An Overview on the Basic Concepts of Multiple Topological Spaces

Bandita Devi, Sandhya S Pai*, Sanjitha R, Baiju T.

***Abstract*—A multiple set is an extended version of a fuzzy set that can handle the uncertainty of an element along with its multiplicity. Multiple sets provide a significant advantage over fuzzy sets by allowing multiple occurrences of elements, each with a finite number of the same or different membership values. Multiple topological space is a generalized version of fuzzy topological space. We try to expand on the ideas based on multiple topological spaces in this study. We will focus on the key ideas of an interior, closure, continuity, open set, closed set, denseness, and multiple points to keep things brief. Additionally, we have proven a few intriguing conclusions based on these topological ideas.**

***Index Terms*—Multiple set, multiple topology, closure, interior, continuity, denseness, neighborhood, multiple point.**

I. Introduction

The fundamental ideas in mathematics are sets. Cantor, a German mathematician who lived from 1845 to 1918, made important advances in set theory, which has since become the official language of science. It is possible to clearly and precisely distinguish between a member and a non-member of any well-defined collection of entities.

The majority of our conventional tools for formal reasoning, modeling, and computation are exact, predictable, and crisp (i.e., dichotomous). These instruments provide clear explanations for simplistic systems. Although they provide clear explanations for simple systems, they fall short when it comes to delivering accurate and meaningful insights into the behavior of complex and varied systems. To address this issue, L.A. Zadeh [1], [2], [3], [4], [5] introduced the theory of fuzzy sets as an extension of dual logic. Fuzzy logic focuses on approximate reasoning, as opposed to strictly deductive reasoning from classical predicate logic. Since Zadeh's development of fuzzy set theory in the 1960s [1], there has been increasing recognition of how human uncertainty can influence scientific problems. The combination of fuzzy logic with expert systems has become one of the most well-known and widely applied approaches. Until recently, fuzzy set theory operating "numerically" in engineering applications has not received much attention. Artificial intelligence, computer science, medicine, robotics, control engineering, decision theory, expert systems, logic, management science, and operation research are just a few fields in which this theory finds applications.

The idea of a multiset is a generalization of the concept of a set in which members are permitted to appear more than once. In 1971, Cerf et al. introduced the multiset theory. Yager and Peterson contributed even more to it. In 1989, Blizard formalized the concept of multisets, which have since become a widely used method for implementing relations in database systems. Yager later introduced fuzzy multisets [7] as a generalization of traditional multisets. In fuzzy multisets, an element of a set X can appear multiple times, each with the same or different membership values. Yager introduced the concept of a fuzzy multiset in 1986 as an extension of a fuzzy set. In a fuzzy multiset, fuzzy membership values are assigned to each identical copy of an object. The main advantage of a fuzzy multiset over a fuzzy set is its ability to handle multiple instances of objects. However, it can only manage one attribute of the object at a time. On the other hand, a multi-fuzzy set is an extension of a fuzzy set that provides fuzzy membership values for multiple attributes of an object. In 2010, Sebastian and Ramakrishnan [8] proposed the multi-fuzzy set. The key benefit of a multi-fuzzy set over a fuzzy set is its capacity to manage several unknown attributes of an object simultaneously, though it cannot address the multiplicity of the object itself.

In practical applications, representing an object's multiplicity and many aspects may be crucial. A new mathematical structure called multiple sets was proposed by Shijina et al [9], [10] to concurrently represent numerous uncertain features together with multiplicity and developed an in-depth study on multiple sets. To illustrate incomplete knowledge, several sets are introduced, from which all the previously described instances can be inferred as particular situations. Multiple sets may include different iterations of the same element with a limited amount of distinct or similar membership values. In other words, for every element x in the universal set X, a multiple set of order (n, k) assigns nk membership grades. Previous research has focused on the theoretical development of multiple sets along with a rudimentary introduction to aggregation operators [11], [12], relations [13], similarity measures[14], and the topological structure of multiple sets [15]. They included the basic notion of a multiple topological space, base of a multiple topology, subbase, interior, closure, and subspace of a multiple topological space in their study on the topological structure of multiple sets. This work aims to extend the work done by Shijina et al. in comparison with the study made by Chang on fuzzy topological spaces [16], [17]. We have done a comparative study with [17] and expanded the theory of open sets, closed

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sets, interior, closure, and denseness on multiple topological spaces. A few basic results concerning these concepts are investigated. We have extended the work to define multiple continuous mappings by defining neighbourhood and multiple point, We have explored the results based on multiple continuous mappings using multiple point.

II. Preliminaries and basic definitions

Definition 1: [1] Let X be a universal set a fuzzy set $\tilde{A}$ over X is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$ where $\mu_{\tilde{A}}$ is called the membership function or grade of membership of x in $\tilde{A}$ that maps X to the membership space M. The range of the membership function is a subset of the non-negative real numbers, with a finite supremum. When M contains only 0 and 1, $\tilde{A}$ is non-fuzzy, and $\mu_{\tilde{A}}(x)$ is identical to the characteristic function of a non-fuzzy set.

Definition 2: [6] A multiset M drawn from X(non-empty set) is represented by a count function $C_M : X \to N \cup \{0\}$ where N is the set of positive integers. For each $x \in X, C_M(x)$ indicates the number of occurrences of the element x in M. Then a multiset M can be represented as $\{C_M(x)|x; x \in X\}$.

Definition 3: [7] For $x \in X$, the membership sequence of x is defined as the non-increasing sequence of membership values of x and is denoted by $(\mu_A^1(x), \mu_A^2(x), ..., \mu_A^k(x))$, such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^k(x)$ where μ_A is a membership function and μ_A^j, j=1,2,...,k are values (same or different) of membership function μ_A. A fuzzy multi-set is a collection of all x together with its membership sequence.

Definition 4: [8] Let X be a non-empty set and let $\{L_i, i \in N\}$ be a family of complete lattices where N is the set of positive integers. A multifuzzy set A in X is a set of ordered sequences A = $\{(x, \mu_1(x), \mu_2(x), ...); x \in X\}$ where $\mu_i \in L_i^X$ for $i \in N$. The function μ_A = $(\mu_1, \mu_2,...)$ is called a membership function of multifuzzy set A.

Definition 5: [9] Let X be a non-empty set. A multiple set $\mathbf{A}$ drawn from X is an object of the form $\{(x, \mathbf{A}(x)); x \in X\}$ where for each $x \in X$, its membership value is an $n \times k$ matrix,

$$\mathbf{A}(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \cdots & A_1^k(x) \\ A_2^1(x) & A_2^2(x) & \cdots & A_2^k(x) \\ \vdots & \ddots & \vdots & \vdots \\ A_n^1(x) & A_n^2(x) & \cdots & A_n^k(x) \end{bmatrix}$$

where A_i i=1,2,...,n are membership functions. For each i=1,2,...,n, $A_i^j(x)$, j=1,2,...,k are membership values of the membership function A_i for the element $x \in X$, written in decreasing order. Then A is called a multiple set of orders (n,k).

The collection of all multiple sets of order (n, k) is denoted by $MS_{(n,k)}(X)$.

Example 1: [9] Suppose X = $\{x_1, x_2, x_3\}$ is the universal set of students under consideration. There is a panel consisting of three experts evaluating the students under the criteria of intelligence, extra-curricular activities, communication skills, and personality. The membership functions $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}$ and $\mathbf{A_4}$ represent criteria intelligence, extra-curricular activities, communication skill, and personality respectively. For each i=1,2,3,4, membership values $A_i^1(x), A_i^2(x)$ and $A_i^3(x)$ of the membership function A_i for the element $x \in X$ are given by the three experts, written in the decreasing order. Then the performance of students can be represented by a multiple set of order (4,3) as follows: $\mathbf{A} = \{(x, \mathbf{A}(x_1)), (x, \mathbf{A}(x_2)), (x, \mathbf{A}(x_3))\}$ where $A(x_i)$ for i=1,2,3 are 4×3 matrices given as follows:

$$\mathbf{A}(x_1) = \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.6 & 0.5 & 0.4 \\ 0.7 & 0.5 & 0.3 \\ 0.9 & 0.9 & 0.8 \end{bmatrix}, \ \mathbf{A}(x_2) = \begin{bmatrix} 0.8 & 0.6 & 0.6 \\ 0.6 & 0.5 & 0.4 \\ 0.7 & 0.5 & 0.4 \\ 0.9 & 0.8 & 0.7 \end{bmatrix},$$

$$\mathbf{A}(x_3) = \begin{bmatrix} 0.8 & 0.7 & 0.5 \\ 0.7 & 0.6 & 0.4 \\ 0.7 & 0.4 & 0.4 \\ 0.8 & 0.8 & 0.7 \end{bmatrix}$$

Definition 6: [9] Let X be a universal set. Let $\mathbf{A}, \mathbf{B} \in MS_{(n,k)}(X)$ and ,

$$\mathbf{A}(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \cdots & A_1^k(x) \\ A_2^1(x) & A_2^2(x) & \cdots & A_2^k(x) \\ \vdots & \ddots & \vdots & \vdots \\ A_n^1(x) & A_n^2(x) & \cdots & A_n^k(x) \end{bmatrix},$$

$$\mathbf{B}(x) = \begin{bmatrix} B_1^1(x) & B_1^2(x) & \cdots & B_1^k(x) \\ B_2^1(x) & B_2^2(x) & \cdots & B_2^k(x) \\ \vdots & \ddots & \vdots & \vdots \\ B_n^1(x) & B_n^2(x) & \cdots & B_n^k(x) \end{bmatrix}$$

be the membership matrices for x in $\mathbf{A}$ and $\mathbf{B}$ respectively.

- Subset: $\mathbf{A} \subseteq \mathbf{B}$ iff $A_i^j(x) \leq B_i^j(x)$ for every $x \in X$, i=1,2,...,n and j=1,2,...,k.
- Equality: $\mathbf{A} = \mathbf{B}$ iff $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$. That is, $A_i^j(x) = B_i^j(x)$ for every $x \in X$, i=1,2,...,n and j=1,2,...,k.
- Standard Union: The union of $\mathbf{A}$ and $\mathbf{B}$, denoted as $\mathbf{A} \cup \mathbf{B}$, is a multiple set whose membership matrix for every $x \in X$, i=1,2,...,n and j=1,2,...,k is given by $(A \cup B)_i^j(x) = max\{A_i^j(x), B_i^j(x)\}$.
- Standard Intersection: The intersection of $\mathbf{A}$ and $\mathbf{B}$, denoted as $\mathbf{A} \cap \mathbf{B}$, is a multiple set whose membership matrix for every $x \in X$, i=1,2,...,n and j=1,2,...,k is given by $(A \cap B)_i^j(x) = min\{A_i^j(x), B_i^j(x)\}$.
- Standard Complement: The standard complement of $\mathbf{A}$ is denoted as $\bar{\mathbf{A}}$, is a multiple set whose membership matrix for each $x \in X$, is given by $\bar{A}_i^j(x) = 1 - A_i^{k-j+1}(x)$ for every i=1,2,...,n and j=1,2,...,k.

Definition 7: [17] On a set X, a fuzzy topology is a family F=$\{\mu : \mu$ is a fuzzy set in X$\}$ of fuzzy subsets that satisfies the following three axioms:

- 0,1 $\in$ F.
- $\mu_1, \mu_2 \in$F, then $\mu_1 \wedge \mu_2 \in$F.
- If $\{\mu_i : i \in J\} \subset F$ where J denotes an index set, then $\vee \mu_i \in$F.

F is described as a fuzzy topology for X and (X, F) is named as a fuzzy topological space or fts. The members of F are defined as an F-open fuzzy set.

If the complement of ρ denoted by ρ^c, is F-open, then an element $\rho \in [0, 1]^X$ is said to be an F-closed fuzzy set.

III. Multiple topological spaces

Definition 8: A multiple topology on a set X is a family of multiple subsets that satisfy the following three axioms:

- $\phi, X \in M$.
- $\mathbf{P}_1, \mathbf{P}_2 \in M$ then $\mathbf{P}_1 \cap \mathbf{P}_2 \in M$.
- If $\{\mathbf{P}_i : i \in J\} \subset M$ then $\cup \mathbf{P}_i \in M$.

M is described as a multiple topology for X and the pair (X, M) is named a multiple topological space.

Definition 9: Let, (X, M) be a multiple topological space then the members of M are defined as M-open multiple set or open set .
A multiple set $\mathbf{P}$ is said to be closed in a multiple topological space (X, M) iff $\mathbf{P}^c \in M$.

Example 2: $\mathbf{A}$ $=\{(x_1, \mathbf{A}(x_1)), (x_2, \mathbf{A}(x_2)), (x_3, \mathbf{A}(x_3))\}$ and $\mathbf{B}$ $=\{(x_1, \mathbf{B}(x_1)), (x_2, \mathbf{B}(x_2)), (x_3, \mathbf{B}(x_3))\}$ be two multiple sets of order (2,2) defined over $X = \{x_1, x_2, x_3\}$ where

$\mathbf{A}(x_1)$= $\begin{bmatrix} 0.9 & 0.7 & 0.6 \\ 0.8 & 0.5 & 0.2 \\ 0.8 & 0.7 & 0.3 \end{bmatrix}$, $\mathbf{A}(x_2)$= $\begin{bmatrix} 0.6 & 0.7 & 0.3 \\ 0.3 & 0.2 & 0.1 \\ 0.7 & 0.4 & 0.3 \end{bmatrix}$,
$\mathbf{A}(x_3)$= $\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.1 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$

$\mathbf{B}(x_1)$= $\begin{bmatrix} 0.6 & 0.5 & 0.4 \\ 0.5 & 0.5 & 0.2 \\ 0.3 & 0.2 & 0.1 \end{bmatrix}$, $\mathbf{B}(x_2)$= $\begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.1 \\ 0.4 & 0.4 & 0.1 \end{bmatrix}$,
$\mathbf{B}(x_3)$= $\begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.2 \end{bmatrix}$

Let, M = $\{\phi, \mathbf{A}, \mathbf{B}, X\}$.
$\mathbf{A}(x_1) \cap \mathbf{B}(x_1) = \mathbf{B}(x_1)$,
$\mathbf{A}(x_2) \cap \mathbf{B}(x_2) = \mathbf{B}(x_2)$,
$\mathbf{A}(x_3) \cap \mathbf{B}(x_3) = \mathbf{B}(x_3)$,
$\mathbf{A}(x_1) \cup \mathbf{B}(x_1) = \mathbf{A}(x_1)$,
$\mathbf{A}(x_2) \cup \mathbf{B}(x_2) = \mathbf{A}(x_2)$,
$\mathbf{A}(x_3) \cup \mathbf{B}(x_3) = \mathbf{A}(x_3)$,
That is $\mathbf{A} \cup \mathbf{B} \in \mathbf{M}$ and $\mathbf{A} \cap \mathbf{B} \in \mathbf{M}$. Clearly, (X,M) forms a multiple topological space.

Example 3: Consider the collection $\mathbf{M} = \{X, \phi\}$. Clearly $\mathbf{M}$ defines a multiple topology over X called indiscrete multiple topology.

Definition 10: Discrete multiple topology of order (n, k) over X is a collection that contains all the multiple sets of order (n, k) defined over X.

Definition 11: Let (X, M) be a multiple topological space and $\mathbf{P} \subset X$. The closure of $\mathbf{P}$ is denoted by $\bar{\mathbf{P}}$ and is defined as the smallest closed multiple set containing $\mathbf{P}$.
Equivalently, $\bar{\mathbf{P}}$ is defined as following way:

$$\bar{\mathbf{P}} = \cap\{\mathbf{Q} : \mathbf{Q} \text{ is M-closed and } \mathbf{P} \subset \mathbf{Q}\}.$$

Obviously then $\bar{\mathbf{P}}$ is always M-closed.
Multiple topology generated by the closure operator is denoted by M_X and is defined by,

$$M_X = \{\mathbf{P} \in MS_{(n,k)}(X) \text{ where } \overline{1 - \mathbf{P}} = 1 - \mathbf{P}\}.$$

Then (X, M_X) is called the closure of multiple topological space generated by the closure operator.

Theorem 1: A map $\mathbf{P} \rightarrow \bar{\mathbf{P}}$ defined over the multiple topological space (X,M) is said to be a closure operator if it satisfies the following properties:

1) $\mathbf{P} \leq \bar{\mathbf{P}}$.
2) $\bar{\bar{\mathbf{P}}} = \bar{\mathbf{P}}$.
3) For any multiple set $\mathbf{Q}, \overline{\mathbf{P} \cup \mathbf{Q}}$ = $\bar{\mathbf{P}} \cup \bar{\mathbf{Q}}$.
4) $\bar{\phi} = \phi$.

Proof: 1. From the definition of closure operator, $\overline{\mathbf{P}}$=$\cap\{\mathbf{R} : \mathbf{R}$ is closed and $\mathbf{R} \geq \mathbf{P}\}$.
therefore $\mathbf{P} \leq \overline{\mathbf{P}}$.
2. Since $\overline{\overline{\mathbf{P}}}$ is the smallest closed set containing $\overline{\mathbf{P}}$ and $\overline{\mathbf{P}}$ itself is closed.
Then $\overline{\overline{\mathbf{P}}}$=$\overline{\mathbf{P}}$.
3. Clearly, $\overline{\mathbf{P}} \cup \overline{\mathbf{Q}}$ is closed.
Again, $\overline{\mathbf{P}} \cup \overline{\mathbf{Q}} \geq \mathbf{P} \vee \mathbf{Q}$.
$\Rightarrow \overline{\overline{\mathbf{P}} \vee \overline{\mathbf{Q}}} \geq \overline{\mathbf{P} \cup \mathbf{Q}}$.
$\Rightarrow \overline{\mathbf{P}} \vee \overline{\mathbf{Q}} \geq \overline{\mathbf{P} \cup \mathbf{Q}}$
And, $\mathbf{P} \cup \mathbf{Q} \geq \mathbf{P}$.
$\Rightarrow \overline{\mathbf{P} \cup \mathbf{Q}} \geq \overline{\mathbf{P}}$.
therefore$\overline{\mathbf{P} \cup \mathbf{Q}} \geq \overline{\mathbf{P}} \cup \overline{\mathbf{Q}}$.
Hence, $\overline{\mathbf{P} \cup \mathbf{Q}}$=$\overline{\mathbf{P}} \cup \overline{\mathbf{Q}}$.
4. Since the whole space X $\in$ M is open, then it's complement, X^c= ϕ is closed.
Also $\overline{\phi}$ is closed.
So we can write $\overline{\phi}$=X. ∎

Example 4: Let **A, B,C** be multiple sets over R, defined as

$$A_i^j(x) = \begin{cases} \frac{2x}{5} + \frac{1}{2};\ \text{x} \in [0,1] & \forall i, j \in N. \\ 0;\ otherwise. \end{cases}$$

$$B_i^j(x) = \begin{cases} \frac{2x}{15} + \frac{2}{3};\ \text{x} \in [0,1] & \forall i, j \in N. \\ 0;\ otherwise. \end{cases}$$

$$C_i^j(x) = \begin{cases} \frac{x}{4};\ \text{x} \in [0,1] & \forall i, j \in N. \\ 0;\ otherwise. \end{cases}$$

$\rho = \{X, \phi, , \mathbf{A}, \mathbf{C}\}$ is a multiple topology on R.

Then $\overline{\mathbf{A}}$=X, $\overline{\mathbf{B}}$=X and $\overline{\mathbf{C}}$=$\mathbf{C}^c$.

Example 5: Let **A,B,C** be multiple sets of order (2,2) over X = $\{1, 2\}$ whose membership values are given by,
A$_i^j(x) = \frac{2x}{15i} + \frac{2}{3j}$; x$\in X$ and for i,j=1,2.
B$_i^j(x) = \frac{x}{4ij}$; x$\in X$ and for i,j=1,2.
The membership matrix is calculated below,
$\mathbf{A}(1) = \begin{bmatrix} 0.8 & 0.46 \\ 0.73 & 0.4 \end{bmatrix}$, $\mathbf{A}(2) = \begin{bmatrix} 0.93 & 0.6 \\ 0.8 & 0.46 \end{bmatrix}$
$\mathbf{B}(1) = \begin{bmatrix} 0.25 & 0.125 \\ 0.125 & 0.06 \end{bmatrix}$, $\mathbf{B}(2) = \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 0.125 \end{bmatrix}$
Here, $\mathbf{A} \cup \mathbf{B} = \mathbf{A}$, $\mathbf{A} \cap \mathbf{B} = \mathbf{B}$.
Then,ρ = $\{\phi, X,$ **A,B**$\}$ is a multiple topology on X, where $\overline{\mathbf{A}}$=X, and $\overline{\mathbf{B}}$=$\mathbf{B}^c$.

Theorem 2: Let (X, M) be a multiple topological space and $\mathbf{P}, \mathbf{Q}$ be two multiple sets in X. Then $\overline{\mathbf{P}} \cap \overline{\mathbf{Q}}$=$\overline{\mathbf{P} \cap \mathbf{Q}}$.

Proof: Clearly, $\bar{\mathbf{P}} \cap \bar{\mathbf{Q}}$ is closed.
Again, $\bar{\mathbf{P}} \cap \bar{\mathbf{Q}} \leq \mathbf{P} \cap \mathbf{Q}$.
$\Rightarrow \overline{\bar{\mathbf{P}} \cap \bar{\mathbf{Q}}} \leq \overline{\mathbf{P} \cap \mathbf{Q}}$.
$\Rightarrow \bar{\mathbf{P}} \cap \bar{\mathbf{Q}} \leq \overline{\mathbf{P} \cap \mathbf{Q}}$.
Again, $\mathbf{P} \cap \mathbf{Q} \leq \mathbf{P}$.
$\Rightarrow \overline{\mathbf{P} \cap \mathbf{Q}} \leq \bar{\mathbf{P}}$.
Similarly, $\overline{\mathbf{P} \cap \mathbf{Q}} \leq \bar{\mathbf{Q}}$.
Therefore, $\overline{\mathbf{P} \cap \mathbf{Q}} \leq \bar{\mathbf{P}} \cap \bar{\mathbf{Q}}$.
Hence, $\overline{\mathbf{P} \cap \mathbf{Q}} = \bar{\mathbf{P}} \cap \bar{\mathbf{Q}}$.

∎

Definition 12: Let (X, M) be a multiple topological space and $\mathbf{Q} \subset X$. The interior of $\mathbf{Q}$ is denoted by $\mathbf{Q}^\circ$ and is defined as the largest open multiple set contained in $\mathbf{Q}$.
Equivalently, $\mathbf{Q}^\circ$ is defined in the following way:

$$\mathbf{Q}^\circ = \cup\{\mathbf{P} : \mathbf{P} \text{ is M-open and } \mathbf{P} \subset \mathbf{Q}\}.$$

Then $\mathbf{Q}^\circ$ is always M-open.

Example 6: Let **A**,**B**, **C** be multiple sets over R, defined as

$$A_i^j(x) = \begin{cases} \frac{2x}{5} + \frac{1}{2}; \text{x} \in [0,1] & \forall i,j \in N. \\ 0;\ otherwise. \end{cases}$$

$$B_i^j(x) = \begin{cases} \frac{2x}{15} + \frac{2}{3}; \text{ x} \in [0,1] & \forall i,j \in N. \\ 0;\ otherwise. \end{cases}$$

$$C_i^j(x) = \begin{cases} \frac{x}{4}; \text{ x} \in [0,1] & \forall i,j \in N. \\ 0;\ otherwise. \end{cases}$$

$\rho = \{X, \phi, A, C\}$ is a multiple topology on R.

Then $\mathbf{A}^\circ = \mathbf{C}$, $\mathbf{B}^\circ = \mathbf{C}$ and $\mathbf{C}^\circ = \phi$.

Example 7: Let **A, B, C** be multiple sets of order (2,2) over X = $\{1, 2\}$ whose membership values are given by,

$A_i^j(x) = \frac{2x}{15i} + \frac{2}{3j}$; x$\in X$ and for i,j=1,2.

$B_i^j(x) = \frac{x}{4ij}$; x$\in X$ and for i,j=1,2.

$\rho = \{X, \phi, \mathbf{A}, \mathbf{B}\}$ is a multiple topology on X.

Then $\mathbf{A}^\circ = \mathbf{B}$ and $\mathbf{B}^\circ = \phi$.

Theorem 3: Let (X, M) be a multiple topological space and $\mathbf{P}, \mathbf{Q}$ be two multiple sets in X. Then

1) $\phi^\circ = \phi, X^\circ = X$.
2) $\mathbf{P}^\circ \subset \mathbf{P}$.
3) $\mathbf{P}^{\circ\circ} = \mathbf{P}^\circ$.
4) $(\mathbf{P} \cap \mathbf{Q})^\circ$=$\mathbf{P}^\circ \cap \mathbf{Q}^\circ$.

Proof: 1. Since the interior of any set is the join of all open subsets contained in this set. Now the empty set ϕ and the whole space X of a multiple topological space is open. Thus ϕ°=ϕ and X°=X.

2. From the definition of interior of a set **P**, the combination of all open subsets is included in **P**, denoted by $\mathbf{P}^\circ$ i.e., $\mathbf{P}^\circ$=$\cup\{\mathbf{Q} : \mathbf{Q}$ is open and $\mathbf{Q} \subset \mathbf{P}\}$.
Hence, $\mathbf{P}^\circ \subset \mathbf{P}$.

3. Since $\mathbf{P}^\circ$ itself is open and $\mathbf{P}^{\circ\circ}$ is the greatest open set contained in $\mathbf{P}^\circ$.
So, evidently $\mathbf{P}^{\circ\circ} = \mathbf{P}^\circ$.

4. Since $(\mathbf{P} \cap \mathbf{Q})^\circ \subset \mathbf{P}^\circ$ and $(\mathbf{P} \cap \mathbf{Q})^\circ \subset \mathbf{Q}^\circ$.
So, $(\mathbf{P} \cap \mathbf{Q})^\circ \subset \mathbf{P}^\circ \cap \mathbf{Q}^\circ$.
On the other hand, $\mathbf{P}^\circ \cap \mathbf{Q}^\circ \subset \mathbf{P} \cap \mathbf{Q}$ of which the open set $\mathbf{P}^\circ \cap \mathbf{Q}^\circ$ hold in $\mathbf{P} \cap \mathbf{Q}$.
Hence $\mathbf{P}^\circ \cap \mathbf{Q}^\circ$ must be contained in the largest open set $(\mathbf{P} \cap \mathbf{Q})^\circ$.
i.e., $\mathbf{P}^\circ \cap \mathbf{Q}^\circ \subset (\mathbf{P} \cap \mathbf{Q})^\circ$.
Therefore, $(\mathbf{P} \cap \mathbf{Q})^\circ$=$\mathbf{P}^\circ \cap \mathbf{Q}^\circ$.

■

Theorem 4: Let (X, M) be a multiple topological space and $\mathbf{P}, \mathbf{Q}$ be two multiple sets in X. Then $\mathbf{P}^\circ \cup \mathbf{Q}^\circ \subseteq (\mathbf{P} \cup \mathbf{Q})^\circ$.

Proof: Clearly, $\mathbf{P}^\circ \cup \mathbf{Q}^\circ$ is open.
Also, $\mathbf{P}^\circ \subseteq \mathbf{P}$ and $\mathbf{Q}^\circ \subseteq \mathbf{Q}$.
Then, $\mathbf{P}^\circ \cup \mathbf{Q}^\circ \subseteq \mathbf{P} \cup \mathbf{Q}$.
$\Rightarrow (\mathbf{P}^\circ \cup \mathbf{Q}^\circ)^\circ \subseteq (\mathbf{P} \cup \mathbf{Q})^\circ$.
$\Rightarrow \mathbf{P}^\circ \cup \mathbf{Q}^\circ \subseteq (\mathbf{P} \cup \mathbf{Q})^\circ$. ■

Definition 13: Suppose (X, M_1) and (Y, M_2) be two multiple topological spaces and let $F : (X, M_1) \to (Y, M_2)$ be a function from X into Y. Then F is said to be continuous at a point $x \in X$ if every M_2 open subset **Q** , $F^{-1}(\mathbf{Q})$ is open in M_1.

Theorem 5: Let (X, M_1) and (Y, M_2) be two multiple topological spaces. Then a function $F : (X, M_1) \to (Y, M_2)$ is continuous if $F(\bar{\mathbf{P}}) \subset \overline{F(\mathbf{P})}$ for all **P** in multiple topological space (X, M_1).

Proof: Let $\mathbf{P}_1 \in \{(y_i, A(y_i)); y_i \in Y\}$ be such that ${\mathbf{P}_1}^c \in M_2$ and put

$$\begin{aligned} \mathbf{P} &= F^{-1}(\mathbf{P}_1), \text{ then} \\ F(\bar{\mathbf{P}}) &\subset \overline{F(\mathbf{P})}. \\ &= \overline{F(F^{-1}(\mathbf{P}_1)}. \\ &\subset \bar{\mathbf{P}}_1. \\ &= \mathbf{P}_1. \end{aligned}$$

But $\bar{\mathbf{P}}$ is such that $\mathbf{P} \subset \bar{\mathbf{P}}$. So, $\mathbf{P} = \bar{\mathbf{P}}$ and **P** is closed. i.e., $F^{-1}(\mathbf{P}_1)$ is closed. Therefore, F is continuous.

■

Definition 14: Let (X,M) be a multiple topological space. Some definitions on this space are given below:

1) A multiple set **P** is said to be everywhere dense iff $\bar{\mathbf{P}}$=X.
2) A multiple set **P** is nowhere dense iff $(\bar{\mathbf{P}})^c$=X.
3) int($\bar{\mathbf{P}}$) = ϕ, that is if the interior of closure of any multiple set **P** is empty, then **P** is called nowhere dense in X.
4) Any multiple set **P** is said to be multiple boundary(M-boundary) iff $\overline{1 - \mathbf{P}}$=X.

Example 8: Let **A,B,C** be multiple sets over R, defined as

$$A_i^j(x) = \begin{cases} \frac{2x}{5} + \frac{1}{2}; \text{ x} \in [0,1] & \forall i,j \in N. \\ 0;\ otherwise. \end{cases}$$

$$B_i^j(x) = \begin{cases} \frac{2x}{15} + \frac{2}{3}; \text{ x} \in [0,1] & \forall i,j \in N. \\ 0;\ otherwise. \end{cases}$$

$$C_i^j(x) = \begin{cases} \frac{x}{4}; \text{ x} \in [0,1] & \forall i,j \in N. \\ 0;\ otherwise. \end{cases}$$

$\rho = \{\phi, X, \mathbf{A}, \mathbf{C}\}$ is a multiple topology on R.

Then $\overline{\mathbf{A}}$=1, $\overline{\mathbf{B}}$=1 and $\overline{\mathbf{C}}$=$\mathbf{C}^c$.
This implies that **A** and **B** is everywhere dense in R.

Result 1: If $\mathbf{P} \subseteq \mathbf{Q}$ and **Q** is everywhere dense then **P** is everywhere dense too.

Proof: Since $\mathbf{P} \subseteq \mathbf{Q}$ and $\mathbf{Q} \subseteq \bar{\mathbf{Q}}$,
$\Rightarrow \mathbf{P} \subseteq \bar{\mathbf{Q}}$.
$\Rightarrow \bar{\mathbf{P}} \subseteq \bar{\bar{\mathbf{Q}}}$.
$\Rightarrow \bar{\mathbf{P}} \subseteq \bar{\mathbf{Q}}$.
Since $\bar{\mathbf{Q}}$ = X, therefore $\bar{\mathbf{P}}$ = X.
Hence, **P** is everywhere dense too. ■

Result 2: If $\mathbf{Q} \subseteq \mathbf{P}$ and **P** is M-boundary, consequently then **Q** is M-boundary too.

Proof: Since $\mathbf{Q} \subseteq \mathbf{P}$,
$\Rightarrow 1 - \mathbf{P} \subseteq 1 - \mathbf{Q}$.
$\Rightarrow \overline{1 - \mathbf{P}} \subseteq \overline{1 - \mathbf{Q}}$.
So, $\overline{1 - \mathbf{P}}$=X.
Thus, $\overline{1 - \mathbf{Q}}$=X.
Hence, **Q** is M-boundary too. ■

Result 3: If **P** is nowhere dense and $\mathbf{Q} \subseteq \mathbf{P}$, then **Q** is nowhere dense too.

Proof: Since $\mathbf{Q} \subseteq \mathbf{P}$ and $\mathbf{P} \subseteq \bar{\mathbf{P}}$,
$\Rightarrow \mathbf{Q} \subset \overline{\mathbf{P}}$.

$\Rightarrow 1 - \mathbf{P} \subseteq 1 - \bar{\mathbf{Q}}$.
$\Rightarrow \overline{1 - \mathbf{P}} \subseteq \overline{1 - \bar{\mathbf{Q}}}$.
Since $\mathbf{P}$ is nowhere dense which implies $1 - \overline{\mathbf{P}}$=X.
Hence, $\mathbf{Q}$ is nowhere dense too. ■

Result 4: If $\mathbf{P}$ is nowhere dense then so is $\bar{\mathbf{P}}$.

Proof: Since $\mathbf{P} \subseteq \bar{\mathbf{P}}$,
$\Rightarrow 1 - \mathbf{P} \supseteq 1 - \bar{\mathbf{P}}$.
$\Rightarrow \overline{1 - \mathbf{P}} \supseteq \overline{1 - \bar{\mathbf{P}}}$.
$\Rightarrow \overline{(\mathbf{P}^c)} \supseteq \overline{(\bar{\mathbf{P}}^c)}$.
Since $\mathbf{P}$ is nowhere dense, so $\overline{\mathbf{P}}^c = X$.
Hence, $\bar{\mathbf{P}}$ is nowhere dense too. ■

Definition 15: Let (X, M) be a multiple topological space and $\mathbf{Q} \subset X$ and $x \in X$. Then $\mathbf{Q}$ is a neighborhood of x if M-open set $\mathbf{A}$ exists such that $x \in \ \mathbf{A} \subset \mathbf{Q}$.

Definition 16: A multiple set $\mathbf{Q}$ in the multiple topological space (X, M) is called a multiple point of y if $M(x) = [0]_{(n\times k)}$ for every $x \in X - \{y\}$.

Definition 17: Let (X, M) and (Y, N) be multiple topological spaces. The mapping $F_G : (X, M) \to (Y, N)$ is called multiple mapping from (X, M) to (Y, N), where $F : X \to Y$ and $G : M \to N$. For each multiple neighborhood $\mathbf{P}$ of $F(q_x)$, if there exists a multiple neighborhood $\mathbf{Q}$ of q_x such that $F_G(\mathbf{Q}) \subset \mathbf{P}$, then F_G is said to be multiple continuous mapping at q_x.
If F_G is multiple continuous mapping at all q_x, then F_G is called multiple continuous mapping.

Definition 18: Let (X, M) and (Y, N) be multiple topological spaces. The mapping $F_G : (X, M) \to (Y, N)$ is called multiple mapping from (X, M) to (Y, N), where $F : X \to Y$ and $G : M \to N$. Then
(1) if the image of each M-open set over X is M-open in Y, then F_G is said to be M-open mapping.
(2) if the image of each M-closed set over X is M-closed in Y, then F_G is said to be M-closed mapping.

Theorem 6: We know that (X, M) and (Y, N) are two multiple lattice topological spaces, $F_G : (X, M) \to (Y, N)$ be a mapping. Then the following conditions are equivalent:
(1) $F_G : (X, M) \to (Y, N)$ is a multiple continuous mapping.
(2) For each M-open set O over Y, $F_G(O)$ is a M-open set over X.
(3) For each M-closed set B over Y, $F_G^{-1}(B)$ is a M-closed set over X.
(4) For each M-open set A over X, $\underline{F_G\overline{(A)}} \subset \overline{F_G(A)}$.
(5) For each M-open set O over Y, $\underline{F_G^{-1}(O)} \subset F_G\overline{(O)}$.
(6) For each M-open set O over Y, $F_G^{-1}((O)^o) \subset (F_G^{-1}(O))^o$.

Proof: $\underline{(1) \Rightarrow (2)}$:
Let G be a M-open set over Y and $Q \in F_G^{-1}(O)$ be an arbitrary M-point. Then $F_G(Q) \in A$.
Since F_G is M-continuous mapping, there exists $Q \in A \in \tau$ such that $F_G(A) \subset O$.
This implies $Q \in A \subset F_G^{-1}(O)$, $F_G^{-1}(O)$ is a M-open set over X.
$\underline{(2) \Rightarrow (1)}$:
Let Q be a M-point and $F(Q) \in O$ be an arbitrary M-neighbourhood. Then
$F(Q) \in F_G^{-1}(O)$ is a M-neighbourhood and $F_G(F_G^{-1}(O)) \subset O$.
$\underline{(2) \Rightarrow (3)}$:
If for each M-open set G over Y, $F_G^{-1}(O)$ is a M-open set X, then for each M-closed set H over Y, $F_G^{-1}(B)$ is a M-closed set over X.
$\underline{(3) \Rightarrow (4)}$:
Let A be a M-open set over X.
Since $F_P^L \subset F_G^{-1}(F_G(A))$ and $F_G(F_P^L) \subset \overline{(F_G(A))}$, we have
$F_P^L \subset F_G^{-1}(F_G(A)) \subset F_G^{-1}\overline{(F_G(A))}$.
By (3), since $F_G^{-1}\overline{(F_G(A))}$ is a M-closed set over X, $\overline{(A)} \subset F_G^{-1}\overline{(F_G(A))}$.
Thus $F_G(\overline{(A)}) \subset F_G(\overline{F_G^{-1}(A)}) \subset \overline{F_G(A)}$ is obtained.
$\underline{(4) \Rightarrow (5)}$:
Let G be a M-open set over Y and $F_G^{-1}(O) = A$.
By (4),
$F_G(\overline{(A)}) = F_G(\overline{F_G^{-1}(O)}) \subset \overline{F_G(F_G^{-1}(O))} \subset \overline{(O)}$.
Then $\overline{F_G^{-1}(O)} = \overline{(A)} \subset F_G^{-1}\overline{(F_G(A))} \subset F_G^{-1}\overline{(O)}$.
$\underline{(5) \Rightarrow (6)}$:
Let G be a M-open set over Y.
Substitute $(A)'$ in (5), then
$\overline{F_G^{-1}((O)')} \subset F_G^{-1}\overline{((O)')}$.
Since $(O)^o = (\overline{(O)'})'$, then we have
$F_G^{-1}((O)^o) = F_G^{-1}((\overline{(O)'})') = (F_G^{-1}((\overline{(O)'})') \subset (\overline{F_G^{-1}(((O)')})') = ((\overline{F_G^{-1}(((O)')})' = (F_G^{-1}(O))^o$.
$\underline{(6) \Rightarrow (2)}$:
Let G be a M-open set over Y.
Then since $(F_G^{-1}((O))^o \subset F_G^{-1}(O) = F_G^{-1}(O)^o) \subset (F_G^{-1}((O))^o$,
$(F_G^{-1}((O))^o = F_G^{-1}(O)$ is obtained.
This implies $F_G^{-1}(O)$ is a M-open set over X. ■

Note 1: M-open, M-closed, and multiple continuous mappings are all independent.

Theorem 7: If $F_G : (X, M) \to (Y, N)$ is a multiple continuous mapping, then for each element x, $F_{Gx} : (X, M_x) \to (Y, N_x)$ is a multiple continuous mapping.

Proof: Let $U \in N_x$. Then there exists a M-open set O over Y such that $U = O(x)$.
Since $F_G : (X, M) \to (Y, N)$ is a multiple continuous mapping, $F_G^{-1}(O)$ is a M-open set over X and $F_G^{-1}(O)(x) = F_G^{-1}(O(x)) = F^{-1}(A)$ is a M-open set.
This implies F_{Gx} is a multiple continuous mapping. ■

Theorem 8: If $(\bar{A})'$ is a M-open set over X, for each M-open set A, then $F_G : (X, M) \to (Y, N)$ is a multiple continuous mapping if and only if $F_{Gx} : (X, M_x) \to (Y, N_x)$ is a multiple continuous mapping for each element x.

Proof: Let $F_{Gx} : (X, M_x) \to (Y, N_x)$ be a multiple continuous mapping for each element x and let A be an arbitrary M-open set over X. Then $F_{Gx}(\bar{A}(x)) \subset \overline{F_{Gx}(\overline{A})(x)}$ is satisfied for each x.
Since $(\bar{A})' \in X$, $(\bar{A})' = \overline{A'}$.
Thus $F_G(\overline{(A)}) = \overline{F_G(A)}$ is obtained.
Let $F_G : (X, M) \to (Y, N)$ be a multiple continuous mapping. Then for each element x and if A be an arbitrary M-open set over X. Then $F_G(\bar{A}(x)) \subset \overline{F_G(\overline{A})(x)}$ is satisfied for each element x.
Thus $F_{Gx} : (X, M_x) \to (Y, N_x)$ is obtained.
This implies F_{Gx} is a multiple continuous mapping for each element x. ■

Conclusion

A detailed study has been made on the topological structure of multiple sets. The analysis proceeds as a generalization of fuzzy topology by Chang. The theory of interior, closure, denseness, open and closed sets, neighbourhood, multiple point, and continuous mapping in multiple topological spaces have been extensively studied. The theory has been illustrated with numerical examples combined with graphical representations. We can extend the theory of multiple continuous mapping and explore some interesting results with proof.

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