Ordered almost (m, n)-ideals and Fuzzy ordered almost (m, n)-ideals in Ordered Semigroups

Pannawit Khamrot, Suchada Saekuai, Thawathchai Phongthai, Thiti Gaketem

***Abstract*—The ordered semigroups are algebraic systems consisting of a nonempty set, an associative binary operation, and a partial order compatible with this binary operation. Grosek and Stako studied almost ideals in semigroups in 1980. In 2019, S. Suebsung et al. studied almost (m, n)-ideals in semigroups. Later, in 2022, S. Suebsung et al. introduced ordered almost ideals in ordered semigroups. This paper aims to define ordered almost (m, n)-ideals and fuzzy ordered almost (m, n)-ideals in ordered semigroups. We prove the union of ordered almost (m, n)-ideals, including ordered almost (m, n)-ideals in ordered semigroups. In class, fuzzifications are the same. We connected the relation and ordered almost (m, n)-ideals, and the fuzzy ordered almost (m, n)-ideals in ordered semigroups. Finally, we study bipolar fuzzy prime almost interior ideals in semigroups.**

***Index Terms*—ordered almost (m, n)-ideals, Fuzzy ordered almost (m, n)-ideals, Prime ordered almost (m, n)-ideals, Ordered semigroups**

I. Introduction

ORDERED semigroups are an algebraic structure in a binary operation satisfying the associative property and a partial order with compatibility. Dealing with various problems related to uncertain conditions by fuzzy sets by Zadeh in 1965, [1]. These concepts were applied in many areas, such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, and topology. Rosenfeld studied the concentration of fuzzy subgroups and fuzzy ideals. In 1981, Kuroki studied the types of fuzzy subsemigroups. In the same year, Satko and Grosek [2] discussed the concept of an almost-ideal (A-ideal) in a semilattice. And S. Bogdanovic [3] gave the concept of almost bi-ideals in semigroups. In 2019, S. Suebsung et al. [4] investigated ordered almost ideals and fuzzy almost ideals in ternary semigroups. In 2020, Chinram et al. [5] discussed almost interior ideals and weakly almost interior ideals in semigroups and studied the relationship between almost interior ideals and weakly almost interior ideals in semigroups. The research on almost ideals studied in semihypergroups, such that in 2021, P. Muangdoo et al. [6] studied almost bi-hyperideals and their fuzzification of semihypergroups. W. Nakkhasen et al. [7] discussed fuzzy almost interior hyperideals of semihypergroups. in 2022, S. Suebsung et al. [8] introduced ordered almost ideals in ordered semigroups. In the same year T. Gaketem and P. Khamrot [9] explored the concept of almost ideals within the framework of bipolar fuzzy sets, specifically focusing on bipolar fuzzy almost bi-ideals in semigroups. In 2023, T. Gaketem and P. Khamrot [10] studied bipolar fuzzy almost interior ideals in semigroups. In 2024, T. Gaketem and P. Khamrot [11], [12] discussed bipolar fuzzy almost ideals and quasi ideals in semigroups. In addition, ordered almost ideal's work also has many studies, such as ordered almost ideals in ordered semigroup [13], ordered almost ideals in semirings [14], ordered almost Ideals in ternary semiring [15], etc. In 2025, P. Khamrot et al. [16] studied fuzzy (m, n)-ideals and n-interior ideals in ordered semigroups.

In this paper, we extend the definition of ordered almost (m, n)-ideals in semigroups to ordered semigroups. We discussed the union of ordered almost (m, n)-ideals, including ordered almost (m, n)-ideals in ordered semigroups. In class, fuzzifications are the same. We connected the relation and ordered almost (m, n)-ideals, and the fuzzy ordered almost (m, n)-ideals in ordered semigroups. Finally, we study bipolar fuzzy prime ordered almost interior ideals in semigroups.

II. Preliminaries

Now, we recall the concept of ordered semigroups and fuzzy sets. Additionally, their preliminary results are provided.

Definition 2.1. *[17]. Let $\mathfrak{T}$ be a set with a binary opeation $\cdot$ and a binary opeation relation $\leq$. Then $(\mathfrak{T}, \cdot, \leq)$ is called an* ordred semigroup *if*

(1) $(\mathfrak{T}, \cdot)$ *is a semigroup,*
(2) $(\mathfrak{T}, \leq)$ *is a partially ordered set,*
(3) *for all* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{T}$*, we have* $\mathfrak{a} \leq \mathfrak{b}$ *then* $\mathfrak{ac} \leq \mathfrak{bc}$ *and* $\mathfrak{ca} \leq \mathfrak{cb}$.

For a nonempty subset $\mathfrak{X}$ and $\mathfrak{Y}$ of ordered semigroup $\mathfrak{T}$, we write
$(\mathfrak{X}] := \{\mathfrak{a} \in \mathfrak{T} \mid \mathfrak{a} \leq \mathfrak{b}$ for some $\mathfrak{b} \in \mathfrak{X}\}$ and $\mathfrak{X}\mathfrak{Y} := \{\mathfrak{x}\mathfrak{y} \mid \mathfrak{x} \in \mathfrak{X}$ and $\mathfrak{y} \in \mathfrak{Y}\}$.

It is observed that

(1) $\mathfrak{X} \subseteq (\mathfrak{X}]$,
(2) if $\mathfrak{X} \subseteq \mathfrak{Y}$, then $(\mathfrak{X}] \subseteq (\mathfrak{Y}]$,
(3) $((\mathfrak{X}]] = (\mathfrak{X}]$,
(4) $(\mathfrak{X}](\mathfrak{Y}] \subseteq (\mathfrak{X}\mathfrak{Y}]$,
(5) $((\mathfrak{X}](\mathfrak{Y}]] = (\mathfrak{X}\mathfrak{Y}]$,
(6) $(\mathfrak{X} \cup \mathfrak{Y}] = (\mathfrak{X}] \cup (\mathfrak{Y}]$,
(7) $(\mathfrak{X} \cap \mathfrak{Y}] = (\mathfrak{X}] \cap (\mathfrak{Y}]$.

Manuscript received March 29, 2025; revised May 21, 2025.

This research was supported by the School of Science, University of Phayao, Phayao, Thailand.

P. Khamrot is a lecturer at the Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University Technology Lanna Phitsanulok, Phitsanulok, Thailand. (e-mail: pk_g@rmutl.ac.th).

S. Saekuai is an undergraduate student at the Department of Mathematics, School of Education, University of Phayao, Phayao, Thailand. (e-mail: 65202768@up.ac.th).

T. Phongthai is an undergraduate student at the Department of Mathematics, School of Education, University of Phayao, Phayao, Thailand. (e-mail: 65202375@up.ac.th).

T. Gaketem is a lecturer at the Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand.(corresponding author to provide email: thiti.ga@up.ac.th).

Let $(\mathfrak{T}, \cdot, \leq)$ be an ordered semigroup, $(\emptyset \neq \mathcal{K} \subseteq \mathfrak{T})$ is called a *subsemigroup* such that $\mathcal{K}^2 \subseteq \mathcal{K}$. A *left (right) ideal* of a ordered semigroup $(\mathfrak{T}, \cdot, \leq)$ is a non-empty set $\mathcal{K}$ of $\mathfrak{T}$ such that $\mathfrak{T}\mathcal{K} \subseteq \mathcal{K}$ $(\mathcal{K}\mathfrak{T} \subseteq \mathcal{K})$ and $(\mathcal{K}]$. By an *ideal* of an ordered semigroup $(\mathfrak{T}, \cdot, \leq)$, we mean a non-empty set of $\mathfrak{T}$ which is both a left and a right ideal of $\mathfrak{T}$.

Definition 2.2. *[18] A subsemigroup $\mathcal{K}$ of an ordered semigroup $(\mathfrak{T}, \cdot, \leq)$ is called an (m,n)-*ideal *of $\mathfrak{T}$ if $\mathcal{K}$ satisfies the following conditions:*

(1) $\mathcal{K}^m\mathfrak{T}\mathcal{K}^n \subseteq \mathcal{K}$.
(2) $\mathcal{K} = (\mathcal{K}]$, *that is for $x \in \mathcal{K}$ and $y \in \mathfrak{T}$, $y \leq x$ implies $y \in \mathcal{K}$.*

where m, n are non-negative integers.

Definition 2.3. *[19] A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called a* left ordered almost ideal *(LOAI) of $\mathcal{S}$ if $(s\mathcal{I}] \cap \mathcal{I} \neq \emptyset$ for all $s \in \mathcal{S}$.*

Definition 2.4. *[19] A nonempty subset of $\mathcal{I}$ an ordered semigroup $\mathcal{S}$ is called a* right ordered almost ideal *(ROAI) of $\mathcal{S}$ if $(\mathcal{I}s] \cap \mathcal{I} \neq \emptyset$ for all $s \in \mathcal{S}$.*

Definition 2.5. *[20] A nonempty set $\mathcal{I}$ of an ordered semigroup $\mathcal{S}$ is called an* ordered almost n-interior ideal *(OA-n-II) of $\mathcal{S}$ if $(a\mathcal{I}^n b] \cap \mathcal{I} \neq \emptyset$, for all $a, b \in \mathcal{S}$ and $n \in \mathbb{N}_0$.*

For any $h_i \in [0,1]$, $i \in \mathcal{F}$, define

$$\underset{i\in\mathcal{F}}{\vee} h_i := \sup_{i\in\mathcal{F}}\{h_i\} \quad \text{and} \quad \underset{i\in\mathcal{F}}{\wedge} h_i := \inf_{i\in\mathcal{F}}\{h_i\}.$$

We see that for any $h, r \in [0,1]$, we have

$$h \vee r = \max\{h, r\} \quad \text{and} \quad h \wedge r = \min\{h, r\}.$$

A *fuzzy set* ϑ in a nonempty set $\mathfrak{T}$ is a function from $\mathfrak{T}$ into the unit closed interval $[0,1]$ of real numbers, i.e., $\vartheta : \mathfrak{T} \to [0,1]$.

For any two fuzzy sets ϑ and ξ of a non-empty set $\mathfrak{T}$, define the symbol as follows:

(1) $\vartheta \leq \xi \Leftrightarrow \vartheta(h) \leq \xi(h)$ for all $h \in \mathfrak{T}$,
(2) $\vartheta = \xi \Leftrightarrow \vartheta \leq \xi$ and $\xi \leq \vartheta$,
(3) $(\vartheta \wedge \xi)(h) = \min\{\vartheta(h), \xi(h)\} = \vartheta(h) \wedge \xi(h)$ for all $h \in \mathfrak{T}$,
(4) $(\vartheta \vee \xi)(h) = \max\{\vartheta(h), \xi(h)\} = \vartheta(h) \vee \xi(h)$ for all $h \in \mathfrak{T}$,
(5) the *support* of ϑ instead of $\text{supp}(\vartheta) = \{h \in \mathfrak{T} \mid \vartheta(h) \neq 0\}$.
For the symbol $\vartheta \geq \xi$, we mean $\xi \leq \vartheta$.

If $\mathfrak{K} \subseteq \mathfrak{T} \neq \emptyset$, then the characteristic function $\chi_{\mathfrak{K}}$ of $\mathfrak{T}$ is a function from $\mathfrak{T}$ into $\{0,1\}$ defined as follows:

$$\chi_{\mathfrak{K}}(\mathfrak{x}) = \begin{cases} 1 & \text{if } \mathfrak{x} \in \mathfrak{K} \\ 0 & \text{otherwise.} \end{cases}$$

for all $\mathfrak{x} \in \mathfrak{T}$

Lemma 2.6. *If $\mathfrak{T}$ and $\mathfrak{L}$ are nonempty subsets of an oredred semigroup $\mathfrak{T}$, then the following are true:*

(1) $\chi_{\mathfrak{T}} \wedge \chi_{\mathfrak{L}} = \chi_{\mathfrak{T}\cap\mathfrak{L}}$.
(2) *If $\mathfrak{T} \subseteq \mathfrak{L}$, then $\chi_{\mathfrak{T}} \leq \chi_{\mathfrak{L}}$.*
(3) $\chi_{\mathfrak{T}} \circ \chi_{\mathfrak{L}} = \chi_{\mathfrak{T}\mathfrak{L}}$.

Definition 2.7. *Let $\mathfrak{T}$ be an ordered semigroup and F be a non-empty subset of $\mathfrak{T}$, we define the set F_u by*

$$F_u := \{(x,y) \in \mathfrak{T} \times \mathfrak{T} \mid u \leq xy\}.$$

Definition 2.8. *Let ϑ and η be fuzzy sets of an ordered semigroup $\mathfrak{T}$. The product of fuzzy subsets ϑ and η of $\mathfrak{T}$ is defined as follow, for all $u \in \mathfrak{T}$*

$$(\vartheta \circ \eta)(u) = \begin{cases} \bigvee\limits_{(x,y)\in F_u} \{\vartheta(x) \wedge \eta(y)\} & \text{if} \quad F_u \neq \emptyset, \\ 0 & \text{if} \quad F_u = \emptyset. \end{cases}$$

For $u \in \mathfrak{T}$ and $t \in (0,1]$, a fuzzy point x_t of a set $\mathfrak{T}$ is a fuzzy set of $\mathfrak{T}$ defined by

$$x_t(e) = \begin{cases} t & \text{if} \quad e = u, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathfrak{k} \in \mathbb{N}$, let $\vartheta^n := \underbrace{\vartheta \circ \vartheta \circ \cdots \circ \vartheta}_{n\text{-times}}$.

Lemma 2.9. *[13] If φ, ν and ξ are fuzzy sets of an ordered semigroup S, then the following are true:*

(1) *If $\varphi \preceq \nu$, then $\varphi^n \preceq \nu^n$*
(2) *If $\varphi \preceq \nu$, then $\varphi \circ \xi \preceq \nu \circ \xi$.*
(3) *If $\varphi \preceq \nu$, then $\varphi \vee \xi \preceq \nu \vee \xi$.*
(4) *If $\varphi \preceq \nu$, then $\varphi \wedge \xi \preceq \nu \wedge \xi$.*
(5) *If $\varphi \preceq \nu$, then $\text{supp}(\varphi) \preceq \text{supp}(\nu)$.*

For a fuzzy set φ of an ordered semigroup S, we define $(\varphi] : S \to [0,1]$ by $(\varphi] := \sup\limits_{a \leq b} \varphi(b)$ for all $a \in S$.

Lemma 2.10. *[13] If φ, ν and ξ are fuzzy sets of an ordered semigroup $\mathcal{S}$, then the following are true:*

(1) $\varphi \preceq (\varphi]$.
(2) *If $\varphi \preceq \nu$, then $(\varphi] \preceq (\xi]$.*
(3) *If $\varphi \preceq \nu$, then $(\varphi \circ \xi] \preceq (\nu \circ \xi]$ and $(\xi \circ \varphi] \preceq (\xi \circ \nu]$.*

Lemma 2.11. *[13] If φ is a fuzzy set of an ordered semigroup $\mathcal{S}$, then the following are equivalent.*

(1) *If $a \leq b$, then $\varphi(a) \preceq \varphi(b)$.*
(2) $(\varphi] = \varphi$.

Definition 2.12. *[13] A fuzzy set δ of a semigroup $\mathfrak{T}$ is said to be a fuzzy ideal of $\mathfrak{T}$ if $\delta(\mathfrak{u}\mathfrak{v}) \geq \delta(\mathfrak{u}) \vee \delta(\mathfrak{v})$ for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{T}$.*

Definition 2.13. *[17] A fuzzy subsemigroup δ of a ordered semigroup $\mathfrak{T}$ is said to be a* fuzzy (m,n)-ideal *of $\mathfrak{T}$ if*

(1) $\delta(\mathfrak{u}_1\mathfrak{u}_2\cdots\mathfrak{u}_m\mathfrak{z}\mathfrak{v}_1\mathfrak{v}_2\cdots\mathfrak{v}_n) \geq \delta(\mathfrak{u}_1) \wedge \delta(\mathfrak{u}_2) \wedge \ldots \wedge \delta(\mathfrak{u}_m) \wedge \delta(\mathfrak{v}_1) \wedge \delta(\mathfrak{v}_2) \wedge \ldots \wedge \delta(\mathfrak{v}_n)$ *for all* $\mathfrak{u}_1, \mathfrak{u}_2, \ldots, \mathfrak{u}_m, \mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n, \mathfrak{z} \in \mathfrak{T}$ *and* $m, n \in \mathbb{N}$.
(2) *If $\mathfrak{u}_1 \leq \mathfrak{u}_2$, then $\delta(\mathfrak{u}_1) \geq \delta(\mathfrak{u}_2)$, for all $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{T}$.*

Definition 2.14. *A fuzzy set φ of an ordered semigroup $\mathcal{S}$ is called*

(1) *a* fuzzy subsemigroup *(FSG) of $\mathcal{S}$ if $\varphi(ab) \leq \varphi(a) \wedge \varphi(b)$ for all $a, b \in \mathcal{S}$,*
(2) *a* fuzzy left ideal *(FLI) of $\mathcal{S}$ if $\varphi(ab) \leq \varphi(b)$ and if $a \leq b$, then $\varphi(a) \leq \varphi(b)$ for all $a, b \in \mathcal{S}$,*
(3) *a* fuzzy right ideal *(FRI) of $\mathcal{S}$ if $\varphi(ab) \leq \varphi(a)$ and if $a \leq b$, then $\varphi(a) \leq \varphi(b)$ for all $a, b \in \mathcal{S}$,*
(4) *a* fuzzy ideal *(FI) of $\mathcal{S}$ if it is both a FLI and FRI of $\mathcal{S}$,*
(5) *a* fuzzy left ordered almost ideal *(FLOAI) of $\mathcal{S}$ if $(x_t \circ \varphi] \wedge \varphi \neq 0$ for all fuzzy point r_β.*

(6) *a* fuzzy right ordered almost ideal *(FROAI) of $\mathcal{S}$ if* $(\varphi \circ x_t] \wedge \varphi \neq 0$ *for all fuzzy point r_β.*
(7) *a* fuzzy ordered almost ideal *(FOAI) of $\mathcal{S}$ if it is both a FLOAI and FROAI of $\mathcal{S}$,*

III. Main Results

In this section, we define the ordered almost (m,n)-ideals and fuzzy ordered almost (m,n)-ideals in an ordered semigroup. We prove some basic interesting properties of ordered almost (m,n)-ideals and fuzzy ordered almost (m,n)-ideals in an ordered semigroup.

Definition 3.1. *A non-empty subset $\mathfrak{B}$ on an ordered semigroup $\mathfrak{T}$ is called an* ordered almost (m,n)-ideal *(OA-(m,n)-I)of $\mathfrak{T}$ if $(\mathfrak{B}^m\mathfrak{t}\mathfrak{B}^n] \cap \mathfrak{B} \neq \emptyset$ for all $\mathfrak{t} \in \mathfrak{T}$ where $m,n \in \{1,2,...,n\}$.*

Example 3.2. (1) *An OA-$(1,0)$-I of an ordered semigroup $\mathfrak{T}$ is a ROAI of $\mathfrak{T}$.*
(2) *An OA $(0,1)$-I of an ordered semigroup $\mathfrak{T}$ is a LOAI of $\mathfrak{T}$.*
(3) *Consider the ordered semigroup $\mathbb{Z}_6$ under the usual addition and the partial ordered $\leq := \{(\bar{\mathfrak{a}},\bar{\mathfrak{a}}) \mid \bar{\mathfrak{a}} \in \mathbb{Z}_6\}$. We have $\mathfrak{A} = \{\bar{1},\bar{4},\bar{5}\}$ is an OA-$(1,0)$-I of $\mathbb{Z}_6$.*

Theorem 3.3. *Every (m,n)-ideal of an ordered semigroup $\mathfrak{T}$ is an OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Assume that $\mathfrak{B}$ is an (m,n)-ideal of $\mathfrak{T}$ and let $\mathfrak{t} \in \mathfrak{T}$. Then $(\mathfrak{B}^m\mathfrak{t}\mathfrak{B}^n] \subseteq (\mathfrak{B}^m\mathfrak{T}\mathfrak{B}^n]$ $\mathfrak{B}$. Thus $(\mathfrak{B}^m\mathfrak{t}\mathfrak{B}^n] \cap \mathfrak{B} \neq \emptyset$. We conclude that $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. ■

Theorem 3.4. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two non-empty subsets of an ordered semigroup $\mathfrak{T}$ such that $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$. If $\mathfrak{B}_1$ is an OA-(m,n)-I of $\mathfrak{T}$, then $\mathfrak{B}_2$ is also an OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Let $\mathfrak{B}_1$ is an OA-(m,n)-I of $\mathfrak{T}$ with $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ and let $\mathfrak{t} \in \mathfrak{T}$ Then $(\mathfrak{B}_1^m\mathfrak{t}\mathfrak{B}_1^n] \subseteq (\mathfrak{B}_2^m\mathfrak{t}\mathfrak{B}_2^n]$ Thus, $(\mathfrak{B}_2^m\mathfrak{t}\mathfrak{B}_2^n] \cap \mathfrak{B}_2 \neq \emptyset$. Hence, $\mathfrak{B}_2$ is an OA-(m,n)-I of $\mathfrak{T}$. ■

Corollary 3.5. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be OA-(m,n)-Is of an ordered semigroup $\mathfrak{T}$. Thus $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is also an OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are subsets of $\mathfrak{B}_1 \cup \mathfrak{B}_2$, by Theorem 3.4, $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is an OA-(m,n)-I of $\mathfrak{T}$. ■

Theorem 3.6. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be nonempty subsets of an ordered semigroup $\mathfrak{T}$. If $\mathfrak{B}_1$ is an OA-(m,n)-I of $\mathfrak{T}$, then $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is an OA-(m,n)-I of $\mathfrak{T}$.*

Proof: By Theorem 3.4, and $\mathfrak{B}_1 \subseteq \mathfrak{B}_1 \cup \mathfrak{B}_2$. Thus, $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is an OA-(m,n)-I of $\mathfrak{T}$. ■

Corollary 3.7. *The finite union of OA-(m,n)-Is of an ordered semigroup $\mathfrak{T}$ is an OA-(m,n)-I of $\mathfrak{T}$.*

Example 3.8. *Consider the ordered semigroup $\mathbb{Z}_6$ under the usual addition and the partial ordered $\leq := \{(\bar{\mathfrak{a}},\bar{\mathfrak{a}}) \mid \bar{\mathfrak{a}} \in \mathbb{Z}_6\}$. We have $\mathfrak{A} = \{\bar{1},\bar{4},\bar{5}\}$ and $\mathfrak{B} = \{\bar{1},\bar{2},\bar{5}\}$ are ordered almost $(1,0)$-ideals of $\mathbb{Z}_6$ but $\mathfrak{A} \cap \mathfrak{B} = \{\bar{1},\bar{5}\}$ is not an ordered almost $(1,0)$-ideal of $\mathbb{Z}_6$.*

Definition 3.9. *A fuzzy set ϑ on an ordered semigroup $\mathfrak{T}$ is called a* fuzzy OA-(m,n)-I *of $\mathfrak{T}$ if $(\vartheta^m \circ x_t \circ \vartheta^n] \wedge \vartheta \neq 0$ for any fuzzy point $x_t \in \mathfrak{T}$ where $m,n \in \{1,2,...,n\}$.*

Theorem 3.10. *If ϑ is a fuzzy OA-(m,n)-I of an ordered semigroup $\mathfrak{T}$ and ξ is a fuzzy subset of $\mathfrak{T}$ such that $\vartheta \leq \xi$, then ξ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Suppose that ϑ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$ and ξ is a fuzzy subset of $\mathfrak{T}$ such that $\vartheta \leq \xi$. Then for any fuzzy points $x_t \in \mathfrak{T}$, we obtain that $(\vartheta^m \circ x_t \circ \vartheta^n] \wedge \vartheta \neq 0$. Thus,

$$(\vartheta^m \circ x_t \circ \vartheta^n] \wedge \vartheta \leq (\xi^m \circ x_t \circ \xi^n] \wedge \xi \neq 0.$$

Hence, $(\xi^m \circ x_t \circ \xi^n] \wedge \xi \neq 0$. Therefore, ξ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. ■

The following results is an obvious of Theorem 3.10.

Theorem 3.11. *Let ϑ and ξ be fuzzy OA-(m,n)-Is of an ordered semigroup $\mathfrak{T}$. Then $\vartheta \vee \xi$ is also a fuzzy OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Since $\vartheta \leq \vartheta \vee \xi$, by Theorem 3.10, $\vartheta \vee \xi$ is also a fuzzy OA-(m,n)-I of $\mathfrak{T}$. ■

Theorem 3.12. *If ϑ fuzzy OA-(m,n)-I of an ordered semigroup $\mathfrak{T}$. and ξ is a fuzzy set, then $\vartheta \vee \xi$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$.*

Proof: By Theorem 3.10, and $\vartheta \leq \vartheta \vee \xi$. Thus, $\vartheta \vee \xi$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. ■

Corollary 3.13. *Let $\mathfrak{T}$ be an ordered semigroup. Then the finite maximum of fuzzy OA-(m,n)-Is of $\mathfrak{T}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$.*

Example 3.14. *Consider $n = 1, m = 0$ and the ordered semigroup $\mathbb{Z}_6$ under the usual addition and the partial ordered $\leq := \{(\bar{\mathfrak{a}},\bar{\mathfrak{a}}) \mid \bar{\mathfrak{a}} \in \mathbb{Z}_6\}$. $\vartheta : \mathbb{Z}_6 \to [0,1]$ is defined by $\vartheta(\bar{0}) = 0$, $\vartheta(\bar{1}) = 0.2$, $\vartheta(\bar{2}) = 0$, $\vartheta(\bar{3}) = 0$, $\vartheta(\bar{4}) = 0.3$, $\vartheta(\bar{5}) = 0.5$ and $\nu : \mathbb{Z}_6 \to [0,1]$ is defined by $\nu(\bar{0}) = 0$, $\nu(\bar{1}) = 0.8$, $\nu(\bar{2}) = 0.4$, $\nu(\bar{3}) = 0.3$, $\nu(\bar{4}) = 0$, $\nu(\bar{5}) = 0.3$. We have ϑ and ν are fuzzy ordered almost $(1,0)$-ideals of $\mathbb{Z}_6$ but $\vartheta \wedge \nu$ is not a fuzzy ordered almost $(1,0)$-ideal of $\mathbb{Z}_6$.*

Lemma 3.15. *Let $\mathfrak{A}$ be a subset of $\mathfrak{T}$ and $n \in \mathbb{N} \cup \{0\}$. Then $(\chi_{\mathfrak{A}})^n = \chi_{\mathfrak{A}^n}$*

Theorem 3.16. *Let $\mathfrak{B}$ be a nonempty subset of an ordered semigroup $\mathfrak{T}$. Then $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Suppose that $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Then $(\mathfrak{B}^m\mathfrak{t}\mathfrak{B}^n] \cap \mathfrak{B} \neq \emptyset$ for all $\mathfrak{t} \in \mathfrak{T}$ and $m,n \in \{1,2,...,n\}$. Thus there exists $\mathfrak{c} \in \mathfrak{T}$ such that $\mathfrak{c} \in (\mathfrak{B}^m\mathfrak{t}\mathfrak{B}^n]$ and $\mathfrak{c} \in \mathfrak{B}$. Let $x_t \in \mathfrak{T}$ and $t \in (0,1]$. Then $((\chi_{\mathfrak{B}^m} \circ x_t \circ \chi_{\mathfrak{B}^n}])(\mathfrak{c}) \neq 0$ and $\chi_{\mathfrak{B}}(\mathfrak{c}) = 1$ and $m,n \in \{1,2,...,n\}$. Thus, $(\chi_{\mathfrak{B}^m} \circ x_t \circ \chi_{\mathfrak{B}^n}] \wedge \chi_{\mathfrak{B}}(\mathfrak{c}) = ((\chi_{\mathfrak{B}})^m \circ x_t \circ (\chi_{\mathfrak{B}}^n)] \wedge \chi_{\mathfrak{B}}(\mathfrak{c}) \neq 0$ and $m,n \in \{1,2,...,n\}$. So $(\chi_{\mathfrak{B}^m} \circ x_t \circ \chi_{\mathfrak{B}^n}] \wedge \chi_{\mathfrak{B}} \neq 0$ and $m,n \in \{1,2,...,n\}$. Hence, $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$ and let $x_t \in \mathfrak{T}$ and $t \in (0,1]$ where $m,n \in \{1,2,...,n\}$. Then $(\chi_{\mathfrak{B}^m} \circ x_t \circ \chi_{\mathfrak{B}^n}] \wedge \chi_{\mathfrak{B}} \neq 0$. Thus, there exists $\mathfrak{c} \in \mathfrak{B}$ such that $((\chi_{\mathfrak{B}^m} \circ x_t \circ \chi_{\mathfrak{B}^n}] \wedge \chi_{\mathfrak{B}})(\mathfrak{c}) \neq 0$. It implies that $(\chi_{\mathfrak{B}^m} \circ x_t \circ \chi_{\mathfrak{B}^n}](\mathfrak{c}) \neq 0$ and $\chi_{\mathfrak{B}}(\mathfrak{c}) \neq 0$. Hence $\mathfrak{c} \in \mathfrak{B}^m\mathfrak{t}\mathfrak{B}^n$ and $\mathfrak{c} \in \mathfrak{B}$ So $(\mathfrak{B}^m\mathfrak{t}\mathfrak{B}^n] \cap \mathfrak{B} \neq \emptyset$. We conclude that $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. ■

Theorem 3.17. *Let ϑ be a fuzzy subset of an ordered semigroup $\mathfrak{T}$. Then ϑ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$ if and only if* $\mathrm{supp}(\vartheta)$ *is an OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Assume that ϑ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$ and let $x_t \in \mathfrak{T}$ and $t \in (0,1]$. Then $(\vartheta^m \circ x_t \circ \vartheta^n] \wedge \vartheta \neq 0$ for all $m,n \in \{1,2,...,n\}$. Thus, there exists $\mathfrak{z} \in \mathfrak{T}$ such that $((\vartheta^m \circ x_t \circ \vartheta^n] \wedge \vartheta)(\mathfrak{z}) \neq 0$. So $\vartheta(\mathfrak{z}) \neq 0$ and $\mathfrak{z} = \mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_m\mathfrak{x} = \mathfrak{x}\mathfrak{b}_1\mathfrak{b}_2\cdots\mathfrak{b}_n$ for some $\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_m, \mathfrak{b}_1\mathfrak{b}_2\cdots\mathfrak{b}_n \in \mathfrak{T}$ such that $\vartheta(\mathfrak{a}_1) \neq 0$, $\vartheta(\mathfrak{a}_2) \neq 0, \cdots, \vartheta(\mathfrak{a}_m) \neq 0$ $\vartheta(\mathfrak{b}_1) \neq 0$, $\vartheta(\mathfrak{b}_2) \neq 0, \cdots \vartheta(\mathfrak{b}_n) \neq 0$. Thus, $\mathfrak{a}_1\mathfrak{a}_2 \cdot \mathfrak{a}_m, \mathfrak{b}_1\mathfrak{b}_2\cdots\mathfrak{b}_n \in \text{supp}(\vartheta)$. Thus, $((\chi_{\text{supp}(\vartheta)^m} \circ x_t \circ \chi_{\text{supp}(\vartheta)^n}] \wedge \chi_{\text{supp}(\vartheta)})(\mathfrak{z}) \neq 0$. Hence, $(\chi_{\text{supp}(\vartheta)^m} \circ x_t \circ \chi_{\text{supp}(\vartheta)^n}] \wedge \chi_{\text{supp}(\vartheta)} \neq 0$ for all $m,n \in \{1,2,...,n\}$. Therefore, $\chi_{\text{supp}(\vartheta)}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. By Theorem 4.5, $\text{supp}(\vartheta)$ is an OA-(m,n)-I of $\mathfrak{T}$.

Conversely, suppose that $\text{supp}(\vartheta)$ is an OA-(m,n)-I of $\mathfrak{T}$. By Theorem 4.5, $\chi_{\text{supp}(\vartheta)}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. Then for any fuzzy point $x_t \in \mathfrak{T}$ and $m,n \in \{1,2,...,n\}$, we have $(\chi_{\text{supp}(\vartheta)^m} \circ x_t \circ \chi_{\text{supp}(\vartheta)^n}] \wedge \chi_{\text{supp}(\vartheta)} \neq 0$. Thus, there exists $\mathfrak{z} \in \mathfrak{T}$ such that $((\chi_{\text{supp}(\vartheta)^m} \circ x_t \circ \chi_{\text{supp}(\vartheta)^n}] \wedge \chi_{\text{supp}(\vartheta)})(\mathfrak{z}) \neq 0$. Hence, $((\vartheta^m \circ x_t \circ \vartheta^n] \wedge \vartheta)(\mathfrak{z}) \neq 0$. So $\vartheta(\mathfrak{z}) \neq 0$ and $\mathfrak{z} = \mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_m\mathfrak{x} = \mathfrak{x}\mathfrak{b}_1\mathfrak{b}_2\cdots\mathfrak{b}_n$ for some $\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_m, \mathfrak{b}_1\mathfrak{b}_2\cdots\mathfrak{b}_n \in \mathfrak{T}$ such that $\vartheta(\mathfrak{a}_1) \neq 0$, $\vartheta(\mathfrak{a}_2) \neq 0, \cdots, \vartheta(\mathfrak{a}_m) \neq 0$ $\vartheta(\mathfrak{b}_1) \neq 0$, $\vartheta(\mathfrak{b}_2) \neq 0, \cdots \vartheta(\mathfrak{b}_n) \neq 0$. Thus, $\mathfrak{a}_1\mathfrak{a}_2 \cdot \mathfrak{a}_m, \mathfrak{b}_1\mathfrak{b}_2\cdots\mathfrak{b}_n \in \text{supp}(\vartheta)$. Thus, $((\vartheta^m \circ x_t \circ \vartheta^n] \wedge \vartheta)(\mathfrak{z}) \neq 0$. So $(\vartheta^m \circ x_t \circ \vartheta^n] \wedge \vartheta \neq 0$. Therefore, ϑ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. ■

Next, we investigate the connection between minimal, maximal OA-(m,n)-Is and minimal, maximal fuzzy OA-(m,n)-Is of ordered semigroups.

Definition 3.18. *An OA-(m,n)-I $\mathfrak{B}$ of an ordered semigroup $\mathfrak{T}$ is called*

(1) *a* minimal *if for any OA-(m,n)-I $\mathfrak{R}$ of $\mathfrak{T}$ if whenever $\mathfrak{R} \subseteq \mathfrak{B}$, then $\mathfrak{R} = \mathfrak{B}$,*
(2) *a* maximal *if for any OA-(m,n)-I $\mathfrak{M}$ of $\mathfrak{T}$ if whenever $\mathfrak{B} \subseteq \mathfrak{R}$, then $\mathfrak{R} = \mathfrak{B}$.*

Definition 3.19. *A fuzzy OA-(m,n)-I ϑ of an ordered semigroup $\mathfrak{T}$ is called*

(1) *a* minimal *if for any fuzzy OA-(m,n)-I ξ of $\mathfrak{T}$ if whenever $\xi \leq \vartheta$, then $\text{supp}(\xi) = \text{supp}(\vartheta)$,*
(2) *a* maximal *if for any fuzzy OA-(m,n)-I ξ of $\mathfrak{T}$ if whenever $\vartheta \leq \xi$, then $\text{supp}(\xi) = \text{supp}(\vartheta)$.*

Theorem 3.20. *Let $\mathfrak{B}$ be a nonempty subset of an ordered semigroup $\mathfrak{T}$. Then*

(1) *$\mathfrak{B}$ is a minimal OA-(m,n)-I of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a minimal fuzzy OA-(m,n)-I of $\mathfrak{T}$.*
(2) *$\mathfrak{B}$ is a maximal OA-(m,n)-I of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a maximal fuzzy OA-(m,n)-I of $\mathfrak{T}$.*

Proof:

(1) Assume that $\mathfrak{B}$ is a minimal OA-(m,n)-I of $\mathfrak{T}$. Then $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Thus by Theorem 4.5, $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$.
Let ξ be a fuzzy OA-(m,n)-I of $\mathfrak{T}$ such that $\xi \leq \chi_{\mathfrak{B}}$. Then by Theorem 4.6, $\text{supp}(\xi)$ is an OA-(m,n)-I of $\mathfrak{T}$ such that $\text{supp}(\xi) \subseteq \text{supp}(\chi_{\mathfrak{B}}) = \mathfrak{B}$. Since $\mathfrak{B}$ is minimal we have $\text{supp}(\xi) = \mathfrak{B} = \text{supp}(\chi_{\mathfrak{B}})$. Therefore, $\chi_{\mathfrak{B}}$ is minimal.
Conversely, suppose that $\chi_{\mathfrak{B}}$ is a minimal fuzzy OA-(m,n)-I of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. Thus by Theorem 4.5, $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Let $\mathfrak{R}$ be an OA-(m,n)-I of $\mathfrak{T}$ such that $\mathfrak{R} \subseteq \mathfrak{B}$. Then $\chi_{\mathfrak{R}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$ such that $\chi_{\mathfrak{R}} \leq \chi_{\mathfrak{B}}$. Hence, $\mathfrak{R} = \text{supp}(\chi_{\mathfrak{R}}) = \text{supp}(\chi_{\mathfrak{B}}) = \mathfrak{B}$. Therefore, $\mathfrak{B}$ is minimal.
(2) Assume that $\mathfrak{B}$ is a maximal OA-(m,n)-I of $\mathfrak{T}$. Then $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Thus by Theorem 4.5, $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. Let ξ be a fuzzy OA-(m,n)-I of $\mathfrak{T}$ such that $\chi_{\mathfrak{B}} \leq \xi$. Then by Theorem 4.6, $\text{supp}(\xi)$ is an OA-(m,n)-I of $\mathfrak{T}$ such that $\mathfrak{B} = \text{supp}(\chi_{\mathfrak{B}}) \subseteq \text{supp}(\xi)$. Since $\mathfrak{B}$ is maximal we have $\text{supp}(\xi) = \mathfrak{B} = \text{supp}(\chi_{\mathfrak{B}})$. Therefore, $\chi_{\mathfrak{B}}$ is maximal.
Conversely, suppose that $\chi_{\mathfrak{B}}$ is a maximal fuzzy OA-(m,n)-I of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. By Theorem 4.5, $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Let $\mathfrak{R}$ be an OA-(m,n)-I of $\mathfrak{T}$ such that $\mathfrak{B} \subseteq \mathfrak{R}$. Then $\chi_{\mathfrak{R}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$ such that $\chi_{\mathfrak{B}} \leq \chi_{\mathfrak{R}}$. Since $\chi_{\mathfrak{B}}$ is a maximal we have $\mathfrak{R} = \text{supp}(\chi_{\mathfrak{R}}) = \text{supp}(\chi_{\mathfrak{B}}) = \mathfrak{B}$. Therefore, $\mathfrak{B}$ is maximal.

■

Corollary 3.21. *Let $\mathfrak{T}$ be an ordered semigroup. Then $\mathfrak{T}$ has no proper OA-(m,n)-I if and only if $\text{supp}(\vartheta) = \mathfrak{T}$ for every fuzzy OA-(m,n)-I ϑ of $\mathfrak{T}$.*

Next, we give definition of prime (resp., semiprime, strongly prime) OA-(m,n)-Is and prime (resp., semiprime, strongly prime) fuzzy OA-(m,n)-Is. We study the relationships between prime (resp., semiprime strongly prime) OA-(m,n)-Is and their fuzzification of ordered semigroups.

Definition 3.22. *Let $\mathfrak{B}$ be an OA-(m,n)-I of an ordered semigroup $\mathfrak{T}$. Then we called*

(1) *$\mathfrak{B}$ is a **prime** if for any two OA-(m,n)-Is $\mathfrak{N}$ and $\mathfrak{H}$ of $\mathfrak{T}$ such that $\mathfrak{N}\mathfrak{H} \subseteq \mathfrak{B}$ implies that $\mathfrak{N} \subseteq \mathfrak{B}$ or $\mathfrak{H} \subseteq \mathfrak{B}$.*
(2) *$\mathfrak{B}$ is a **semiprime** if for any OA-(m,n)-I $\mathfrak{N}$ of $\mathfrak{T}$ such that $\mathfrak{N}^2 \subseteq \mathfrak{N}$ implies that $\mathfrak{N} \subseteq \mathfrak{B}$.*
(3) *$\mathfrak{B}$ is a **strongly prime** if for any OA-(m,n)-Is $\mathfrak{N}$ and $\mathfrak{H}$ of $\mathfrak{T}$ such that $\mathfrak{N}\mathfrak{H} \cap \mathfrak{H}\mathfrak{N} \subseteq \mathfrak{B}$ implies that $\mathfrak{N} \subseteq \mathfrak{B}$ or $\mathfrak{H} \subseteq \mathfrak{B}$.*

Definition 3.23. *A fuzzy OA-(m,n)-I ϑ on an ordered semigroup $\mathfrak{T}$. Then we called*

(1) *ϑ is a **prime** if for any two fuzzy OA-(m,n)-Is ξ and ν of $\mathfrak{T}$ such that $\xi \circ \nu \leq \vartheta$ implies that $\xi \leq \vartheta$ or $\nu \leq \vartheta$.*
(2) *ϑ is a **semiprime** if for any fuzzy OA-(m,n)-I ξ of $\mathfrak{T}$ such that $\xi \circ \xi \leq \xi$ implies that $\xi \leq \vartheta$.*
(3) *ϑ is a **strongly prime** if for any two fuzzy OA-(m,n)-Is ξ and ν of $\mathfrak{T}$ such that $(\xi \circ \nu) \wedge (\nu \circ \xi) \leq \vartheta$ implies that $\xi \leq \vartheta$ or $\nu \leq \vartheta$.*

It is clearly, every fuzzy strongly prime OA-(m,n)-I of an ordered semigroup is a fuzzy prime OA-(m,n)-I, and every fuzzy prime OA-(m,n)-I of an ordered semigroup is a fuzzy semiprime OA-(m,n)-I.

Theorem 3.24. *Let $\mathfrak{B}$ be a nonempty subset of an ordered semigroup $\mathfrak{T}$. Then $\mathfrak{B}$ is a prime OA-(m,n)-I of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a prime fuzzy OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Suppose that $\mathfrak{B}$ is a prime OA-(m,n)-I of $\mathfrak{T}$. Then $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Thus by Theorem 4.5, $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. Let ϑ and ξ be fuzzy OA-(m,n)-Is such that $\vartheta \circ \xi \leq \chi_{\mathfrak{B}}$. Assume that $\vartheta \not\leq \chi_{\mathfrak{B}}$ and $\xi \not\leq \chi_{\mathfrak{B}}$. Then there exist $\mathfrak{h}, \mathfrak{r} \in \mathfrak{T}$ such that $\vartheta(\mathfrak{h}) \neq 0$ and $\xi(\mathfrak{r}) \neq 0$. While $\chi_{\mathfrak{B}}(\mathfrak{h}) = 0$ and $\chi_{\mathfrak{B}}(\mathfrak{r}) = 0$. Thus, $\mathfrak{h} \in \text{supp}(\vartheta)$ and $\mathfrak{r} \in \text{supp}(\xi)$, but $\mathfrak{h}, \mathfrak{r} \notin \mathfrak{B}$. So $\text{supp}(\vartheta) \not\subseteq$

$\mathfrak{B}$ and $\text{supp}(\xi) \nsubseteq \mathfrak{B}$. Since $\text{supp}(\vartheta$ and $\text{supp}(\xi)$ are OA-(m,n)-Is of $\mathfrak{T}$ we have $\text{supp}(\vartheta)\,\text{supp}(\xi) \nsubseteq \mathfrak{B}$. Thus there exists $\mathfrak{m} = \mathfrak{p}\mathfrak{q}$ for some $\mathfrak{p} \in \text{supp}(\vartheta)$ and $\mathfrak{q} \in \text{supp}(\xi)$ such that $\mathfrak{m} \in \mathfrak{B}$. Hence $\chi_{\mathfrak{B}}(\mathfrak{m}) = 0$ implies that $(\vartheta \circ \xi)(\mathfrak{m}) = 0$. Since $\vartheta \circ \xi \leq \chi_{\mathfrak{B}}$. we have $\mathfrak{p} \in \text{supp}(\vartheta)$ and $\mathfrak{q} \in \text{supp}(\xi)$. Thus $\vartheta(\mathfrak{p}) \neq 0$, and $\xi(\mathfrak{q}) \neq 0$. It implies that

$$(\vartheta \circ \xi)(\mathfrak{m}) = \bigvee_{(\mathfrak{p},\mathfrak{q}) \in F_{\mathfrak{m}}} \{\vartheta(\mathfrak{p}) \wedge \xi(\mathfrak{q})\} \neq 0$$

It is a contradiction so $\vartheta \leq \chi_{\mathfrak{B}}$ or $\xi \leq \chi_{\mathfrak{B}}$. Therefore, $\chi_{\mathfrak{B}}$ is a prime fuzzy OA-(m,n)-I of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a prime fuzzy OA-(m,n)-I of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. Thus by Theorem 4.5, $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Let $\mathfrak{N}$ and $\mathfrak{H}$ be OA-(m,n)-I of $\mathfrak{T}$ such that $\mathfrak{N}\mathfrak{H} \subseteq \mathfrak{B}$. Then $\chi_{\mathfrak{B}}$ and $\chi_{\mathfrak{H}}$ are fuzzy OA-(m,n)-Is of $\mathfrak{T}$. By Lemma 2.6 $\chi_{\mathfrak{N}} \circ \chi_{\mathfrak{H}} = \chi_{\mathfrak{N}\mathfrak{H}} \leq \chi_{\mathfrak{B}}$. By assumption, $\chi_{\mathfrak{N}} \leq \chi_{\mathfrak{B}}$ or $\chi_{\mathfrak{H}} \leq \chi_{\mathfrak{B}}$. Thus $\mathfrak{N} \subseteq \mathfrak{B}$ or $\mathfrak{H} \subseteq \mathfrak{B}$. We conclude that $\mathfrak{B}$ is a prime ordered almost (m,n)-ideal of $\mathfrak{T}$. ■

Theorem 3.25. *Let $\mathfrak{B}$ be a nonempty subset of an ordered semigroup $\mathfrak{T}$. Then $\mathfrak{B}$ is a semiprime OA-(m,n)-I of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a semiprime fuzzy OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Suppose that $\mathfrak{B}$ is a semiprime OA-(m,n)-I of $\mathfrak{T}$. Then $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Thus by Theorem 4.5, $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. Let ϑ be a fuzzy OA-(m,n)-I such that $\vartheta \circ \vartheta \leq \chi_{\mathfrak{B}}$. Assume that $\vartheta \nleq \chi_{\mathfrak{B}}$. Then there exist $\mathfrak{h} \in \mathfrak{T}$ such that $\vartheta(\mathfrak{h}) \neq 0$. While $\chi_{\mathfrak{B}}(\mathfrak{h}) = 0$. Thus, $\mathfrak{h} \in \text{supp}(\vartheta)$, but $\mathfrak{h} \notin \mathfrak{B}$. So $\text{supp}(\vartheta) \nsubseteq \mathfrak{B}$. Since $\text{supp}(\vartheta)$ is an OA-(m,n)-I of $\mathfrak{T}$ we have $\text{supp}(\vartheta) \nsubseteq \mathfrak{B}$. Thus, $\mathfrak{h} \in \text{supp}(\vartheta)$ such that $\mathfrak{m} \in \mathfrak{B}$. Hence, $\chi_{\mathfrak{B}}(\mathfrak{m}) = 0$ implies that $(\vartheta \circ \vartheta)(\mathfrak{m}) = 0$. Since $\vartheta \circ \vartheta \leq \chi_{\mathfrak{B}}$. we have $\mathfrak{p} \in \text{supp}(\vartheta)$. Thus, $\vartheta(\mathfrak{p}) \neq 0$. It implies that

$$(\vartheta \circ \vartheta)(\mathfrak{m}) = \bigvee_{(\mathfrak{p},\mathfrak{b}) \in F_{\mathfrak{m}}} \{\vartheta(\mathfrak{p}) \wedge \vartheta(\mathfrak{b})\} \neq 0$$

It is a contradiction so $\vartheta \leq \chi_{\mathfrak{B}}$. Therefore, $\chi_{\mathfrak{B}}$ is a semiprime fuzzy OA-(m,n)-I of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a semiprime fuzzy OA-(m,n)-I of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. Thus by Theorem 4.5, $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Let $\mathfrak{N}$ be an OA-(m,n)-I of $\mathfrak{T}$ such that $\mathfrak{N}^2 \subseteq \mathfrak{B}$. Then $\chi_{\mathfrak{N}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. By Lemma 2.6 $\chi_{\mathfrak{N}} \circ \chi_{\mathfrak{N}} = \chi_{\mathfrak{N}^2} \leq \chi_{\mathfrak{B}}$. By assumption, $\chi_{\mathfrak{N}} \leq \chi_{\mathfrak{B}}$. Thus $\mathfrak{N} \subseteq \mathfrak{B}$. We conclude that $\mathfrak{B}$ is a semiprime ordered almost (m,n)-ideal of $\mathfrak{T}$. ■

Theorem 3.26. *Let $\mathfrak{B}$ be a nonempty subset of an ordered semigroup $\mathfrak{T}$. Then $\mathfrak{B}$ is a strongly prime OA-(m,n)-I of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{B}}$ is a fuzzy strongly prime OA-(m,n)-I of $\mathfrak{T}$.*

Proof: Suppose that $\mathfrak{B}$ is a strongly prime OA-(m,n)-I of $\mathfrak{T}$. Then $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Thus by Theorem 4.5, $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. Let ϑ and ξ be fuzzy OA-(m,n)-Is of $\mathfrak{T}$ such that $(\vartheta \circ \xi) \wedge (\xi \circ \vartheta) \leq \chi_{\mathfrak{B}}$. Assume that $\vartheta \nleq \chi_{\mathfrak{B}}$ and $\xi \nleq \chi_{\mathfrak{B}}$. Then there exist $\mathfrak{h}, \mathfrak{b} \in \mathfrak{T}$ such that $\vartheta(\mathfrak{h}) \neq 0$ and $\xi(\mathfrak{b}) \neq 0$. While $\chi_{\mathfrak{B}}(\mathfrak{h}) = 0$ and $\chi_{\mathfrak{B}}(\mathfrak{b}) = 0$. Thus, $\mathfrak{h} \in \text{supp}(\vartheta)$ and $\mathfrak{b} \in \text{supp}(\xi)$, but $\mathfrak{h}, \mathfrak{b} \notin \mathfrak{B}$. So, $\text{supp}(\vartheta) \nsubseteq \mathfrak{B}$ and $\text{supp}(\xi) \nsubseteq \mathfrak{B}$. Hence, there exists $m \in [\text{supp}(\vartheta)\,\text{supp}(\xi)] \cap (\text{supp}(\vartheta)\,\text{supp}(\xi))$ such that $\mathfrak{m} \notin \mathfrak{B}$. Thus, $\chi_{\mathfrak{B}}(\mathfrak{m}) = 0$. Since $m \in \text{supp}(\vartheta)\,\text{supp}(\xi)$ and $m \in \text{supp}(\xi)\,\text{supp}(\vartheta)$ we have $\mathfrak{m} = \mathfrak{d}\mathfrak{k}$ and $\mathfrak{m} = \mathfrak{g}\mathfrak{q}$ for some $\mathfrak{d}, \mathfrak{q} \in \text{supp}(\vartheta)$, and for some $\mathfrak{k}, \mathfrak{g} \in \text{supp}(\xi)$. we have

$$(\vartheta \circ \xi)(\mathfrak{m}) = \bigvee_{(\mathfrak{d},\mathfrak{k}) \in F_{\mathfrak{m}}} \{\vartheta(\mathfrak{d}) \wedge \xi^p(\mathfrak{k})\} \neq 0.$$

Similarly

$$(\xi \circ \vartheta)(\mathfrak{m}) = \bigvee_{(\mathfrak{g},\mathfrak{q}) \in F_{\mathfrak{m}}} \{\xi(\mathfrak{g}) \wedge \vartheta(\mathfrak{q})\}.$$

So $(\vartheta \circ \xi)(\mathfrak{m}) \wedge (\xi \circ \vartheta)(\mathfrak{m}) \neq 0$. It is a contradiction so, $\vartheta \leq \chi_{\mathfrak{B}}$ or $\xi \leq \chi_{\mathfrak{B}}$. Therefore, $\chi_{\mathfrak{B}}$ is a fuzzy strongly prime OA-(m,n)-I of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{B}}$ is a fuzzy strongly prime OA-(m,n)-I of $\mathfrak{T}$. Then $\chi_{\mathfrak{B}}$ is a fuzzy OA-(m,n)-I of $\mathfrak{T}$. Thus, by Theorem 4.5, $\mathfrak{B}$ is an OA-(m,n)-I of $\mathfrak{T}$. Let $\mathfrak{N}$ and $\mathfrak{H}$ be OA-(m,n)-Is of $\mathfrak{T}$ such that $\mathfrak{N}\mathfrak{H} \cap \mathfrak{H}\mathfrak{N} \subseteq \mathfrak{B}$. Then $\chi_{\mathfrak{N}}$ and $\chi_{\mathfrak{H}}$ are fuzzy OA-(m,n)-Is of $\mathfrak{T}$. By Lemma 2.6 $\chi_{\mathfrak{N}\mathfrak{H}} = \chi_{\mathfrak{N}} \circ \chi_{\mathfrak{H}}$ and $\chi_{\mathfrak{H}\mathfrak{N}} = \chi_{\mathfrak{H}} \circ \chi_{\mathfrak{N}}$. Thus $(\chi_{\mathfrak{N}} \circ \chi_{\mathfrak{H}}) \wedge (\chi_{\mathfrak{H}} \circ \chi_{\mathfrak{N}}) = \chi_{\mathfrak{N}\mathfrak{H}} \wedge \chi_{\mathfrak{H}\mathfrak{N}} = \chi_{\mathfrak{N}\mathfrak{H} \cap \mathfrak{H}\mathfrak{N}} \leq \chi_{\mathfrak{B}}$. By assumption, $\chi_{\mathfrak{N}} \leq \chi_{\mathfrak{B}}$ and $\chi_{\mathfrak{N}} \leq \chi_{\mathfrak{B}}$. Thus $\mathfrak{N} \subseteq \mathfrak{B}$ or $\mathfrak{H} \subseteq \mathfrak{B}$. We conclude that $\mathfrak{B}$ is a strongly prime OA-(m,n)-I of $\mathfrak{T}$. ■

IV. Bipolar fuzzy almost Prime interior ideals in Semigroups

Before, we will define the bipolar fuzzy almost interior ideals and bipolar fuzzy almost weakly interior ideals in semigroups.

An *almost interior ideal* $\mathfrak{K}$ of a semigroup $\mathfrak{T}$ if $t\mathfrak{K}r \cap \mathfrak{K} \neq \emptyset$ for all $t, r \in \mathfrak{T}$. A *weakly almost ideal* $\mathfrak{K}$ of a semigroup $\mathfrak{T}$ if $t\mathfrak{K}t \cap \mathfrak{K} \neq \emptyset$ and for all $t \in \mathfrak{T}$ [5].

Definition 4.1. *[21] A* ***bipolar fuzzy set*** *(BF set) ϑ on a non-empty set $\mathfrak{T}$ is an object having the form*

$$\vartheta := \{(h, \vartheta^p(h), \vartheta^n(h)) \mid h \in \mathfrak{T}\},$$

where $\vartheta^p : \mathfrak{T} \to [0,1]$ and $\vartheta^n : \mathfrak{T} \to [-1,0]$.

Remark 4.2. *For the sake of simplicity we shall use the symbol $\vartheta = (\mathfrak{T}; \vartheta^p, \vartheta^n)$ for the BF set $\vartheta = \{(h, \vartheta^p(h), \vartheta^n(h)) \mid h \in E\}$.*

Definition 4.3. *[22] A BF set $\vartheta = (\mathfrak{T}; \vartheta^p, \vartheta^n)$ on a semigroup $\mathfrak{T}$ is called a* BF almost interior ideal *of $\mathfrak{T}$ if $(x_t^p \circ \vartheta^p \circ y_{t'}^p) \wedge \vartheta^p \neq 0$ and $(x_s^n \circ \vartheta^n \circ y_{s'}^n) \vee \vartheta^n \neq 0$ for any BF point $x_t^p, y_{t'}^p, x_s^n, y_{s'}^n \in \mathfrak{T}$.*

Definition 4.4. *[22] A BF set $\vartheta = (\mathfrak{T}; \vartheta^p, \vartheta^n)$ on a semigroup $\mathfrak{T}$ is called a* BF weakly almost interior ideal *of $\mathfrak{T}$ if $(x_t^p \circ \vartheta^p \circ x_{t'}^p) \wedge \vartheta^p \neq 0$ and $(x_s^n \circ \vartheta^n \circ x_{s'}^n) \vee \vartheta^n \neq 0$ for any BF point $x_t^p, x_{t'}^p, x_s^n, x_{s'}^n \in \mathfrak{T}$.*

It is clarly every BF almost interior ideal of a semigroup $\mathfrak{T}$ is a BF weakly almost interior ideal of $\mathfrak{T}$.

Theorem 4.5. *Let $\mathfrak{K}$ be a nonempty subset of a semigroup $\mathfrak{T}$. Then $\mathfrak{K}$ is an almost interior ideal (weakly almost interior ideal) of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi_{\mathfrak{K}}^p, \chi_{\mathfrak{K}}^n)$ is a BF almost interior ideal (weakly almost interior ideal) of $\mathfrak{T}$.*

Proof: Suppose that $\mathfrak{K}$ is an almost interior ideal of a semigroup $\mathfrak{T}$. Then $x\mathfrak{K}y \cap \mathfrak{K} \neq \emptyset$ for all $x, y \in \mathfrak{T}$. Thus there exists $c \in \mathfrak{T}$ such that $c \in x\mathfrak{K}y$ and $c \in \mathfrak{K}$. Let $x, y \in \mathfrak{T}$ and $t, t' \in (0,1]$ and $s, s' \in [-1,0)$. Then $(x_t^p \circ \chi_{\mathfrak{K}}^p \circ y_{t'}^p)(c) \neq 0$, $(x_s^n \circ \chi_{\mathfrak{K}}^n \circ y_{s'}^n)(c) \neq 0$ and $\chi_{\mathfrak{K}}^p(c) = 1$ and $\chi_{\mathfrak{K}}^n(c) = -1$.

Thus, $(x_t^p \circ \chi_{\mathfrak{K}}^p \circ y_{t'}^p) \neq 0$, $(x_s^n \circ \chi_{\mathfrak{K}}^n \circ y_{s'}^n) \neq 0$.
Hence, $\chi_{\mathfrak{K}} = (E; \chi_{\mathfrak{K}}^p, \chi_{\mathfrak{K}}^n)$ is a BF almost interior ideal of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi_{\mathfrak{K}}^p, \chi_{\mathfrak{K}}^n)$ is a BF almost interior ideal of $\mathfrak{T}$ and let $x, y \in \mathfrak{T}$ and $t, t' \in (0,1]$ and $s, s' \in [-1, 0)$. Then $(x_t^p \circ \chi_{\mathfrak{K}}^p \circ y_{t'}^p) \neq 0$ and $(x_s^n \circ \chi_{\mathfrak{K}}^n \circ y_{s'}^n) \neq 0$. Thus there exists $c \in \mathfrak{T}$ such that $(x_t^p \circ \chi_{\mathfrak{K}}^p \circ y_{t'}^p)(c) \neq 0$ and $(x_s^n \circ \chi_{\mathfrak{K}}^n \circ y_{s'}^n)(c) \neq 0$. Hence $c \in x\mathfrak{K}y \cap \mathfrak{K}$. So $x\mathfrak{K}y \cap \mathfrak{K} \neq \emptyset$. We conclude that $\mathfrak{K}$ is an almost interior ideal of $\mathfrak{T}$. ■

Theorem 4.6. *Let $\vartheta = (\mathfrak{T}; \vartheta^p, \vartheta^n)$ be a fuzzy subset of a semigroup $\mathfrak{T}$. Then $\vartheta = (\mathfrak{T}; \vartheta^p, \vartheta^n)$ is a BF almost interior ideal (weakly almost interior ideal) of $\mathfrak{T}$ if and only if* $\text{supp}(\vartheta)$ *is an almost interior ideal (weakly almost interior ideal) of $\mathfrak{T}$.*

Proof: Assume that $\vartheta = (\mathfrak{T}; \vartheta^p, \vartheta^n)$ is a BF almost interior ideal of a semigroup $\mathfrak{T}$ and let $x, y \in \mathfrak{T}$ and $t, t' \in (0,1]$ and $s, s' \in [-1, 0)$. Then $(x_t^p \circ \vartheta^p \circ y_{t'}^p) \neq 0$, $(x_s^n \circ \vartheta^n \circ y_{s'}^n) \neq 0$. Thus there exists $z \in \mathfrak{T}$ such that $(x_t^p \circ \vartheta^p \circ y_{t'}^p)(z) \neq 0$, $(x_s^n \circ \vartheta^n \circ y_{s'}^n)(z) \neq 0$. So $\vartheta^p(z) \neq 0$, $\vartheta^n(z) \neq 0$ and $\vartheta^n(z) \neq 0$, $\vartheta^n(z) \neq 0$ there exists $w \in \mathfrak{T}$ such that $z = xwy$ and $\vartheta^p(w) \neq 0$, and $\vartheta^n(w) \neq 0$. Thus $(x_t^p \circ \chi_{\text{supp}(\vartheta)}^p \circ y_{t'}^p)(z) \neq 0$ and $(x_s^n \circ \chi_{\text{supp}(\vartheta)}^n \circ y_{s'}^n)(z) \neq 0$. Hence, $(x_t^p \circ \chi_{\text{supp}(\vartheta)}^p \circ y_{t'}^p) \neq 0$ and $(x_s^n \circ \chi_{\text{supp}(\vartheta)}^n \circ y_{s'}^n) \neq 0$. Therefore, $\chi_{supp(\vartheta)}^p$ is a BF almost interior ideal of $\mathfrak{T}$. By Theorem 4.5, $\text{supp}(\vartheta)$ is an almost interior ideal of $\mathfrak{T}$.

Conversely, suppose that $\text{supp}(\vartheta)$ is an almost interior ideal of $\mathfrak{T}$. By Theorem 4.5, $\chi_{\text{supp}(\vartheta)}^p$ is a BF almost interior ideal of $\mathfrak{T}$. Then for any BF points $x_t^p, y_{t'}^p, x_s^n, y_{s'}^n \in \mathfrak{T}$, we have $(x_t^p \circ \chi_{\text{supp}(\vartheta)}^p \circ y_{t'}^p) \wedge \chi_{\text{supp}(\vartheta)}^p \neq 0$ and $(x_s^n \circ \chi_{\text{supp}(\vartheta)}^n \circ y_{s'}^n) \wedge \chi_{\text{supp}(\vartheta)}^n \neq 0$. Thus there exists $c \in \mathfrak{T}$ such that $[(x_t^p \circ \chi_{\text{supp}(\vartheta)}^p \circ y_{t'}^p) \wedge \chi_{\text{supp}(\vartheta)}^p](c) \neq 0$ and $[(x_s^p \circ \chi_{\text{supp}(\vartheta)}^n \circ y_{s'}^n) \vee \chi_{\text{supp}(\vartheta)}^n](c) \neq 0$. Hence, $(x_t^p \circ \chi_{\text{supp}(\vartheta)}^p \circ y_{t'}^p)(c) = 0$, $\chi_{\text{supp}(\vartheta)}^p(c) \neq 0$ and $(x_s^n \circ \chi_{\text{supp}(\vartheta)}^n \circ y_{s'}^n)(c) = 0$, $\chi_{\text{supp}(\vartheta)}^n(c) \neq 0$. Then there exists $b \in \text{supp}(\vartheta)$ such that $c = xby$. Thus $\vartheta^p(c) \neq 0, \vartheta^p(b) \neq 0$ and $\vartheta^n(c) \neq 0, \vartheta^n(b) \neq 0$. So $(x_t^p \circ \vartheta^p \circ y_{t'}^p) \neq 0$, $(x_s^n \circ \vartheta^n \circ y_{s'}^n) \neq 0$. Therefore, ϑ is a BF almost interior ideal of $\mathfrak{T}$. ■

Next, we give definition of prime (resp., semiprime, strongly prime) almost interior ideals (weakly almost interior ideals) and prime (resp., semiprime strongly prime) BF almost interior ideals (weakly almost interior ideals). We study the relationships between prime (resp., semiprime strongly prime) almost interior ideals (weakly almost interior ideals) and their bipolar fuzzification of semigroups.

Definition 4.7. *Let $\mathfrak{K}$ be an almost ideal of semigroup $\mathfrak{T}$. Then we called*

1) *$\mathfrak{K}$ is a **prime** if for any almost interior ideals $\mathfrak{M}$ and $\mathfrak{L}$ of $\mathfrak{T}$ such that $\mathfrak{M}\mathfrak{L} \subseteq \mathfrak{K}$ implies that $\mathfrak{M} \subseteq \mathfrak{K}$ or $\mathfrak{L} \subseteq \mathfrak{K}$.*
2) *$\mathfrak{K}$ is a **semiprime** if for any almost interior ideal $\mathfrak{M}$ of $\mathfrak{T}$ such that $\mathfrak{M}^2 \subseteq \mathfrak{K}$ implies that $\mathfrak{M} \subseteq \mathfrak{K}$.*
3) *$\mathfrak{K}$ is a **strongly prime** if for any almost interior ideals $\mathfrak{M}$ and $\mathfrak{L}$ of $\mathfrak{T}$ such that $\mathfrak{M}\mathfrak{L} \cap \mathfrak{L}\mathfrak{M} \subseteq \mathfrak{K}$ implies that $\mathfrak{M} \subseteq \mathfrak{K}$ or $\mathfrak{L} \subseteq \mathfrak{K}$.*

Definition 4.8. *A BF almost interior ideal $\vartheta = (\mathfrak{T}; \vartheta^p, \vartheta^n)$ on a semigroup $\mathfrak{T}$. Then we called*

1) *ϑ is a **prime** if for any two BF almost interior ideals $\xi = (\mathfrak{T}; \xi^p, \xi^n)$ and $\nu = (\mathfrak{E}; \nu^p, \nu^n)$ of $\mathfrak{T}$ such taht $\xi^p \circ \nu^p \leq \vartheta^p$ and $\xi^n \circ \nu^n \geq \vartheta^n$ implies that $\xi^p \leq \vartheta^p$ and $\xi^n \geq \vartheta^n$ or $\nu^p \leq \vartheta^p$ and $\nu^n \geq \vartheta^n$.*
2) *ϑ is a **semiprime** if for any BF almost interior ideal $\xi = (\mathfrak{T}; \xi^p, \xi^{n)}$ of $\mathfrak{T}$ such taht $\xi^p \circ \xi^p \leq \vartheta^p$ and $\xi^n \circ \xi^n \leq \vartheta^n$ implies that $\xi^p \leq \vartheta^p$ or $\xi^n \geq \vartheta^n$.*
3) *ϑ is a **strongly prime** if for any two BF almost interior ideals $\xi = (\mathfrak{T}; \xi^p, \xi^{n)}$ and $\nu = (\mathfrak{T}; \nu^p, \nu^{n)}$ of $\mathfrak{T}$ such taht $(\xi^p \circ \nu^p) \wedge (\nu^p \circ \xi^p) \leq \vartheta^p$ and $(\xi^n \circ \nu^n) \vee (\nu^n \circ \xi^n) \leq \vartheta^n$ implies that $\xi^p \subseteq \vartheta^p$ and $\xi^n \geq \vartheta^n$ or $\nu^p \leq \vartheta^p$ and $\nu^n \geq \vartheta^n$.*

It is clarly, every BF strongly prime almost interior ideal of a semigroup is a BF prime almost interior ideal, and every BF prime almost interior ideal of a semigroup is a BF semiprime almost interior ideal.

Theorem 4.9. *Let $\mathfrak{K}$ be a nonempty subset of a semigroup $\mathfrak{T}$. Then $\mathfrak{K}$ is a prime (resp., semiprime) almost interior ideal of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi_{\mathfrak{K}}^p, \chi_{\mathfrak{K}}^n)$ is a prime (resp., semiprime) BF almost interior ideal of $\mathfrak{T}$.*

Proof: Suppose that $\mathfrak{K}$ is a prime almost interior ideal of a semigroup $\mathfrak{T}$. Then $\mathfrak{K}$ is an almost interior ideal of $\mathfrak{T}$. Thus by Theorem 4.5, $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi_{\mathfrak{K}}^p, \chi_{\mathfrak{K}}^n)$ is a BF almost interior ideal of $\mathfrak{T}$. Let $\vartheta = (\mathfrak{T}; \vartheta_{\mathfrak{K}}^p, \vartheta_{\mathfrak{K}}^n)$ and $\xi = (\mathfrak{T}; \xi_{\mathfrak{K}}^p, \xi_{\mathfrak{K}}^n)$ be BF almost interior ideals such that $\vartheta^p \circ \xi^p \leq \chi_{\mathfrak{K}}^p$ and $\vartheta^n \circ \xi^n \geq \chi_{\mathfrak{K}}^n$. Assume that $\vartheta^p \nleq \chi_{\mathfrak{K}}^p$ and $\vartheta^n \ngeq \chi_{\mathfrak{K}}^n$ or $\xi^p \nleq \chi_{\mathfrak{K}}^p$ and $\xi^n \ngeq \chi_{\mathfrak{K}}^n$. Then there exist $h, r \in \mathfrak{T}$ such that $\vartheta^p(h) \neq 0$, $\vartheta^n(h) \neq 0$ and $\xi^p(r) \neq 0$, $\xi^n(h) \neq 0$. While $\chi_{\mathfrak{K}}^p(h) = 0$, $\chi_{\mathfrak{K}}^n(h) = 0$ and $\chi_{\mathfrak{K}}^p(r) = 0$, $\chi_{\mathfrak{K}}^n(r) = 0$. Thus $h \in \text{supp}(\vartheta)$ and $r \in \text{supp}(\xi)$, but $h, r \notin \mathfrak{K}$. So $\text{supp}\,\vartheta) \nsubseteq \mathfrak{K}$ and $\text{supp}(\xi) \nsubseteq \mathfrak{K}$. Since $\text{supp}(\vartheta)$ and $\text{supp}(\xi)$ are almost interior ideals of $\mathfrak{T}$ we have $\text{supp}(\vartheta)\,\text{supp}(\xi) \nsubseteq \mathfrak{K}$. Thus there exists $m = de$ for some $d \in \text{supp}(\vartheta)$ and $e \in \text{supp}(\xi)$ such that $m \in \mathfrak{K}$. Hence $\chi_{\mathfrak{K}}^p(m) = 0$ and $\chi_{\mathfrak{K}}^n(m) = 0$ implies that $(\vartheta^p \circ \xi^p)(m) = 0$ and $(\vartheta^n \circ \xi^n)(m) = 0$. Since $\vartheta^p \circ \xi^p \leq \chi_{\mathfrak{K}}^p$ and $\vartheta^n \circ \xi^n \leq \chi_{\mathfrak{K}}^n$. we have $d \in \text{supp}(\vartheta)$ and $e \in \text{supp}(\xi)$. Thus $\vartheta^p(d) \neq 0$, $\vartheta^n(d) \neq 0$ and $\xi^p(e) \neq 0$, $\xi^n(e) \neq 0$. It implies that

$$(\vartheta^p \circ \xi^p)(m) = \bigvee_{(de) \in F_m} \{\vartheta^p(d) \wedge \xi^p(e)\} \neq 0$$

and

$$(\vartheta^n \circ \xi^n)(m) = \bigwedge_{(de) \in F_m} \{\vartheta^n(d) \vee \xi^n(e)\} \neq 0.$$

It is a contradiction so $\vartheta^p \leq \chi_{\mathfrak{K}}^p$ and $\vartheta^n \geq \chi_{\mathfrak{K}}^n$ or $\xi^p \leq \chi_{\mathfrak{K}}^p$ and $\xi^n \geq \chi_{\mathfrak{K}}^n$. Therefore, $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi_{\mathfrak{K}}^p, \chi_{\mathfrak{K}}^n)$ is a prime BF almost interior ideal of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi_{\mathfrak{K}}^p, \chi_{\mathfrak{K}}^n)$ is a prime BF almost interior ideal of $\mathfrak{T}$. Then $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi_{\mathfrak{K}}^p, \chi_{\mathfrak{K}}^n)$ is a BF almost interior ideal of $\mathfrak{T}$. Thus by Theorem 4.5, $\mathfrak{K}$ is an almost interior ideal of $\mathfrak{T}$. Let $\mathfrak{M}$ and $\mathfrak{L}$ be almost interior ideals of $\mathfrak{T}$ such that $\mathfrak{M}\mathfrak{L} \subseteq \mathfrak{K}$. Then $\chi_{\mathfrak{M}} = (\mathfrak{T}; \chi_{\mathfrak{M}}^p, \chi_M^n)$ and $\chi_{\mathfrak{L}} = (\mathfrak{T}; \chi_{\mathfrak{L}}^p, \chi_{\mathfrak{L}}^n)$ are BF almost interior ideals of $\mathfrak{T}$. By Lemma 2.6 $\chi_{\mathfrak{M}}^p \circ \chi_{\mathfrak{L}}^p = \chi_{\mathfrak{M}\mathfrak{L}}^p \leq \chi_{\mathfrak{K}}^p$ and $\chi_{\mathfrak{M}}^n \circ \chi_{\mathfrak{L}}^n = \chi_{\mathfrak{M}\mathfrak{L}}^n \geq \chi_{\mathfrak{K}}^n$. By assumption, $\chi_{\mathfrak{M}}^p \leq \chi_{\mathfrak{K}}^p$ and $\chi_{\mathfrak{M}}^n \geq \chi_{\mathfrak{K}}^n$ or $\chi_{\mathfrak{L}}^p \leq \chi_{\mathfrak{K}}^p$ and $\chi_{\mathfrak{L}}^n \geq \chi_{\mathfrak{K}}^n$. Thus, $\mathfrak{M} \subseteq \mathfrak{K}$ or $\mathfrak{L} \subseteq \mathfrak{K}$. We conclude that $\mathfrak{K}$ is a prime almost interior ideal of $\mathfrak{T}$. ■

Theorem 4.10. *Let $\mathfrak{K}$ be a nonempty subset of a semigroup $\mathfrak{T}$. Then $\mathfrak{K}$ is a strongly prime almost interior ideal of $\mathfrak{T}$ if and only if $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi_{\mathfrak{K}}^p, \chi_{\mathfrak{K}}^n)$ is a BF strongly prime almost interior ideal of $\mathfrak{T}$.*

Proof: Suppose that $\mathfrak{K}$ is a strongly prime almost interior ideal of a semigroup $\mathfrak{T}$. Then $\mathfrak{K}$ is an almost interior ideal of $\mathfrak{T}$. Thus by Theorem 4.5, $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi^p_{\mathfrak{K}}, \chi^n_{\mathfrak{K}})$ is a BF almost interior ideal of $\mathfrak{T}$. Let $\vartheta = (\mathfrak{T}; \vartheta^p_{\mathfrak{K}}, \vartheta^n_{\mathfrak{K}})$ and $\xi = (\mathfrak{T}; \xi^p_{\mathfrak{K}}, \xi^n_{\mathfrak{K}})$ be BF almost interior ideals of $\mathfrak{T}$ such that $(\vartheta^p \circ \xi^p) \wedge (\xi^p \circ \vartheta^p) \leq \chi^p_{\mathfrak{K}}$ and $(\vartheta^n \circ \xi^n) \vee (\xi^n \circ \vartheta^n) \geq \chi^n_{\mathfrak{K}}$. Assume that $\vartheta^p \nleq \chi^p_{\mathfrak{K}}$ and $\vartheta^n \ngeq \chi^n_{\mathfrak{K}}$ or $\xi^p \nleq \chi^p_{\mathfrak{K}}$ and $\xi^n \ngeq \chi^n_{\mathfrak{K}}$. Then there exist $h, r \in E$ such that $\vartheta^p(h) \neq 0$, $\vartheta^n(h) \neq 0$ and $\xi^p(r) \neq 0$, $\vartheta^n(h) \neq 0$. While $\chi^p_{\mathfrak{K}}(h) = 0$, $\chi^n_{\mathfrak{K}}(h) = 0$ and $\chi^p_{\mathfrak{K}}(r) = 0$, $\chi^n_{\mathfrak{K}}(r) = 0$. Thus $h \in \operatorname{supp}(\vartheta)$ and $r \in \operatorname{supp}(xi)$, but $h, r \notin \mathfrak{K}$. So $\operatorname{supp}(\vartheta) \nsubseteq \mathfrak{K}$ and $\operatorname{supp}(\xi) \nsubseteq \mathfrak{K}$. Hence, there exists $m \in [\operatorname{supp}(\vartheta) \operatorname{supp}(\xi)] \cap [\operatorname{supp}(\vartheta) \operatorname{supp}(\xi)]$ such that $m \notin \mathfrak{K}$. Thus, $\chi^p_{\mathfrak{K}}(m) = 0$, $\chi^n_{\mathfrak{K}}(m) = 0$. Since $m \in \operatorname{supp}(\vartheta) \operatorname{supp}(\xi)$ and $m \in \operatorname{supp}(\xi) \operatorname{supp}(\vartheta)$ we have $m = d_1 e_1$ and $m = e_2 d_2$ for some $d_1, d_2 \in \operatorname{supp}(\vartheta)$ and for some $e_1, e_2 \in \operatorname{supp}(\xi)$. we have

$$(\vartheta^p \circ \xi^p)(m) = \bigvee_{(d_1 e_1) \in F_m} \{\vartheta^p(d_1) \wedge \xi^p(e_1)\} \neq 0$$

and

$$(\vartheta^n \circ \xi^n)(m) = \bigwedge_{(d_1 e_1) \in F_m} \{\vartheta^n(d_1) \vee \xi^n(e_1)\} \neq 0.$$

Similarly

$$(\xi^p \circ \vartheta^p)(m) = \bigvee_{(e_2 d_2) \in F_m} \{\xi^p(e_2) \wedge \vartheta^p(d_2)\} \neq 0$$

and

$$(\vartheta^n \circ \xi^n)(m) = \bigwedge_{(e_2 d_2) in F_m} \{\xi^n(e_2) \vee \vartheta^n(d_2)\} \neq 0.$$

So $(\vartheta^p \circ \xi^p)(m) \wedge (\vartheta^p \circ \xi^p)(m) \neq 0$ and $(\vartheta^n \circ \xi^n)(m) \vee (\vartheta^n \circ \xi^n)(m) \neq 0$. It is a contradiction so $(\vartheta^p \circ \xi^p)(m) \wedge (\xi^p \circ \vartheta^p)(m) = 0$ and $(\vartheta^n \circ \xi^n)(m) \vee (\xi^n \circ \vartheta^n)(m) = 0$. Hence, $\vartheta^p \leq \chi^p_{\mathfrak{K}}$ and $\vartheta^n \geq \chi^n_{\mathfrak{K}}$ or $\xi^p \leq \chi^p_{\mathfrak{K}}$ and $\xi^n \geq \chi^n_{\mathfrak{K}}$. Therefore $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi^p_{\mathfrak{K}}, \chi^n_{\mathfrak{K}})$ is a BF strongly prime almost interior ideal of $\mathfrak{T}$.

Conversely, suppose that $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi^p_{\mathfrak{K}}, \chi^n_{\mathfrak{K}})$ is a BF strongly prime almost interior ideal of $\mathfrak{T}$. Then $\chi_{\mathfrak{K}} = (\mathfrak{T}; \chi^p_{\mathfrak{K}}, \chi^n_{\mathfrak{K}})$ is a BF almost interior ideal of $\mathfrak{T}$. Thus by Theorem 4.5, $\mathfrak{K}$ is an almost interior ideal of $\mathfrak{T}$. Let $\mathfrak{M}$ and $\mathfrak{L}$ be almost interior ideals of $\mathfrak{T}$ such that $\mathfrak{M}\mathfrak{L} \cap \mathfrak{L}\mathfrak{M} \leq$. Then $\chi_{\mathfrak{M}} = (\mathfrak{T}; \chi^p_{\mathfrak{M}}, \chi^n_{\mathfrak{M}})$ and $\chi_{\mathfrak{L}} = (\mathfrak{T}; \chi^p_{\mathfrak{L}}, \chi^n_{\mathfrak{L}})$ are BF almost interior ideals of $\mathfrak{T}$. By Lemma 2.6 $\chi^p_{\mathfrak{M}\mathfrak{L}} = \chi^p_{\mathfrak{M}} \circ \chi^p_{\mathfrak{L}}, \chi^p_{LM} = \chi^p_{\mathfrak{L}} \circ \chi^p_{\mathfrak{M}}$ and $\chi^n_{\mathfrak{M}\mathfrak{L}} = \chi^n_{\mathfrak{M}} \circ \chi^n_{\mathfrak{L}}, \chi^n_{LM} = \chi^n_{\mathfrak{L}} \circ \chi^n_{\mathfrak{M}}$. Thus $(\chi^p_{\mathfrak{M}} \circ \chi^p_{\mathfrak{L}}) \wedge (\chi^p_{\mathfrak{L}} \circ \chi^p_{\mathfrak{M}}) = \chi^p_{\mathfrak{M}\mathfrak{L}} \wedge \chi^p_{LM} = \chi^p_{ML \cap LM} \leq \chi^p_{\mathfrak{K}}$ and $(\chi^n_{\mathfrak{M}} \circ \chi^n_{\mathfrak{L}}) \vee (\chi^n_{\mathfrak{L}} \circ \chi^n_{\mathfrak{M}}) = \chi^n_{\mathfrak{M}\mathfrak{L}} \vee \chi^n_{LM} = \chi^n_{ML \cup LM} \geq \chi^n_{\mathfrak{K}}$. By assumption, $\chi^p_{\mathfrak{M}} \leq \chi^p_{\mathfrak{K}}$ and $\chi^n_{\mathfrak{M}} \geq \chi^n_{\mathfrak{K}}$ or $\chi^p_{\mathfrak{L}} \leq \chi^p_{\mathfrak{K}}$ and $\chi^n_{\mathfrak{L}} \geq \chi^n_{\mathfrak{K}}$. Thus $\mathfrak{M} \subseteq \mathfrak{K}$ or $\mathfrak{L} \subseteq \mathfrak{K}$. We conclude that $\mathfrak{K}$ is a strongly prime almost interior ideal of $\mathfrak{T}$. ■

V. Conclusion

The aim of the paper is to give the concept of almost (m, n)-ideals in ordered semigroups. The union of two almost (m, n)-ideals is also an almost (m, n)-ideal in ordered semigroups, and the results in class fuzzifications are the same. In Theorems 4.5, 4.6, 3.20, 3.25, and 3.26, we prove the relationship between almost (m, n)-ideals and class fuzzifications. Finally, we study bipolar fuzzy prime almost interior ideals in semigroups. In future work, we can study other kinds of almost ideals and their fuzzifications in an ordered ternary semigroup.

References

[1] L.A. Zadeh "Fuzzy sets," *Information and Control*, vol. 8, pp.338-353, 1965.

[2] L. Satko and O. Grosek, On minimal A-ideals of semigroups, *Semigroup Forum*, vol. 23, pp. 283-295, 1981.

[3] S. Bogdanovic, "Semigroups in which some bi-ideals is a group," *Review of Research Faculty of Science-University of Novi Sad*, vol.11, pp. 261-266, 1981.

[4] S. Suebsung, K. Wattanatripop and R. Chinram, "On almost (m, n)-ideals and fuzzy almost (m, n)-ideals in semigroups," *Journal of Taibah Universtiy for Science*, vol. 13, 897-902, 2019.

[5] N. Kaopusek, T. Kaewnoi and R. Chinram, "On almost interior ideals and weakly almost interior ideals of semigroups," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 23, no. 3 pp. 773-778, 2020.

[6] P. Muangdoo, T. Chuta and W.Nakkhasen, Almost bu-hyperideals and their fuzzification of semihypergroups, *Journal of Mathematics computre Science*, vol. 11, no. 3, pp. 2755-2767, 2021.

[7] W. Nakkhasen, P. Khathipphathi and S. Panmuang, A note on fuzzy almost interior hyperideals of semihypergroups, *International Journal of Mathematics and Computer Science*, vol. 17, No. 3, pp. 1419–1426, 2022.

[8] S. Suebsung, W. Yonthanthum and R. Chinram, " Ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups," Italian Journal of Pure and Applied Mathematics, vol. 48, 2022, pp. 1206–1217.

[9] T. Gaketem and P. Khamrot, "Bipolar fuzzy almost bi-ideal in semigroups," *International Journal of Mathematics and Computer Science*, vol. 17, no. 1, pp. 345-352, 2022.

[10] T. Gaketem, " Bipolar almost interior ideals in semigroups," *ICIC Express Lettes*, vol. 17, no. 4, pp. 381-387, 2023.

[11] P. Khamrot and T. Gaketem, "Applications of bipolar fuzzy almost ideals in semigroups," *International Journal of Analysis and Applications*, vol. 22, no. 8, pp. 1-10, 2024.

[12] P. Khamrot and T. Gaketem, "Bipolar fuzzy almost quasi-ideals in semigroups," *International Journal of Analysis and Applications*, vol. 22, no. 12, pp. 1-10, 2024.

[13] R. Chinramm, S. Baupradist, A. Iampan and P. Singvananda, " Characterizations of ordered almost ideals and fuzzifications in partially ordered ternary semigroups," *ICIC Express Lettes*, vol. 17, no. 6, pp. 631-639, 2023.

[14] R. Rittichuai, A. Iampan, R. Chinram and P. Singavanda, "Almost subsemirings and fuzzificaion, " International Journal of Mathematics and Computer Science, vol. 17, no. 4, pp. 1491–1497. 2022.

[15] N. Sarasit, R. Chinram and A. Rattana, "Applications of fuzzy sets for almostity of ternary subsemirings," International Journal of Applied Mathematics, vol. 17, No. 4, pp. 497–507. 2023.

[16] P. Khamrot, A. Iampan and T. Gaketem, "Fuzzy (m, n)-ideals and n-interior ideals in ordered semigroups," *European Journal of Pure and Applied Mathematics* vol.18, no. 1, Article Number 5596, 2025.

[17] A. Mahboob, M. Al-Tahan and G. Muhiuddin "Characterizations of ordered semigroups in terms of fuzzy (m, n)-substructures," Soft Computing, pp. 1-8, 2024.

[18] T. Changphas, "On (m, n)-ideals of an ordered semigroup," International Journal of Pure and Applied Mathematics, vol. 100, no. 1, pp. 1-5, 2015.

[19] S. Suebsung, W. Yonthanthum and R. Chinram, "Ordered almost ideal and fuzzy ordered almost ideals in ordered semigroups," Italian Journal of Pure and Applied Mathematics, vol.48, pp. 1206-1217, 2022.

[20] P. Khamrot, A. Phukhaengst, and T. Gaketem, "Ordered Almost n-interior ideals in Semigroups Class Fuzzifications," IAENG International Journal of Computer Science, vol. 51, no. 11, pp. 1711-1719, 2024.

[21] K. Lee "Bipolar-valued fuzzy sets and their operations," *In proceeding International Conference on Intelligent Technologies Bangkok*, Thailand, pp. 307-312, 2000.

[22] P. Khamrot and T. Gaketem, " Bipolar fuzzy almost interior ideals in semigroups," *ICIC Express Lettes*, vol. 17, no. 4, pp. 381-387, 2023.