Model of Lagrange Two-dimensional Interpolation Based on Dimensionality Reduction

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***Abstract*—Data interpolation is a common challenge in both scientific research and engineering. Multidimensional interpolation, which encompasses one-dimensional interpolation as a subset, covers a broader range of problems. Notably, interpolation issues in dimensions higher than three can be reduced to a two-dimensional framework through dimensionality reduction, making two-dimensional interpolation a representative paradigm. In this paper, a mathematical model is presented consideration of dimensionality reduction to derive the two-dimensional interpolation polynomial for the tabular function $f(x, y)$. This model is further employed to analyze the interpolation error and determine the remainder term of the polynomial, which is then used to evaluate computational results and perform error estimation. Finally, an engineering example of two-dimensional interpolation is given. While the number of interpolation points is optimized, the proposed algorithm of two-dimensional interpolation yields accurate interpolation outcomes with reduced the complexity of computations.**

***Index Terms*—two-dimensional interpolation, numerical algorithm, bivariate function, remainder term**

I. INTRODUCTION

FUNCTIONS are frequently employed in scientific calculating to address various problems [1]. In numerical analysis, it is essential to substitute the tabular function with a numerical algorithm that can be executed on a computer to facilitate calculations [2], [3]. The Lagrange interpolation method serves to approximate a tabular function using a polynomial function, enabling fundamental interpolation calculations, and is widely utilized in both scientific research and engineering applications [4], [5].

Multi-dimensional interpolation represents a more general situation compared to one-dimensional interpolation [6]. The key advantage of multi-dimensional Lagrange interpolation lies in its ability to provide an exact interpolation on a regular grid and the fact that it can preserve smoothness and continuity in the estimated function. Two-dimensional interpolation serves as a typical problem in multi-dimensional interpolation. Interpolation problems with dimensions greater than three can be effectively transformed into two-dimensional interpolation through the dimensionality reduction. According to the Weierstrass theorem, the space of polynomials is dense within the space of continuous functions, which primarily pertains to two-dimensional interpolation problems. Two-dimensional Lagrange interpolation is a mathematical technique used to calculate unknown values over a two-dimensional space by constructing a polynomial function based on a set of known data points. By leveraging the properties of polynomials, it provides a smooth surface that fits the given data exactly, making it a valuable tool in numerical analysis.

Two-dimensional Lagrange interpolation also plays a significant role in geospatial data modeling, image reconstruction, and surface fitting in engineering and scientific computations. It is widely used in scenarios where accuracy and smoothness are crucial. In computational fluid dynamics, for example, it is employed to reconstruct velocity and pressure fields from discrete simulation data [7]. In geospatial sciences, it enables the creation of 3D terrain models from elevation data, which is critical for geographic information systems (GIS) [7]-[12]. Furthermore, in medical imaging, this method plays a role in constructing 3D models from sparse imaging data, such as in the creation of 3D MRI or CT scans [13]. The ability of the method to handle irregular grids and its straightforward implementation are among its advantages. Additionally, the absence of derivatives in the interpolation formula makes it suitable for datasets where derivative information is unavailable or unreliable [14], [15]. This approach needs consider delivering comparable accuracy and a reduced computational burden. Furthermore, the method can be easily adapted for use in various engineering domains requiring tabular data interpolation or efficient surface fitting.

Despite its strengths, two-dimensional Lagrange interpolation has limitations. One significant drawback is the computational complexity, especially when the number of data points increases, as the number of terms in the polynomial grows exponentially [16], [17]. This makes the method less efficient for large datasets. Another issue is the susceptibility to Runge’s phenomenon, which manifests as oscillations near the edges of the interpolation interval when high-degree polynomials are used [18]. Consequently, the method is used in combination with other techniques to mitigate these issues. Recent advancements in computational power and hybrid methods have enabled more efficient implementations of Lagrange interpolation. For instance,

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adaptive grid refinement techniques have been introduced to minimize computational costs while maintaining accuracy. Additionally, machine learning algorithms are being integrated with Lagrange interpolation to improve performance and address limitations like Runge's phenomenon.

In this study, the two-dimensional interpolation method is investigated. Based on dimensionality reduction, a Lagrange interpolation model is presented for two-dimensional interpolation to solve the computational complexity. The Lagrange two-dimensional interpolation polynomial is deduced with this model. A typical two-dimensional interpolation problem in petroleum product measurement is exemplified to show the optimum process of the numerical algorithm.

II. CONFINING OF TWO-DIMENSIONAL INTERPOLATION

A. One-dimensional interpolation

Given n+1 different values x_0, x_1, ..., x_n in the domain F, and any n+1 values of function $f(x_0)$, $f(x_1)$, ..., $f(x_n)$ that are not all zero, there exists the unique polynomial $p(x)$ of degree no greater than n in $F[x]$ that satisfies

$$p(x_i) = f(x_i), \quad i = 0, 1, ..., n.$$

The Lagrange polynomial $p(x)$ can be expressed as

$$p(x) = \sum_{i=0}^{n} f(x_i) \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})} \cdot \frac{(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_{i+1})\cdots(x_i-x_n)}. \tag{1}$$

Equation (1) guarantees that the interpolating polynomial precisely passes through all known data points, ensuring high accuracy, particularly when the given data is well-distributed.

B. Well-posed two-dimensional interpolation

The tabular function $f(x,y)$ of two variables associated with two-dimensional interpolation can be expressed in Table I. Let $H_{m,n}$ represents the set of polynomial of two variables x, y with degree no higher than m about x and no higher than n degree about y, and D be any domain on the plane R^2. Then the general two-dimensional interpolation problem can be described.

TABLE I
TABULAR FUNCTION $f(x,y)$

x_i \ y_j	y_0	y_1	…	y_n
x_0	$f(x_0,y_0)$	$f(x_0,y_1)$	…	$f(x_0,y_n)$
x_1	$f(x_1,y_0)$	$f(x_1,y_1)$	…	$f(x_1,y_n)$
...	...	...	...	...
x_m	$f(x_m,y_0)$	$f(x_m,y_1)$	…	$f(x_m,y_n)$

Given a point set $E_N = (x_i, y_j)_{i,j=0}^{i=m, j=n}$ consisting of mutually distinct points in the domain D, and any function $f(x,y) \in C(D)$ defined on E_N, N=(m+1)(n+1). The function values are represented as $z_q = f(x_i, y_j)$ for $q = 1, 2, ..., N$. The objective is to find the polynomial $p(x,y) \in H_{m,n}$ that satisfies the interpolation conditions

$$p(x_i, y_j) = z_q = f(x_i, y_j), \quad q = 1, 2, ..., N. \tag{2}$$

$p(x,y)$ denote the interpolation polynomial of $f(x,y)$ defined on the set of interpolation points E_N.

Two-dimensional interpolation presents an uncertain problem. The uniqueness of the interpolation polynomial $p(x,y)$ that satisfies the interpolation (2) cannot be guaranteed. The distribution of interpolation points and the selection of the interpolation space significantly influence the uniqueness of two-dimensional interpolation. Therefore, the polynomial $p(x,y)$ should be confined.

Definition There exists a unique polynomial $p(x,y)$ within the polynomial space $H_{m,n}$ for any set of values $(x_i, y_j)_{i,j=0}^{i=m, j=n}$ defined at the point group E_N. This polynomial $p(x,y)$ satisfies the given (2). Consequently, E_N constitutes a well-posed point group of $H_{m,n}$.

Two-dimensional interpolation differs from one-dimensional interpolation. It can be categorized into well-posed and ill-posed problems. This paper primarily focuses on the well-posed two-dimensional interpolation problem.

In two-dimensional interpolation, piecewise interpolation commonly utilizes basis functions of piecewise Lagrange interpolation in both the x and y directions. These basis functions are then multiplied together to produce a piecewise interpolation basis function of two variables. This resulting function is subsequently multiplied by the corresponding value of interpolation point, and summed. The formula of the piecewise Lagrange interpolation polynomial is obtained.

III. NUMERICAL ALGORITHM FOR TWO-DIMENSIONAL INTERPOLATION

A. Lagrange polynomial of two-dimensional interpolation

Here let the given rectangular domain $D = \{(x,y) \mid a \le x \le b, c \le y \le d\}$. The division is defined as

$$E_{m+1}: a = x_0 < x_1 < ... < x_m = b$$

$$E_{n+1}: c = y_0 < y_1 < ... < y_n = d$$

Then $E_N = \{(x_i, y_j) \mid i = 0, 1, ..., m;\ j = 0, 1, ..., n\} = (E_{m+1}, E_{n+1})$ constitutes a set of rectangular dividing points, i.e., an interpolation point group. Two-dimensional interpolation corresponds to an interpolated curved surface for the tabular function $z = f(x, y)$. The mathematical model of two-dimensional interpolation is shown in Fig. 1.

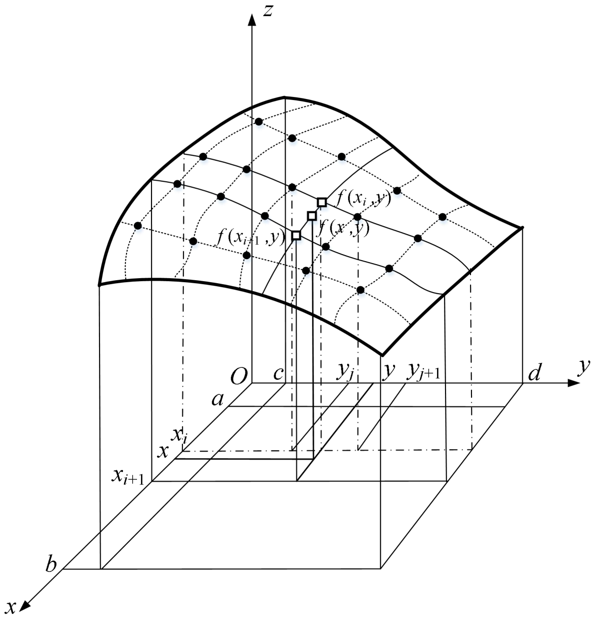

Fig. 1. Model of two-dimensional interpolation

The bivariate function $z = f(x, y)$ is transformed into a unary function of y, expressed as $z = f(x,y)|_{x=x_i}$, where $x = x_i$ for $i = 0, 1, ..., m$. For E_{n+1} defined on the interval $[c, d]$, the points $y_0, y_1, ..., y_n$ represent a given set of mutually distinct points. The construction of an n-th degree polynomial is

$$l_j(y) = \frac{(y-y_0)(y-y_1)\cdots(y-y_{j-1})}{(y_j-y_0)(y_j-y_1)\cdots(y_j-y_{j-1})}\cdot \frac{(y-y_{j+1})\cdots(y-y_n)}{(y_j-y_{j+1})\cdots(y_j-y_n)} = \prod_{\substack{t=0\\ t\neq j}}^{n} \frac{y-y_t}{y_j-y_t}, \quad j = 0, 1, ..., n. \tag{3}$$

And $l_j(y)$ satisfied

$$l_j(y_t) = \delta_{tj} = \begin{cases} 0 & t \neq j \\ 1 & t = j \end{cases}, \quad t, j = 0, 1, ..., n. \tag{4}$$

The Lagrange interpolation polynomial corresponding to the unary function $f(x_i, y)$ is defined as follows

$$L_{0,n}(x_i, y) = \sum_{j=0}^{n} f(x_i, y_j)\cdot l_j(y), \quad i = 0, 1, ..., m. \tag{5}$$

On the interval $[c, d]$, the value of $y = C$ can be chosen arbitrarily. The interpolation value for the points (x_i, y) ($i = 0, 1, ..., m$) can be determined by using (5). Additionally, the bivariate function $z = f(x, y)$ is transformed into a unary function of x, expressed as $z = f(x,y)|_{y=C}$. For E_{m+1}, on the interval $[a, b]$, the points $x_0, x_1, ..., x_m$ represent a given set of mutually distinct points, the polynomial of m-th degree is

$$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})}\cdot \frac{(x-x_{i+1})\cdots(x-x_m)}{(x_i-x_{i+1})\cdots(x_i-x_m)} = \prod_{\substack{s=0\\ s\neq i}}^{m} \frac{x-x_s}{x_i-x_s}, \quad i = 0, 1, ..., m. \tag{6}$$

and $l_i(x)$ satisfied

$$l_i(x_s) = \delta_{si} = \begin{cases} 0 & s \neq i \\ 1 & s = i \end{cases}, \quad s, i = 0, 1, ..., m. \tag{7}$$

The Lagrange interpolation polynomial corresponding to the unary function $f(x, y)|_{y=C}$ is defined as follows

$$L_{m,0}(x, y) = \sum_{i=0}^{m} L_{0,n}(x_i, y)\cdot l_i(x), \quad i = 0, 1, ..., m. \tag{8}$$

By substituting (5) into (8), The Lagrange polynomial is expressed as

$$\begin{aligned} L_{m,n}(x, y) &= \sum_{i=0}^{m}\left(\sum_{j=0}^{n}\left(f(x_i, y_j)\cdot l_j(y)\right)\right)\cdot l_i(x) \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\left(f(x_i, y_j)\cdot l_i(x)\cdot l_j(y)\right). \end{aligned} \tag{9}$$

Let $l_{i,j}(x, y) = l_i(x)\cdot l_j(y)$, which satisfies the interpolation condition

$$l_{i,j}(x_s, y_t) = \delta_{si}\cdot\delta_{tj} = \begin{cases} 0 & \text{else} \\ 1 & s = i \text{ and } t = j \end{cases}, \quad s, i = 0, 1, ..., m;\ t, j = 0, 1, ..., n. \tag{10}$$

Consequently, for any set of interpolation point values $\{f(x_i, y_j)\}_{i,j=0}^{i=m,\,j=n}$ on E_N, the corresponding interpolation polynomial is denoted as

$$L_{m,n}(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n}\left(f(x_i, y_j)\cdot l_{i,j}(x, y)\right). \tag{11}$$

The polynomial $L_{m,n}(x,y)$ represents a bivariate polynomial in x of degree m and in y of degree n, while also satisfying the interpolation condition

$$L_{m,n}(x_i, y_j) = f(x_i, y_j), \quad i = 0, 1, ..., m;\ j = 0, 1, ..., n. \tag{12}$$

B. Interpolation error

In the two-dimensional interpolation, the error associated with using the interpolation polynomial $L_{m,n}(x, y)$ to approximate the bivariate function $f(x, y)$ primarily arises from truncation error. It can be expressed with the interpolation remainder term.

The interpolation remainder term corresponding to the specified interpolation polynomial $L_{0,n}(x_i, y)$ in (5) is given by

$$R_{0,n}(x_i, y) = \frac{f^{(n+1)}(x_i, \eta)}{(n+1)!}\omega_{n+1}(y). \tag{13}$$

where $\eta \in (c, d)$, and it depends on y; $\omega_{n+1}(y) = (y-y_0)(y-y_1)\cdots(y-y_n)$.

The interpolation remainder term associated with the interpolation polynomial $L_{m,n}(x,y)$, as determined by (8), is

$$R_{m,0}(x,y)=\frac{f^{(m+1)}(\xi,y)}{(m+1)!}\omega_{m+1}(x). \tag{14}$$

where $\xi\in(a,\ b)$, and it depends on x. Here, $\omega_{m+1}(x)=(x-x_0)(x-x_1)\cdots(x-x_m)$.

Therefore, the interpolation remainder term of the interpolation polynomial in (11) is expressed as

$$\begin{aligned}R_{m,n}(x,y)&=\sum_{i=0}^{m}R_{0,n}(x_i,y)\cdot l_i(x)+R_{m,0}(x,y)\\&=\sum_{i=0}^{m}\frac{f^{(n+1)}(x_i,\eta)}{(n+1)!}\omega_{n+1}(y)\cdot l_i(x)+\\&\frac{f^{(m+1)}(\xi,y)}{(m+1)!}\omega_{m+1}(x).\end{aligned} \tag{15}$$

The interpolation remainder term $R_{m,n}(x,y)$ is derived by first converting the bivariate function $f(x,\ y)$ into a unary interpolation polynomial in terms of y, followed by obtaining the unary interpolation polynomial in terms of x. Conversely, if the bivariate function $f(x,\ y)$ is initially transformed into a unary interpolation polynomial in terms of x, and subsequently a unary interpolation polynomial in terms of y is generated, the resulting interpolation remainder term is denoted as

$$\begin{aligned}R'_{m,n}(x,y)&=\sum_{j=0}^{n}\frac{f^{(m+1)}(\xi',y_j)}{(m+1)!}\omega_{m+1}(y)\cdot l_j(y)\\&+\frac{f^{(n+1)}(x,\eta')}{(n+1)!}\omega_{n+1}(y).\end{aligned} \tag{16}$$

The interpolation polynomials of the bivariate function $f(x,\ y)$ obtained by the two methods are equal. As the well-posed interpolation problem is discussed, the interpolation remainder terms are $R'_{m,n}(x,y)=R_{m,n}(x,y)$.

IV. Engineering Example

In the measurement of petroleum product, the standard density of petroleum (denoted as z, unit: kg/m³) is determined by measuring the temperature of the petroleum product (denoted as x, unit: °C) and the apparent density (denoted as y, unit: kg/m³). The table of standard density is given by National Institute of Metrology. A portion of the standard density function table is presented in Table II.

To calculate the standard density values, for instance, at the specified points (18.50, 819.0), (20.15, 820.0), (22.25, 826.0), and (24.00, 828.0), a two-dimensional interpolation algorithm is designed by using (11). Taking parameters m=1 and n=1, m=3 and n=3, m=5 and n=5 respectively, following the execution of the program with two-dimensional interpolation algorithm, the numerical results are outputted for each specified point, as detailed in Table III.

Different interpolation points in the set E_N can be selected based on the values of m and n. Specifically, when m=n=1, the number of interpolation points is 4; when m=n=3, the number of interpolation points increases to 16; and when m=n=5, the number of interpolation points reaches 36. As the number of interpolation points increases, the computational demands of the algorithm also rise. Table III illustrates that satisfactory interpolation results can be achieved in all three cases. To reduce the algorithm's complexity and computational load, two improvement measures have been implemented.

TABLE II
Portion of standard densities of petroleum product

x \ y	813.0	815.0	817.0	819.0	821.0	823.0	825.0	827.0	829.0	831.0	833.0
16.00	810.1	812.2	814.2	816.2	818.2	820.2	822.2	824.2	826.2	828.2	830.2
17.00	810.9	812.9	814.9	816.9	818.9	820.9	822.9	824.9	826.9	828.9	830.9
18.00	811.6	813.6	815.6	817.6	819.6	821.6	823.6	825.6	827.6	829.6	831.6
19.00	812.3	814.3	816.3	818.3	820.3	822.3	824.3	826.3	828.3	830.3	832.3
20.00	813.0	815.0	817.0	819.0	821.0	823.0	825.0	827.0	829.0	831.0	833.0
21.00	813.7	815.7	817.7	819.7	821.7	823.7	825.7	827.7	829.7	831.7	833.7
22.00	814.4	816.4	818.4	820.4	822.4	824.4	826.4	828.4	830.4	832.4	834.4
23.00	815.1	817.1	819.1	821.1	823.1	825.1	827.1	829.1	831.1	833.1	835.1
24.00	815.8	817.8	819.8	821.8	823.8	825.8	827.8	829.8	831.8	833.8	835.8
25.00	816.5	818.5	820.5	822.5	824.5	826.5	828.5	830.5	832.5	834.5	836.5
26.00	817.3	819.2	821.2	823.2	825.2	827.2	829.2	831.2	833.2	835.2	837.2

TABLE III
RESULTS OF TWO-DIMENSIONAL LAGRANGE INTERPOLATION

m,*n* \ (*x*,*y*)	(18.5,819)	(20.15,820)	(22.25,826)	(24.0,828)
m=1, *n*=1	818.0	820.1	827.5	830.8
m=3, *n*=3	818.1	820.1	827.6	830.8
m=5, *n*=5	818.0	820.0	827.6	830.9

First, some standard densities of petroleum product listed in Table II can be eliminated. The simplified standard density from Table II is presented in Table IV.

TABLE IV
SIMPLIFIED DATA OF THE STANDARD DENSITIES OF PETROLEUM PRODUCT

x \ *y*	813.0	817.0	821.0	825.0	829.0	833.0
16.00	810.1	814.2	818.2	822.2	826.2	830.2
19.00	812.3	816.3	820.3	824.3	828.3	832.3
22.00	814.4	818.4	822.4	826.4	830.4	834.4
25.00	816.5	820.5	824.5	828.5	832.5	836.5
27.00	818.0	821.9	825.9	829.9	833.9	837.8

Second, for m=1 and n=1, the streamlined standard density interpolations are utilized to calculate the interpolation at the points (18.50, 819.0), (20.15, 820.0), (22.25, 826.0), and (24.00, 828.0), leveraging the standard densities in Table IV. The standard density values at these points are shown in Table V, indicating that satisfactory interpolation results have been obtained.

TABLE V
RESULTS OF THE STANDARD DENSITIES OF PETROLEUM PRODUCT INTERPOLATED WITH THE SIMPLIFIED DATA

m,*n* \ (*x*,*y*)	(18.5,819)	(20.15,820)	(22.25,826)	(24.0,828)
m=1, *n*=1	817.9	820.1	827.6	830.7

In engineering calculations, the measurement standard for petroleum product necessitates the use of four significant figures. When comparing the results of interpolation calculations with actual measured values, the error remains less than the engineering requirements. Notably, in many instances, the number of points in the table of two-dimensional tabular function significantly exceeds the values of m or n. The interpolation domain D can be subdivided into several interpolation sub-domains for the purpose of partitioned interpolation.

V. CONCLUSIONS

This paper presents a novel approach to two-dimensional interpolation by constructing a Lagrange-type interpolation polynomial based on dimensionality reduction. The method targets tabulated functions of two variables and formulates a structured interpolation model that maintains both accuracy and computational efficiency. By reducing the two-variable interpolation into a sequence of one-dimensional interpolations, the approach simplifies the computation while preserving the mathematical integrity of the classical Lagrange framework. Theoretical contributions include the derivation of the interpolation polynomial and the corresponding remainder term, which offers an explicit estimate of the interpolation error.

To validate the practical effectiveness of the method, a real-world engineering problem is introduced to estimate standard density of petroleum based on temperature and apparent density. The results show that the method provides accurate interpolation values with reduced complexity, particularly when the number of interpolation points is optimized. The research can reduce the degree of the polynomial needed to achieve accurate values, enhancing computational feasibility and robustness.

Additionally, the two-dimensional Lagrange interpolation model offers flexibility in handling irregular grids. It can be integrated with other techniques to further enhance performance, making the approach a valuable tool for scientific calculation and engineering applications requiring high precision.

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