a_3 - a_2^2| \leq |\lambda|\tau.$$

V. Fekete-Szegö Inequality for the Class $G_c(\lambda, m)$

Since the bounds of $|a_2|$ and $|a_3|$ are obtained for $f \in G_c(\lambda, m)$, it is easy to determine the sharp bounds of the Fekete-Szegö functional $a_3 - \mu a_2^2$ for $f \in G_c(\lambda, m)$.

Theorem 5.1: For $|m| \leq 1$ but $m \neq 1$, let the function $f \in A$ is given by (1) be in the class G_c (λ, m). Then for some $\mu \in \mathbb{R}$.

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\lambda|\tau}{3(2-m-m^2)}, & |\mu - 1| \leq \left|\frac{(2-3m+m^2)2\lambda^2\tau^2-2(1-m)^2(2\lambda(1+\lambda)\tau^2-\lambda)}{\lambda^2\tau^2}\right| \\ \frac{2\lambda^3\tau^3(1-\mu)}{(2-3m+m^2)\lambda^2\tau^2-(1-m)^2(2\lambda(1+\lambda)\tau^2-\lambda)}, & |\mu - 1| \geq \left|\frac{(2-3m+m^2)2\lambda^2\tau^2-2(1-m)^2(2\lambda(1+\lambda)\tau^2-\lambda)}{\lambda^2\tau^2}\right| \end{cases} \tag{44}$$

Proof: Let $f \in G_c(\lambda, m)$. By using (38) and (41) for some $\mu \in \mathbb{R}$, it follows that

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\left[\frac{[C_1^\lambda(\tau)]^3(r_2+s_2)}{2[(2-3m+m^2)[C_1^\lambda(\tau)]^2 - 4(1-m)^2C_2^\lambda(\tau)]}\right] \\ &\quad + \frac{C_1^\lambda(\tau)(r_2-s_2)}{6(2-m-m^2)} \\ &= C_1^\lambda(\tau)\left([h(\mu) + \frac{1}{6(2-m-m^2)}]r_2 \right. \\ &\quad \left. + [h(\mu) - \frac{1}{6(2-m-m^2)}]s_2\right) \end{aligned}$$

where $h(\mu) = \frac{(1-\mu)[C_1^\lambda(\tau)]^2}{2[(2-3m+m^2)[C_1^\lambda(\tau)]^2-4(1-m)^2C_2^\lambda(\tau)]}$.

Then, it is concluded that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\lambda|\tau}{3(2-m-m^2)}, & |h(\mu)| \leq \frac{1}{6(2-m-m^2)} \\ 4|h(\mu)||\lambda|\tau, & |h(\mu)| \geq \frac{1}{6(2-m-m^2)} \end{cases}$$

This proves Theorem 5.1 ■

By taking $m = 0$ in Theorem 5.1, which was studied by Ala Amourah et al. [7], we obtain the following:

Remark 5.1: Let the function $f \in A$ given by (1) be in the class $G_c(\lambda)$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\lambda|\tau}{3}, & |\mu - 1| \leq \left|\frac{1-2\tau^2}{\lambda\tau^2}\right| \\ \frac{2\lambda|^2\tau^3(1-\mu)}{|1-2\tau^2|}, & |\mu - 1| \geq \left|\frac{1-2\tau^2}{\lambda\tau^2}\right| \end{cases}$$

By putting $\mu = 1$ in theorem 5.1, the following corollary derived.

Corollary 5.1: Let the function $f \in A$ given by (1) be in the class $G_c(\lambda)$. Then

$$|a_3 - a_2^2| \leq \frac{|\lambda|\tau}{3}.$$

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