New Characterizations of the Clifford Torus and the Calabi Torus

Dehe Li and Shujie Zhai

***Abstract*—Let M^n be a compact submanifold minimally immersed into the unit sphere $\mathbb{S}^{n+p}$ with codimension p, and denote by h the second fundamental form. As our main results, we first establish two rigidity theorems in terms of the geometric quantity $\sigma(u) = \|h(u,u)\|^2$ for any unit vector u tangent to M^n, where $\|\cdot\|^2$ denotes the squared norm with respect to the standard metric g on $\mathbb{S}^{n+p}$. Furthermore, we establish an optimal inequality for the conformally flat minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$ with constant scalar curvature, involving the normalized scalar curvature and the squared norms of the traceless Ricci tensor and second fundamental form. In particular, our first theorem related to the hypersurfaces of $\mathbb{S}^{n+1}$ gives a new characterization of the Clifford torus, whereas the other theorems are about the Legendrian submanifolds such that new characterizations of the Calabi torus can be presented.**

***Index Terms*—unit sphere, hypersurface, Legendrian submanifold, Clifford torus, Calabi torus, rigidity theorem.**

I. Introduction

THE study of pinching problems on submanifolds of the unit sphere, closely related to the rigidity phenomena, is always an attractive geometric topic and has been extensively studied by many geometers, under various intrinsic and extrinsic geometric conditions. For the former, a variety of characteristic results were established, see e.g. [5], [6], [35], [36] for pinching of the sectional curvature and [7], [9], [11], [19] for pinching of the Ricci curvature, respectively. In particular, by the minimality it is easily seen that the pinching problem on the scalar curvature of the submanifold M^n in the unit sphere $\mathbb{S}^{n+p}$ of codimension p is equivalent to that on the squared norm S of the second fundamental form h of M^n with respect to the standard metric g on $\mathbb{S}^{n+p}$. Regarding this, Simon [29] obtained the so-called Simons' formula through calculating the Laplacian of S, which states that if $0 \le S \le n/(2-1/p)$ on M^n, then either $S \equiv 0$ or $S \equiv n/(2-1/p) =: c$. Furthermore, such submanifolds attaining $S \equiv c$ were completely determined by Lawson [14] and Chern-do Carmo-Kobayashi [4] and later Li-Li [17] improved the first pinching constant c to $2n/3$.

Let UM^n be the unit tangent bundle on M^n and set $\sigma(u) = \|h(u,u)\|^2$, $u \in UM^n$, where $\|\cdot\|^2$ denotes the squared norm with respect to the standard metric g on $\mathbb{S}^{n+p}$. It is should be pointed out that there have been some papers on studying submanifolds in the unit sphere by taking into account the geometric quantity $\sigma(u)$, especially about its pinching problem (cf. [8], [23], [31], [32] and references therein). Among them, Gauchman [8] investigated the pinching problem of $\sigma(u)$ and proved the following well-known extrinsic rigidity theorem:

Theorem A. *Let M^n be an n-dimensional compact minimal submanifold in the unit sphere $\mathbb{S}^{n+p}$. Then, it holds that*

(1) *assuming that $p = 1$ and n is odd, if $\sigma(u) \le 1/(1-1/n)$ for any $u \in UM^n$, then M^n is totally geodesic with $\sigma(u) \equiv 0$;*

(2) *assuming that $p \ge 2$ and n is odd, if $\sigma(u) \le 1/(3-2/n)$ for any $u \in UM^n$, then M^n is totally geodesic with $\sigma(u) \equiv 0$;*

(3) *assuming that $p = 1$ and n is even, if $\sigma(u) < 1$ for any $u \in UM^n$, then M^n is totally geodesic with $\sigma(u) \equiv 0$, whereas if $\max_{u \in UM^n} \sigma(u) = 1$, then $M^n = \mathbb{S}^{n/2}(\sqrt{1/2}) \times \mathbb{S}^{n/2}(\sqrt{1/2})$ with $\sigma(u) \equiv 1$;*

(4) *assuming that $p \ge 2$ and n is even, if $\sigma(u) < 1/3$ for any $u \in UM^n$, then M^n is totally geodesic with $\sigma(u) \equiv 0$, whereas if $\max_{u \in UM^n} \sigma(u) = 1/3$, then M^n is one of the submanifolds with $\sigma(u) \equiv 1/3$.*

Remark I.1. *It is known from [8] that such submanifolds in $\mathbb{S}^{n+p}$ with $\sigma(u) \equiv 1/3$ are the λ-isotropic minimal ones with parallel second fundamental form and $\lambda = 1/\sqrt{3}$, of which the classification has been obtained by Sakamoto [26].*

However, the pinching constant in Theorem A is not optimal if we consider that $p \ge 1$ and n is odd. This observation combining with the above statement further motivates us to consider the following natural and interesting question:

Question. *For n-dimensional compact minimal submanifolds of the unit sphere $\mathbb{S}^{n+p}$, what is the best possible condition on the geometric quantity $\sigma(u)$ such that submanifolds next to the totally geodesic one can be characterized?*

One of the purposes of this article is to answer the Question, restricted to the hypersurface case of $\mathbb{S}^{n+1}$ for $p = 1$ and $n \ge 2$ and the Legendrian submanifold case of $\mathbb{S}^7$ for $p = 4$ and $n = 3$, respectively. For better illustrating our first result, we review the following compact minimal hypersurface called the *Clifford torus* in $\mathbb{S}^{n+1}$.

Example I.1. *(cf. [2]) The Clifford torus $\mathbf{Cl}_{1,n-1}$ in the unit sphere $\mathbb{S}^{n+1}$.*

The Clifford torus

$$\mathbf{Cl}_{1,n-1} := \mathbb{S}^1(\sqrt{1/n}) \times \mathbb{S}^{n-1}(\sqrt{(n-1)/n})$$

in $\mathbb{S}^{n+1}$ is a compact minimal hypersurface with two distinct constant principal curvatures, one of them being simple:

$$\lambda_1 = \pm\sqrt{n-1}, \quad \lambda_2 = \lambda_3 = \cdots = \lambda_n = \mp\tfrac{1}{\sqrt{n-1}}. \qquad (1)$$

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It follows that the square norm S of the second fundamental form h of $\mathbf{Cl}_{1,n-1}$ satisfies $S = \|h\|^2 = \sum_{i=1}^n \lambda_i^2 = n$.

Consider that M^n is a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$ for $n \geq 2$. Then, the unit tangent bundle is defined by $U_qM^n = \{u \in T_qM^n \mid g(u,u) = 1\}$ for $q \in M^n$, on which there exists a well-defined function given by $f_q(u) := g(A_N u, u)$ on U_qM^n (cf. [11]), where A_N denotes the shape operator of M^n with respect to the unit normal vector field N along M^n. Since U_qM^n is a compact set, we have an element $e \in U_qM^n$ at which it satisfies $f_q(e) = \max_{u\in U_qM^n} f_q(u)$. Thus, with the non-empty set

$$\mathcal{U}_q := \{u \in U_qM^n \mid f_q(u) = f_q(e)\},$$

similar to [11], we can define a function Φ_e on U_qM^n by

$$\Phi_e(u) := [g(A_N e, u)]^2, \quad u \in U_qM^n,$$

where e is any fixed element in $\mathcal{U}_q$.

Now, the first result of this paper can be stated as follows:

Theorem I.1. *Let M^n $(n \geq 2)$ be a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$. If the squared norm $\sigma(u)$ satisfies, at any point $q \in M^n$,*

$$0 \leq \sigma(u) \leq \tfrac{1}{n-1} + \tfrac{n(n-2)}{(n-1)^2}\Phi_e(u) \tag{2}$$

for all $u \in U_qM^n$ and a fixed $e \in \mathcal{U}_q$, then either

(i) *$M^n = \mathbb{S}^n$ is totally geodesic satisfying $\sigma(u) \equiv 0$, or*

(ii) *$M^n = \mathbf{Cl}_{1,n-1}$ is the Clifford torus satisfying $\sigma(u) \equiv \frac{1}{n-1} + \frac{n(n-2)}{(n-1)^2}\Phi_e(u)$.*

Remark I.2. *It is clear that the pinching function in* (2) *is optimal and therefore we obtain a new characterization of the Clifford torus $\mathbf{Cl}_{1,n-1}$ by considering both the case n is odd and the case n is even. In the latter case, different from $\mathbf{Cl}_{1,n-1}$, Theorem A characterized the minimal hypersurface $\mathbb{S}^{n/2}(\sqrt{1/2}) \times \mathbb{S}^{n/2}(\sqrt{1/2})$.*

Recall that, as a real hypersurface of the complex Euclidean space $\mathbb{C}^{n+1}$, the unit sphere $\mathbb{S}^{2n+1}$ naturally admits a Sasakian structure (φ, ξ, η, g) (cf. [30]). Moreover, an m-dimensional submanifold M^m in $\mathbb{S}^{2n+1}$ is said to be C-*totally real* (or equivalently, *integral*) if the contact form η of $\mathbb{S}^{2n+1}$ vanishes when it is restricted to M^m, namely $\eta(X) = 0$ for any $X \in TM^m$. In particular, we call a C-totally real submanifold M^m *Legendrian* if it meets the smallest possible codimension, namely $m = n$ (cf. [33]), and associated with the study of such submanifolds in $\mathbb{S}^{2n+1}$, there are many important results established in the last decades, see e.g. [12], [13], [15], [16], [20], [21], [22], [24], [27], [28], [34], [37].

Before stating the remaining main results, we shall look at the following Legendrian submanifold in $\mathbb{S}^{2n+1}$.

Example I.2. *(cf. [5], [11], [18], [23]) The Calabi torus $\mathbf{Ca}_{1,n-1}$ in the unit sphere $\mathbb{S}^{2n+1}$.*

Let $\gamma = (\gamma_1, \gamma_2) : \mathbb{S}^1 \to \mathbb{S}^3 \subset \mathbb{C}^2$ be a Legendrian curve, defined by

$$\gamma(t) = \left(\sqrt{\tfrac{n}{n+1}}e^{i\frac{1}{\sqrt{n}}t}, \sqrt{\tfrac{1}{n+1}}e^{-i\sqrt{n}t}\right), \tag{3}$$

and $\phi : \mathbb{S}^{n-1} \to \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ be the totally geodesic Legendrian sphere for $n \geq 3$. Then

$$f(t,y) = (\gamma_1\phi, \gamma_2) : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \tag{4}$$

is a minimal Legendrian immersion and $f(\mathbb{S}^1 \times \mathbb{S}^{n-1})$ is called the *Calabi torus*, denoted by $\mathbf{Ca}_{1,n-1}$.

Note from the induced metric of $f(t,y) : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$

$$f^*(g) = (dt)^2 + \tfrac{n}{n+1}[(dy_1)^2 + \cdots + (dy_n)^2]$$

that f is an isometric immersion, where $y = (y_1, \ldots, y_n) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $\sum_{i=1}^n y_i^2 = 1$. Adopting the following local reparametrization

$$\begin{aligned}(y_1, y_2, \ldots, y_n) = (&\sin\theta_1, \cos\theta_1\sin\theta_2, \ldots,\\ &\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-2}\cos\theta_{n-1}),\end{aligned}$$

we then obtain a local orthonormal frame $\{e_i\}_{i=1}^n$ on $f(\mathbb{S}^1 \times \mathbb{S}^{n-1}) =: M^n$ with respect to the metric g, satisfying the relations:

$$\begin{cases} e_1 = -f_t, \quad e_2 = \sqrt{\frac{n+1}{n}}f_{\theta_1}, \\ e_3 = \sqrt{\frac{n+1}{n}}\cos^{-1}\theta_1 f_{\theta_2}, \quad \ldots, \\ e_n = \sqrt{\frac{n+1}{n}}\displaystyle\prod_{\ell=1}^{n-2}\cos^{-1}\theta_\ell\, f_{\theta_{n-1}}. \end{cases} \tag{5}$$

As the unit sphere $\mathbb{S}^{2n+1}$ admits a natural Sasakian structure (φ, ξ, η, g), by definition we see that $\eta(e_i) = 0$ for $1 \leq i \leq n$ and thus f is a Legendrian immersion.

Denote by h the second fundamental form of $f : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{S}^{2n+1}$. Then, direct calculations by using the Gauss formula show that (cf. [10])

$$\begin{cases} \nabla_{e_i}e_j = -\sqrt{\frac{n+1}{n}}\frac{\sin\theta_{j-1}}{\prod_{k=1}^{j-1}\cos\theta_k}e_i, & 2 \leq j < i \leq n, \\ \nabla_{e_i}e_i = \sqrt{\frac{n+1}{n}}\displaystyle\sum_{\ell=2}^{i-1}\frac{\sin\theta_{\ell-1}}{\prod_{k=1}^{\ell-1}\cos\theta_k}e_\ell, & 3 \leq i \leq n, \\ \nabla_{e_i}e_j = 0, & \text{otherwise}, \end{cases} \tag{6}$$

where ∇ is the Levi-Civita connection of the metric g, and

$$\begin{aligned} &h(e_1, e_1) = \tfrac{n-1}{\sqrt{n}}\varphi e_1, \quad h(e_1, e_i) = -\tfrac{1}{\sqrt{n}}\varphi e_i, \\ &h(e_i, e_j) = -\tfrac{1}{\sqrt{n}}\delta_{ij}\varphi e_1, \quad 2 \leq i,j \leq n. \end{aligned} \tag{7}$$

It is obvious that such an immersion f is a compact minimal Legendrian submanifold. Combining with (6) and (7), we get $(\bar\nabla^\xi h)(e_i, e_j, e_k) = 0$ for $1 \leq i,j,k \leq n$, i.e., the immersion f is of C-parallel second fundamental form (cf. Section II).

For the Riemannian curvature tensor of M^n, applying (6) again, we obtain that

$$\begin{aligned} &R(e_1, e_i)e_1 = R(e_1, e_i)e_j = R(e_i, e_j)e_1 = 0, \\ &R(e_i, e_j)e_k = \tfrac{n+1}{n}(\delta_{jk}e_i - \delta_{ik}e_j), \quad 2 \leq i,j,k \leq n. \end{aligned} \tag{8}$$

Therefore, by definition we deduce from (8) that

$$\begin{aligned} &\mathrm{Ric}(e_1, e_1) = \mathrm{Ric}(e_1, e_i) = 0, \\ &\mathrm{Ric}(e_i, e_j) = \tfrac{(n-2)(n+1)}{n}\delta_{ij}, \quad 2 \leq i,j \leq n. \end{aligned} \tag{9}$$

Remark I.3. *According to [23], the Calabi torus $\mathbf{Ca}_{1,n-1}$ in the unit sphere $\mathbb{S}^{2n+1}$ can be viewed as the minimal Calabi product Legendrian immersion of one point and the totally geodesic Legendrian sphere.*

Consider that M^3 is a compact minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7$ with the standard contact metric structure $\{\varphi, \xi, \eta, g\}$. Due to [1], we have such a function $F_q(u) = g(h(u,u), \varphi u)$ defined on U_qM^3 for $q \in M^3$. Similarly, there exists an element $e \in U_qM^3$ such that $F_q(e) = \max_{u \in U_qM^3} F_q(u)$. Hence, we put

$$\mathcal{V}_q := \{u \in U_qM^3 \mid F_q(u) = F_q(e)\},$$

and according to [11], a well-defined function Ψ_e on U_qM^3 can be obtained by

$$\Psi_e(u) = [g(h(e,e), \varphi u)]^2, \quad u \in U_qM^3,$$

where e is any fixed element in $\mathcal{V}_q$.

Next, the second result of this paper can be stated as follows:

Theorem I.2. *Let M^3 be a compact minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7$. If the squared norm $\sigma(u)$ satisfies, at any point $q \in M^3$,*

$$0 \le \sigma(u) \le \tfrac{1}{3} + \tfrac{3}{4}\Psi_e(u) \tag{10}$$

for all $u \in U_qM^3$ and a fixed $e \in \mathcal{V}_q$, then either

(i) *$M^3 = \mathbb{S}^3$ is totally geodesic satisfying $\sigma(u) \equiv 0$, or*

(ii) *$M^3 = \mathbf{Ca}_{1,2}$ is the Calabi torus satisfying $\sigma(u) \equiv \frac{1}{3} + \frac{3}{4}\Psi_e(u)$.*

Remark I.4. *From the view of intrinsic geometry, the characterizations of Calabi torus in $\mathbb{S}^7$ were presented by Dillen-Vrancken [5] for pinching of sectional curvature and by Hu-Xing [11] for pinching of Ricci curvature, respectively. In this paper, Theorem I.2 states that a new characterization of the Calabi torus can be obtained by considering the pinching of the extrinsic geometric quantity $\sigma(u)$.*

Finally, we can prove the following theorem:

Theorem I.3. *Let M^n $(n \ge 3)$ be a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with constant scalar curvature. Then the traceless Ricci tensor $\tilde{\mathrm{Ric}}$ of M^n satisfies*

$$\|\tilde{\mathrm{Ric}}\|^2 \ge \tfrac{(n-2)(n+1)}{n+2} S\chi, \tag{11}$$

where S and χ are respectively the squared norm $\|\cdot\|^2$ of the second fundamental form and the normalized scalar curvature of M^n. Moreover, the equality in (11) holds identically if and only if M^n is locally congruent to one of the following three examples:

(i) *$M^n = \mathbb{S}^n$ is totally geodesic;*

(ii) *$M^n = \mathbb{T}^n$ is the flat Clifford torus;*

(iii) *$M^n = \mathbf{Ca}_{1,n-1}$ is the Calabi torus.*

Remark I.5. *Recall that the Riemannian manifold (M^n, g) is said to be conformally flat if around each point of M^n there exists a neighborhood which can be conformally immersed into the Euclidean space $\mathbb{R}^n$. When $n \ge 4$, it is known that (M^n, g) is conformally flat if and only if its Weyl curvature tensor vanishes. When $n = 3$, we should remark that the Weyl curvature tensor vanishes automatically, and (M^3, g) is conformally flat if and only if its Schouten tensor is a Codazzi tensor.*

Remark I.6. *Recently, the Calabi torus $\mathbf{Ca}_{1,n-1}$ has been characterized by Luo-Sun-Yin [23] from the view of extrinsic geometry and by Li-Xing-Yin [18] from the view of intrinsic geometry. In particular, it was conjectured in [22] that, for a closed minimal Legendrian submanifold M^n in the unit sphere $\mathbb{S}^{2n+1}$, if $0 \le S \le (n+2)(n-1)/n$, then M^n is either the totally geodesic sphere with $S = 0$, or the Calabi torus with $S = (n+2)(n-1)/n$. It is worth mentioning that Theorem I.3 corresponds to Theorem 1.2 of Cheng-Hu [3].*

II. Preliminaries

In this section, we briefly review some basic facts on submanifolds in the unit sphere $\mathbb{S}^{n+p}$ with codimension p, and then present some useful lemmas, associated to the hypersurface case for $p = 1$ and $n \ge 2$ (cf. [25]) as well as the Legendrian submanifold case for $p = 4$ and $n = 3$ (cf. [11]), which we need in the proofs of Theorems I.1–I.3.

Let M^n be an n-dimensional submanifold in the unit sphere $\mathbb{S}^{n+p}$ equipped with the standard metric g. For simplicity, we denote also by g the induced metric on M^n. Let ∇ and $\bar\nabla$ be the Levi-Civita connections of M^n and $\mathbb{S}^{n+p}$, respectively. Then, for the immersion $M^n \hookrightarrow \mathbb{S}^{n+p}$, we have the Gauss and Weingarten formulas:

$$\begin{aligned} \bar\nabla_X Y &= \nabla_X Y + h(X,Y), \\ \bar\nabla_X N &= -A_N X + \nabla^\perp_X N \end{aligned} \tag{12}$$

for any tangent vector fields $X, Y \in TM^n$ and normal vector field $N \in T^\perp M^n$. Here, $\nabla^\perp$ denotes the normal connection in the normal bundle $T^\perp M^n$ and h (resp. A_N) denotes the second fundamental form (resp. the shape operator with respect to N) of $M^n \hookrightarrow \mathbb{S}^{n+p}(1)$. Applying (12), we derive the relation:

$$g(h(X,Y), N) = g(A_N X, Y). \tag{13}$$

A. Hypersurfaces in the unit sphere $\mathbb{S}^{n+1}$

In this subsection, we always assume that M^n is a minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$. For the sake of simplicity, we adopt the notations of Peng-Terng [25] to give the well-known result:

Lemma II.1. *Let M^n be a minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$. Then it holds that*

$$\tfrac{1}{2}\Delta\|h\|^2 = \|\bar\nabla h\|^2 + \|h\|^2(n - \|h\|^2). \tag{14}$$

For later's purpose, by means of (13) we easily derive the following lemma:

Lemma II.2. *Let M^n be a minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$ with unit normal vector field N. Then, for any point $q \in M^n$, there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_qM^n and numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that the second fundamental form h of M^n satisfies that*

$$h(e_1,e_1) = \lambda_1 N, \quad h(e_m,e_m) = -\lambda_m N, \quad h(e_i,e_j) = 0 \tag{15}$$

for $2 \le m \le n$ and $1 \le i \ne j \le n$, where there holds

$$\begin{aligned} \lambda_1 = \sum_{m=2}^{n} \lambda_m &= \max\{\lambda_1, -\lambda_2, ..., -\lambda_n\} \\ &= \max_{u \in U_qM^n} g(A_N u, u). \end{aligned} \tag{16}$$

B. Legendrian submanifolds in the unit sphere $\mathbb{S}^{2n+1}$

In this subsection, we always assume that M^n is a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ admitting a Sasakian structure (φ, ξ, η, g). It is known that, associated to $\bar\nabla$ and ξ, a covariant differentiation $\bar\nabla^\xi$ can be defined such that it acts on h as (cf. [13], [18])

$$(\bar\nabla^\xi h)(X,Y,Z) = (\bar\nabla h)(X,Y,Z) - g(h(Y,Z),\varphi X)\xi \quad (17)$$

for any vector fields X, Y, Z tangent to M^n, where there holds

$$(\bar\nabla h)(X,Y,Z) = \nabla_X^\perp h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (18)$$

In particular, the second fundamental h is called *C-parallel* if it satisfies $\bar\nabla^\xi h = 0$ on M^n.

The *Legendre frame* $\{e_1,\dots,e_n,e_{1^*},\dots,e_{n^*},e_{2n+1}\}$ on M^n can be chosen so that, restricted to M^n, the vector fields $e_1,\dots,e_n$ are orthonormal and tangent to M^n, whereas $\{e_{1^*} = \varphi e_1,\dots,e_{n^*} = \varphi e_n, e_{2n+1} = \xi\}$ are orthonormal normal vector fields of M^n in the unit sphere $\mathbb{S}^{2n+1}$. Set $h_{ij}^{k^*} = g(h(e_i,e_j),\varphi e_k)$ and $h_{ij}^{2n+1} = g(h(e_i,e_j),e_{2n+1})$ and make the following convention on range of indices:

$$i,j,k,\ell = 1,\dots,n; \quad \alpha = 1,\dots,n+1,$$
$$i^*,j^*,k^*,\ell^* = n+1,\dots,2n; \quad \alpha^* = \alpha + n.$$

From now on, we assume that $n=3$ and it therefore follows from the notations given in Chern-do Carmo-Kobayashi [4] and Hu-Yin [13] that (cf. also Lemma 2.1 of [23] or Lemma 2.3 of [11])

Lemma II.3. *Let M^3 be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7$. Then, in terms of $H_i = (h_{jk}^{i^*})$, we have the Laplacian of $\|h\|^2$ as below:*

$$\tfrac{1}{2}\Delta\|h\|^2 = \|\bar\nabla^\xi h\|^2 + 4\|h\|^2 - \sum_{i,j} N(H_iH_j - H_jH_i) - \sum_{i,j}(S_{ij})^2, \quad (19)$$

where $\|\bar\nabla^\xi h\|^2 = \sum_{i,j,k,\ell}(h_{ij,k}^{\ell^*})^2$, $S_{ij} = \mathrm{trace}(H_iH_j)$ *and* $N(A) = \sum_{i,j}(a_{ij})^2$ *for* $A = (a_{ij})$.

Finally, we also need the following three useful lemmas that were presented in [11].

Lemma II.4. *Let M^3 be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7$. Then, for each point $q \in M^3$, there exists an orthonormal basis $\{e_1,e_2,e_3\}$ of T_qM^3 and numbers $\{\lambda_1,\lambda_2,\mu_1,\mu_2\}$ such that the second fundamental form h of M^3 takes the following form:*

$$\begin{cases} h(e_1,e_1) = (\lambda_1+\lambda_2)\varphi e_1,\\ h(e_1,e_2) = -\lambda_1\varphi e_2,\\ h(e_1,e_3) = -\lambda_2\varphi e_3,\\ h(e_2,e_2) = -\lambda_1\varphi e_1 + \mu_1\varphi e_2 + \mu_2\varphi e_3,\\ h(e_2,e_3) = \mu_2\varphi e_2 - \mu_1\varphi e_3,\\ h(e_3,e_3) = -\lambda_2\varphi e_1 - \mu_1\varphi e_2 - \mu_2\varphi e_3, \end{cases} \quad (20)$$

where, for $F_q(u) = g(h(u,u),\varphi u)$ defined on U_qM^3, it satisfies that

$$\begin{cases} \lambda_1+\lambda_2 = \max\limits_{u\in U_qM^3} F_q(u) \ge 0,\\ \lambda_1+\lambda_2 \ge -2\lambda_1, \quad \lambda_1+\lambda_2 \ge -2\lambda_2,\\ -(\lambda_1+\lambda_2) \le \mu_i \le \lambda_1+\lambda_2, \quad i=1,2. \end{cases} \quad (21)$$

Lemma II.5. *If* (20) *holds, then by the notations of Lemma II.3, we have*

$$\|h\|^2 = \sum_{i,j,\alpha}(h_{ij}^{\alpha^*})^2 = 4\lambda_1^2 + 4\lambda_2^2 + 2\lambda_1\lambda_2 + 4\mu_1^2 + 4\mu_2^2. \quad (22)$$

Lemma II.6. *If* (20) *holds, then by the notations of Lemma II.3, we have*

$$\begin{aligned} &\sum_{i,j} N(H_iH_j - H_jH_i) + \sum_{i,j}(S_{ij})^2\\ =\ & 24(\lambda_1^4 + \lambda_1^3\lambda_2 + \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2^3 + \lambda_2^4)\\ &- 36\lambda_1\lambda_2(\mu_1^2+\mu_2^2) + 24(\mu_1^2+\mu_2^2)^2\\ &+ 18(\lambda_1^2+\lambda_2^2)(\mu_1^2+\mu_2^2). \end{aligned} \quad (23)$$

III. Proof of Theorem I.1

Let M^n $(n \ge 2)$ be a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$. At an arbitrary point $q \in M^n$, choosing the orthonormal basis $\{e_1,\dots,e_n\}$ of T_qM^n as stated in Lemma II.2, by definition we have

$$\begin{aligned} &\sigma(e_1) = \lambda_1^2 = \Big(\sum_{m=2}^n \lambda_m\Big)^2,\\ &\sigma(e_m) = \lambda_m^2, \quad 2 \le m \le n. \end{aligned} \quad (24)$$

Moreover, by setting $e = e_1 \in \mathcal{U}_q$ and using Lemma II.2, we obtain from the assumption of Theorem I.1 that

$$0 \le \sigma(v) \le \tfrac{1}{n-1} + \tfrac{n(n-2)}{(n-1)^2}\Phi_{e_1}(u) \quad (25)$$

for all $u \in U_qM^n$, where $\Phi_{e_1}(e_1) = \lambda_1^2$ and $\Phi_{e_1}(e_m) = 0$ for $2 \le m \le n$. In what follows, we shall divide the remaining proof of Theorem I.1 into two cases: $n \ge 3$ and $n = 2$.

Assume that $n \ge 3$. Then, combining (24) with (25) and interchanging e_r and e_s for $2 \le r < s \le n$ if necessary, we may assume without loss of generality that

$$\begin{cases} \lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_2,\\ 0 \le \lambda_1^2 \le n-1,\\ 0 \le \lambda_m^2 \le \frac{1}{n-1}, \quad 2 \le m \le n. \end{cases} \quad (26)$$

Applying the fact $\|h\|^2 = \sum_{i,j=1}^n h_{ij}^2$ and Lemma II.2, we further have

$$\begin{aligned} &\|h\|^2(n - \|h\|^2)\\ =& -\Big(\sum_{i=1}^n \lambda_i^2\Big)\Big(\sum_{i=1}^n \lambda_i^2 - n\Big)\\ =& -\Big(\sum_{i=1}^n \lambda_i^2\Big)\Big(2\sum_{m=2}^n \lambda_m^2 + 2\sum_{r<s}^n \lambda_r\lambda_s - n\Big). \end{aligned} \quad (27)$$

In particular, it is easily seen that the following relation holds:

$$\begin{aligned} &2\sum_{m=2}^n \lambda_m^2 + 2\sum_{r<s}^n \lambda_r\lambda_s - n\\ =& \tfrac{n}{n-1}\sum_{m=2}^n \big[(n-1)\lambda_m^2 - 1\big] - \sum_{r<s}^n (\lambda_r - \lambda_s)^2. \end{aligned} \quad (28)$$

Consequently, it follows from (26)–(28) that

$$\|h\|^2(n - \|h\|^2) \ge 0, \quad (29)$$

where it is obvious that the equality sign in (29) holds if and only if either $\lambda_i = 0$ for $1 \le i \le n$, or $\lambda_1 = \sqrt{n-1}$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = \sqrt{1/(n-1)}$ such that $\|h\|^2 = n$.

Assume that $n = 2$. It is known from (24) and (25) that $\lambda_1^2 = \lambda_2^2 \le 1$. Therefore, we conclude that in this case (29) still holds and the corresponding equality sign holds if and only if either $\lambda_1 = \lambda_2 = 0$, or $\lambda_1 = \lambda_2 = 1$ such that $\|h\|^2 = 2$.

Since M^n is compact, Lemma II.1 and the divergence theorem show that

$$\int_{M^n} \Big\{\|\bar{\nabla} h\|^2 + \|h\|^2(n - \|h\|^2)\Big\} dV_{M^n} = 0, \tag{30}$$

where dV_{M^n} denotes the volume element of the induced metric g on M^n. Together with (29) and the arbitrariness of $q \in M^n$, we get

$$\|\bar{\nabla} h\|^2 = \|h\|^2(n - \|h\|^2) \equiv 0 \tag{31}$$

on M^n. Finally, according to the well-known results of Lawson [14] and Chern-do Carmo-Kobayashi [4], we conclude that either M^n is totally geodesic and $M^n = \mathbb{S}^n$, or M^n is the Clifford torus $\mathbf{Cl}_{1,n-1}$ with $\|h\|^2 = n$. This completes the proof of Theorem I.1. □

IV. Proof of Theorem I.2

Let M^3 be a compact minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7$. Then, for an arbitrary point $q \in M^3$, we can choose the orthonormal basis $\{e_1, e_2, e_3\}$ of T_qM^3 as in Lemma II.4 such that $e = e_1 \in \mathcal{V}_q$ and $\{\sigma(e_1), \sigma(e_2), \sigma(e_3)\}$ take the following forms:

$$\begin{aligned} &\sigma(e_1) = (\lambda_1 + \lambda_2)^2, \\ &\sigma(e_2) = \lambda_1^2 + \mu_1^2 + \mu_2^2, \\ &\sigma(e_3) = \lambda_2^2 + \mu_1^2 + \mu_2^2. \end{aligned} \tag{32}$$

Under the assumption of Theorem I.2, we deduce from Lemma II.4 that, at $q \in M^3$, the squared norm $\sigma(u)$ satisfies that

$$0 \le \sigma(u) \le \tfrac{1}{3} + \tfrac{3}{4}\Psi_{e_1}(u) \tag{33}$$

for all $u \in U_qM^3$, where $\Psi_{e_1}(e_1) = (\lambda_1 + \lambda_2)^2$ and $\Psi_{e_1}(e_j) = 0$ for $j = 2, 3$.

Then, taking $u = e_1$, e_2 and e_3 in (33), respectively, we easily see from (32) that

$$\begin{cases} (\lambda_1 + \lambda_2)^2 \le \frac{1}{3} + \frac{3}{4}(\lambda_1 + \lambda_2)^2, \\ \lambda_1^2 + \mu_1^2 + \mu_2^2 \le \frac{1}{3}, \\ \lambda_2^2 + \mu_1^2 + \mu_2^2 \le \frac{1}{3}. \end{cases} \tag{34}$$

Interchanging e_2 and e_3 if necessary, we can assume that $\lambda_2^2 \le \lambda_1^2$. This combining with (21) and (34) yields that

$$\begin{aligned} &0 \le \lambda_1 \le \tfrac{\sqrt{3}}{3},\ \lambda_2 \le \lambda_1, \\ &-\tfrac{1}{3}\lambda_1 \le \lambda_2,\ \mu_1^2 + \mu_2^2 \le \tfrac{1}{3} - \lambda_1^2. \end{aligned} \tag{35}$$

On the other hand, using the compactness of M^3, we can integrate (19) and thus the combination of Lemma II.5, Lemma II.6 and the divergence theorem gives

$$\begin{aligned} 0 = \int_{M^3} \Big\{ &\|\bar{\nabla}^\xi h\|^2 - \big\{24[\lambda_1^4 + \lambda_2^4 + \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2)] \\ &+ 18(\mu_1^2 + \mu_2^2)(\lambda_1 - \lambda_2)^2 + 24(\mu_1^2 + \mu_2^2)^2 \\ &- 16(\mu_1^2 + \mu_2^2) - 16(\lambda_1^2 + \lambda_2^2 + \tfrac{1}{2}\lambda_1\lambda_2)\big\}\Big\}\, dV_{M^3}, \end{aligned} \tag{36}$$

where dV_{M^3} denotes the volume element of the induced metric g on M^3. Furthermore, according to the proof of Theorem 1.2 of [11], we can also set

$$\begin{aligned} \Lambda := \ & 24[\lambda_1^4 + \lambda_2^4 + \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2)] \\ &+ 18(\mu_1^2 + \mu_2^2)(\lambda_1 - \lambda_2)^2 + 24(\mu_1^2 + \mu_2^2)^2 \\ &- 16(\mu_1^2 + \mu_2^2) - 16(\lambda_1^2 + \lambda_2^2 + \tfrac{1}{2}\lambda_1\lambda_2). \end{aligned} \tag{37}$$

From now on, we assume that M^3 is not totally geodesic. So it is sufficient to consider the point $q \in M^3$ at which $h \neq 0$. In this case, it holds that $\lambda_1 > 0$. By means of (35) we obtain that

$$3\lambda_1 + 5\lambda_2 = \tfrac{4}{3}\lambda_1 + 5(\tfrac{1}{3}\lambda_1 + \lambda_2) > 0, \tag{38}$$

and moreover the expression of (37) can be rewritten as

$$\begin{aligned} \Lambda = \ & 24\big[2(\mu_1^2 + \mu_2^2) + 2\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_2^2\big](\mu_1^2 + \mu_2^2 + \lambda_1^2 - \tfrac{1}{3}) \\ &- 3\big[2(\mu_1^2 + \mu_2^2) + 2\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_2^2\big](3\lambda_1 + 5\lambda_2)(\lambda_1 - \lambda_2) \\ &- 24(\mu_1^2 + \mu_2^2)^2 - 3(\lambda_1 - \lambda_2)^2(2\lambda_1^2 + 2\lambda_2^2 - 3\lambda_1\lambda_2) \\ &- 12(5\lambda_1^2 + 5\lambda_2^2 + 4\lambda_1\lambda_2)(\mu_1^2 + \mu_2^2), \end{aligned} \tag{39}$$

which together with (35) and (38) implies that $\Lambda \le 0$, where the equality holds if and only if $\lambda_1 = \lambda_2 = \sqrt{3}/3$ and $\mu_1 = \mu_2 = 0$. By virtue of the integral identity (36) and the arbitrariness of $q \in M^3$, the fact $\Lambda \le 0$ implies that M^3 is a Legendrian submanifold with C-parallel second fundamental form (i.e., $\bar{\nabla}^\xi h = 0$). Consequently, we conclude that either it is totally geodesic and $M^3 = \mathbb{S}^3(1)$, or by continuity it satisfies the relations $\lambda_1 = \lambda_2 = \sqrt{3}/3$ and $\mu_1 = \mu_2 = 0$ such that $\|h\|^2 = 10/3$ hold identically on M^3.

In the latter case, with (20) and the Gauss equation, a direct calculation shows that the Ricci curvature Ric of M^3 satisfies the following relation:

$$\mathrm{Ric}(u) \ge \tfrac{4}{3} - \Psi_{e_1}(u) \tag{40}$$

for $e_1 \in \mathcal{V}_q$ and all $u \in U_qM^3$, where $\mathrm{Ric}(u) = \mathrm{trace}\{X \mapsto R(X, u)u\}/\|u\|^2$ with R the Riemannian curvature of M^3 (cf. [11]). Finally, with the arbitrariness of $q \in M^3$, by applying Theorem 1.2 of [11] we find that in this case M^3 is congruent to the Calabi torus $\mathbf{Ca}_{1,2}$. This completes the proof of Theorem I.2. □

V. Proof of Theorem I.3

Let M^n ($n \ge 3$) be a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with constant scalar curvature. According to Lemma 4.1 of [18], we immediately have

$$\tfrac{1}{2}\Delta S = \|\bar{\nabla}^\xi h\|^2 - \|\mathrm{Rie}\|^2 - \|\mathrm{Ric}\|^2 + n(n^2 - 1)\chi, \tag{41}$$

where $\|\mathrm{Rie}\|^2$ denotes the squared norm of the Riemannian curvature tensor of M^n. Under the Legendre frame as in Section II, we denote $R_{ijk\ell} = g\big(R(e_i, e_j)e_\ell, e_k\big)$ and $R_{ij} = \sum_k g(R(e_i, e_k)e_k, e_j)$ for $1 \le i, j, k, \ell \le n$. Recall that the components of the Weyl curvature tensor W of M^n satisfy

$$\begin{aligned} W_{ijk\ell} = \ & R_{ijk\ell} + \tfrac{n\chi}{n-2}(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) \\ &- \tfrac{1}{n-2}(\delta_{ik}R_{j\ell} + \delta_{j\ell}R_{ik} - \delta_{i\ell}R_{jk} - \delta_{jk}R_{i\ell}), \end{aligned} \tag{42}$$

and thus it holds that (cf. [18], [33]):

$$\|\mathrm{Rie}\|^2 = \|W\|^2 + \tfrac{4}{n-2}\|\mathrm{Ric}\|^2 - \tfrac{2n^2(n-1)}{n-2}\chi^2. \tag{43}$$

Here, $\|\mathrm{Rie}\|^2 = \sum_{i,j,k,\ell}(R_{ijk\ell})^2$, $\|\mathrm{Ric}\|^2 = \sum_{i,j}(R_{ij})^2$ and $\|W\|^2 = \sum_{i,j,k,\ell}(W_{ijk\ell})^2$. From $\tilde{R}_{ij} = R_{ij} - (n-1)\chi\delta_{ij}$, we easily see that

$$\|\mathrm{Ric}\|^2 = \|\tilde{\mathrm{Ric}}\|^2 + n(n-1)^2\chi^2, \tag{44}$$

where $\|\tilde{\mathrm{Ric}}\|^2 = \sum_{i,j}(\tilde{R}_{ij})^2$ and $\tilde{\mathrm{Ric}}$ is the traceless part of Ric. Substituting (43) and (44) into (41) immediately gives

$$\tfrac{1}{2}\Delta S = \|\bar{\nabla}^{\xi}h\|^2 - \|W\|^2 - \tfrac{n+2}{n-2}\|\tilde{\mathrm{Ric}}\|^2 + (n+1)S\chi, \tag{45}$$

where we used the relation $n(n-1)\chi = n(n-1) - S$.

Now, as M^n is conformally flat and χ is constant, we can derive from (45) that

$$\begin{aligned} 0 = &\ \|\bar{\nabla}^{\xi}h\|^2 - \tfrac{n+2}{n-2}\|\tilde{\mathrm{Ric}}\|^2 + (n+1)S\chi \\ \geq &\ -\tfrac{n+2}{n-2}\|\tilde{\mathrm{Ric}}\|^2 + (n+1)S\chi, \end{aligned} \tag{46}$$

by which we then obtain (11) and find that the equality holds identically if and only if $\bar{\nabla}^{\xi}h = 0$ on M^n, i.e., M^n is of C-parallel second fundamental form. Finally, applying Theorem 1.3 of Li-Xing-Yin [18], we can conclude that M^n is locally congruent to one of the examples (i)–(iii). This completes the proof of Theorem I.3. □

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