Cotton Tensor on Para Kenmotsu 3-Manifold Admitting Etta-Ricci Solitons

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Abstract **–The aim of this paper is to examine certain properties of Cotton tensor on a para-Kenmotsu manifold of dimension 3 admitting η-Ricci solitons. The concept of Cotton pseudo-symmetric manifolds is also introduced in this paper. Some results based on Cotton flat para-Kenmotsu manifolds that admits η-Ricci solitons are also derived. Furthermore, it is investigated whether some geometric properties of the Cotton tensor do not exist in para-Kenmotsu 3-manifolds. Besides these, we studied Codazzi type of Cotton tensor. Some applications of Cotton tensor in fluid space time are also discussed in the paper.**

Index Terms**--Cotton tensor, η-Ricci soliton, para-Kenmotsu manifold, Codazzi type.**

1. INTRODUCTION

FIRST introduced by Sato [31] in 1976, the idea of para-contact manifolds. Subsequently Takahashi [32] defined Almost contact manifolds that are associated with a pseudo-Riemannian metric. In 1985, the concept of Almost para-contact structure that has been demonstrated by Kaneyuki and Williams on pseudo-Riemannian manifolds of dimension $2n+1$. Concept of para-Kenmotsu manifolds first presented by Welyczko [35]. Subsequent research on Para-Kenmotsu manifolds and their special variants has been conducted by Blaga [6]. Additionally, the study of $(LCS)_{2n+1}$-manifolds was undertaken by S.K. Yadav, D. L. Suthat and others [7], [12], and [14]. Hamilton in 1982 introduced that, Ricci flow turned out to be an effective method of simplifying the structure manifolds [15] and is defined as

$$\frac{\partial \varpi}{\partial t} = -2\varsigma,$$

In this context, ϖ represents a Riemannian metric, Ricci curvature tensor is represented by ς, time is signified by t.

A Ricci soliton serves as one of the natural extension of Einstein metric. A structure (ϖ, V, λ) of Ricci soliton is characterized in manifold (M, ϖ), which is pseudo-Riemannian, by

$$(\mathcal{L}_V\varpi)(A,B) + 2\varsigma(A,B) + 2\lambda\varpi(A,B) = 0, \quad (1)$$

where, ς represents the Ricci tensor associated with ϖ, a Riemannian metric, the Lie derivative of ϖ is indicated by $\mathcal{L}_V\varpi$ along with the V vector field, a constant λ, A and B are some vector fields in the manifold M.

Manuscript received August 25, 2024; revised June 18, 2025.

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Ricci solitons has been classified as steady, shrinking and expanding based on whether λ = zero, negative or positive correspondingly. Its links to string theory have recently attracted the attention of theoretical physicists. The Ricci flow is powerful tool for studying some geometrical and topological concepts of Riemannian manifolds. A Ricci soliton is recognized as self-similar soliton within the context of the Ricci flow. Further, Ricci solitons are studied broadly in frame of reference of pseudo-Riemannian Geometry. In various contexts on Kahler manifolds, α-Sasakian manifolds on contact and Lorentzian concircular structure manifolds, in Sasakian manifolds, in k-contact manifolds, in Kenmotsu manifolds, in f-Kenmotsu manifolds, in LP-Sasakian manifolds, in para-Kenmotsu manifolds, etc.

U. C. De obtained findings related to ϕ-recurrent Kenmotsu manifolds [10]. C.S. Bagewadi, S. R. Ashoka and G. Ingalahalli contributed to derivations of certain specific results [1] concerning Ricci soliton with α-Sasakian manifold. Additionally M. Crasmareanu and C. L. Bejan [3] established a connection between Ricci solitons and quasi-constant curvature.

A broader concept of Ricci soliton i.e an η-Ricci soliton is indicated by Kimura and Cho [9] is defined as a structure $(\varpi, V, \lambda, \mu)$

$$(\mathcal{L}_V\varpi)(A,B) + 2\varsigma(A,B) + 2\lambda\varpi(A,B) + 2\mu\eta(A)\eta(B) = 0, \quad (2)$$

where, V on manifold M is used for vector filed, constants are λ and μ and Riemannian (or pseudo-Riemannian) metric is denoted by ϖ.

Numerous generalizations of Ricci solitons exists, including almost Yamabe Ricci soliton, h-almost Yamabe Ricci soliton, $*$-Ricci solitons, Yamabe Ricci soliton, among others. A novel type of soliton referred to as the h-almost Yamabe Ricci soliton, expands upon the concept of almost Yamabe Ricci soliton was presented by C. Ozgur, A. Sardar and U. C. De [11]. Additionally, C. R . Premalatha and H. G. Nagaraja [22] have presented findings on Ricci solitons. Related to para Kenmotsu manifolds, which pave the ways to research scholars to explore new topics and derive further results in various other manifolds. Idea of almost Ricci soliton is recently extended to h-almost Ricci soliton in complete Riemannian manifold by Gomes et al. [13]. Findings on Ricci solitons are recently derived in Contact metric manifolds by Pankaj Pandey and Kamakshi Sharma [24] in LP Sasakian manifold and some other results in generalized form of Z-Ricci soliton, on weakly cyclic Z-symmetric generalized manifolds and on Lorentzian Complex Space Form and Ricci

Yamabe soliton and its cosmological models are also presented by Pankaj Pandey, BB Chaturvedi and K. Sharma [25], [26], [27], [28], [29] and [30] that form the basis for deriving new generalizations of Ricci soliton in different types of manifolds in Differential geometry. Researchers have discovered new examples of Ricci solitons in various dimensions and geometries.

A unique form of conformal invariant, known as the cotton tensor in differential geometry can be characterized by the condition that a Weyl's conformal curvature tensor is zero in pseudo-Riemannian manifold of three dimensions. A Cotton tensor $\mathfrak{C}$, that is a tensor of type (1, 2) described by

$$\mathfrak{C}(A,B) = (\nabla_A Q)B - (\nabla_B Q)A - \frac{1}{4}\{(Ar)B - (Br)A\}, \quad (3)$$

for smooth vector fields A and B.

Eisenhart proposed that in pseudo-Riemannian manifold of three-dimensions Cotton tensor becomes zero conformally flat metric. In recent years, numerous authors have found the Cotton tensor in three-dimensional spaces to be a compelling subject, particularly because of its connection with energy-momentum tensor of matter and Ricci tensor within its Einstein's equations. Pradip Majhi and Debabrata Kar [20] applied some elementary results to find some results based on Cotton tensor that accommodates η-Ricci soliton within Sasakian 3-manifolds.

II. PRELIMINARIES

Definition 2.1 Consider (M^n, ϖ), as a smooth manifold of dimension three that possesses almost para-contact metric structure characterized by components $(\phi, \xi, \eta, \varpi)$. Here, ϕ represents type (1, 1) tensor, a vaetor field is denoted by ξ, one form is represented by η and symbol ϖ is a pseudo-Riemannian metric that satisfies:

$$\phi^2 A = A - \eta(A)\xi, \qquad \eta(\xi) = 1,$$

$$\phi\xi = 0, \qquad \eta(\phi A) = 0, \quad (4)$$

$$\varpi(\phi A, \phi B) = -\varpi(A,B) + \eta(A)\eta(B), \quad (5)$$

$$\varpi(A,\xi) = \eta(A), \quad (6)$$

$$\varpi(\phi A, B) = -\varpi(A, \phi B) \quad (7)$$

for all vector fields $A, B \in TM^n$. Then this manifold denoted by (M^n, ϖ) described by Para-contact metric manifold.

When the following equation is satisfied by almost para-contact metric manifold

$$(\nabla_A \phi)B = \varpi(A,B)\xi - \eta(B)\phi A,$$

for all vector fields $A, B \in TM^n$, then (M^n, ϖ) is referred to as almost para-Kenmotsu manifold.

Standard nearly para-Kenmotsu manifold, known as para-Kenmotsu manifold. A framework of para-Kenmotsu structure with dimension three normal structure has been presented by Welyczko [35]. Results of preceding equation helps us to obtain

$$\nabla_A \xi = A - \eta(A)\xi \text{ and}$$

$$(\nabla_A \eta)B = \varpi(A,B) - \eta(A)\eta(B) \quad (8)$$

In para-Kenmotsu manifold following relations holds:

$$R(A,B,C,W) = \varpi(A,C)\varpi(B,W) - \varpi(B,C)\varpi(A,W),$$

$$R(\xi,A)B = \{-\varpi(A,B)\xi + \eta(B)A\}, \quad (9)$$

$$R(\xi,A)\xi = \{-\eta(B)\xi + B\},$$

$$\varsigma(A,B) = -(n-1)\varpi(A,B),$$

$$\varsigma(\xi,\xi) = -(n-1),$$

$$QA = -(n-1)A.$$

Definition 2.2 Defining curvature tensor of a Riemannian manifold in 3-dimensions by

$$R(A,B)C = [\varsigma(B,C)A - \varsigma(A,C)B + \varpi(B,C)QA - \varpi(A,C)QB] - \frac{r}{2}[\varpi(B,C)A - \varpi(A,C)B], \quad (10)$$

where, the symbol Q is used for Ricci operator and is defined by

$$\varpi(QA,B) = \varsigma(A,B).$$

A para-Kenmotsu three-dimensional manifold's Ricci tensor is provided by

$$\varsigma(A,B) = \frac{1}{2}[(r+2)\varpi(A,B) - (r+6)\eta(A)\eta(B). \quad (11)$$

Definition 2.3 Within the context of Riemannian n-dimensional manifold, a concircular curvature tensor categorized as type (1, 3), is defined as follows:

$$\mathcal{Z}(A,B)C = R(A,B)C - \frac{r}{n(n-1)}[\varpi(B,C)A - \varpi(A,C)B]. \quad (12)$$

In the realm of three dimensional Riemannian manifolds, the formulation of concircular curvature tensor is represented by:

$$\mathcal{Z}(A,B)C = R(A,B)C - \frac{r}{6}[\varpi(B,C)A - \varpi(A,C)B]. \quad (13)$$

Definition 2.4 A Riemannian curvature tensor is labeled as concircularly flat when the concircular curvature tensor is null. Considering (M, ϖ), a Riemannian manifold along with its associated connection ∇ (Levi-Civita connection). If equation $\nabla R = 0$, is satisfied, then Riemannian manifold is identified as locally symmetric, where R is denoting Riemannian curvature tensor in manifold (M, ϖ). Moreover on semi-Riemannian curvature, Riemannian metric manifold (M^n, ϖ), where n is greater than or equal to 3 termed as semisymmetric if it meets the criterion $R.R = 0$. Metric of semi-Riemannian curvature manifold $(M^n, \varpi), n \geq 3$ called Ricci semi-symmetric when it fulfills condition $R.\varsigma = 0$ on the manifold M.

Definition 2.5 Another classification of vector field V as conformal killing vector field when relationship given below is satisfied:

$$\mathcal{L}_V \varpi(A,B) = 2\Omega\varpi(A,B)$$

Here, Ω in the coordinates denotes a function known as a conformal scalar. A conformal killing vector field V referred to as proper when Ω isn't constant. In contrast, if the value of Ω have been constant, V recognized as a homothetic vector field. When non zero value is taken by constant Ω, V is then called a proper homothetic vector field. Furthermore when Ω is equal to zero in equation provided, V is classified as a killing vector field.

III. PROPERTIES OF COTTON TENSOR

This section focuses on a skew-symmetric (1, 2) type tensor within the context of para-Kenmotsu 3-manifold, known as a Cotton tensor $\mathfrak{C}$, might be represented as

$$\mathfrak{C}(A,B) = (\nabla_A Q)B - (\nabla_B Q)A - \frac{1}{4}\{(Ar)B - (Br)A\}, \quad (14)$$

$$\mathfrak{C}(A,B) = (\mu - 1)[\varpi(\phi A, \phi B)\xi + A\eta(B) - \eta(A)\eta(B)\xi] - \frac{1}{4}[(Ar)B - (Br)A]. \quad (15)$$

$\mathfrak{C}(A,B) = (\mu - 1)\{-\varpi(A,B) + A\eta(B)\} - \frac{1}{4}\{(Ar)B - (Br)A\}.$ (16)

Furthermore, as seen below, Cotton tensor may additionally be written as as a tensor of (0, 3) type

$\mathfrak{C}(A,B,C) = \varpi(\mathfrak{C}(A,B),C).$ (17)

By virtue of equations (3.2) and (3.4), it gives us

$\mathfrak{C}(A,B,C) = (\mu - 1)\{\varpi(\phi A,\phi B)\eta(C) + \eta(B)\varpi(A,C) - \eta(A)\eta(B)\varpi(\xi,C)\}.$ (18)

As a consequence of (15), we get

$\mathfrak{C}(A,\xi) = (\mu - 1)(A - \eta(A)\xi - \frac{1}{4}(Ar)\xi.$ (19)

$\eta\big(\mathfrak{C}(A,B)\big) = (\mu - 1)[\varpi(\phi A,\phi B) - \frac{1}{4}\{(Ar)\eta(B) - (Br)\eta(A)\},$ (20)

Taking $B = \xi$ in (20), we get the following equation

$\eta\big(\mathfrak{C}(A,\xi)\big) = -\frac{1}{4}\,(Ar).$ (21)

equation (15) also implies

$\mathfrak{C}(\phi A,B) = (\mu - 1)\{\varpi(A,\phi B)\xi + \phi A\eta(B)\} - \frac{1}{4}\,\{\big((\phi A)r\big)B - (Br)\phi A\big\}.$ (22)

$\eta(\mathfrak{C}(\phi A,B)) = (\mu - 1)\{\varpi(A,\phi B)\eta(\xi) + \eta(\phi A)\eta(B)\} - \frac{1}{4}\,\{\big((\phi A)r\big)\eta(B) - (Br)\eta(\phi A)\big\}.$ (23)

$\eta\big(\mathfrak{C}(\phi A,B)\big) = (\mu - 1)\{\varpi(A,\phi B)\} - \frac{1}{4}\,\{\big((\phi A)r\big)\eta(\phi B).$

$\eta\big(\mathfrak{C}(\phi A,\phi B)\big) = (\mu - 1)\varpi(A,\phi B).$ (24)

$\eta\big(\mathfrak{C}(\phi A,B)\big) = (\mu - 1)\eta(\phi A)\eta(\xi) - \frac{1}{4}\,\{\big((\phi A)r\big)\eta(\xi) - (\xi r)\eta(\phi A).$

$\eta\big(\mathfrak{C}(\phi A,\xi)\big) = -\frac{1}{4}(\phi A)r.$ (25)

IV. MAIN RESULTS

This section focuses on analysis of Cotton flat para-Kenmotsu manifold that supports η-Ricci solitons. As a result for the vector fields A, B and C, we have

$\mathfrak{C}(A,B,C) = 0.$ (26)

Equation (26) and (18) implies that

$(\mu - 1)\{\varpi(\phi A,\phi B)\eta(C) + \eta B\varpi(A,C) - \eta(A)\eta(B)\eta(C)\} - \frac{1}{4}\{(Ar)\varpi(B,C) - (Br)\varpi(A,C)\} = 0.$ (27)

Replacing C by ξ in above equation, we get

$(\mu - 1)\varpi(\phi A,\phi\xi) = \frac{1}{4}\,\{(Ar)\eta(\xi) - (\xi r)\eta(A)\}.$ (28)

$0 = \frac{1}{4}(Ar)$ which implies

$Ar = 0\,.$

Hence, r becomes constant.

From (4.3) we get

$(\mu - 1)\varpi(\phi A,\phi B) = 0,$

which implies $\mu = 1$.

Lemma 4.1 *Th structure of Ricci tensor in para-Kenmotsu manifold of three dimension that possess an η-Ricci soliton is is given as follows:*

$\varsigma(A,B) = -(1 + \lambda)\varpi(A,B) - (\mu - 1)\eta(A)\eta(B).$ (29)

As a consequence, above lemma implies

$\varsigma(A,\xi) = \varsigma(\xi,A) = -(\mu + \lambda)\eta(A).$

Reference [21] addresses a para-contact manifold situated in a (2n+1)-dimensional framework.

$\varsigma(A,\xi) = -(\dim(M) - 1)\eta(A) = -2n\eta(A),$

for 3-dimensional para-Kenmotsu manifold, we get

$\lambda + \mu = 6\,.$ (30)

using $\mu = 1$ in (30) we get

$\lambda = 5.$ (31)

Hence, following theorem can be stated:

Theorem 4.1: *An η-Ricci soliton associated with cotton flat three dimensional para-Kenmotsu manifold is classified as expanding soliton.*

By contracting (29), we get

$\varsigma(\aleph,\ \aleph\,) = -(\lambda + 1)\varpi(\aleph,\aleph) - (\mu - 1)\eta(\aleph)\eta(\aleph).$ (32)

$r = -3\lambda - 2 - \mu.$ (33)

Using $\mu = 1$ in (31) and (32) we get

$r = -18.$

As a result, the following theorem can be declared:

Theorem 4.2*: Three dimensional para-Kenmotsu manifold which was Cotton flat also and supports an η-Ricci soliton, exhibits a scalar curvature -18.*

Theorem 4.3 A three dimensional Cotton flat para-Kenmotsu manifold that accommodates η-Ricci soliton is recognized as η-Einstein manifold.

Proof: It is known that Ricci tensor of 3-dimensional para-Kenmotsu manifold is given by

$\varsigma(A,B) = \frac{1}{2}\{(r + 2)\varpi(A,B) - (r + 6)\eta(A)\eta(B)\}.$ (34)

as a consequence of above equation and by using $r = -18$, we get

$\varsigma(A,B) = \frac{1}{2}\{-16\varpi(A,B) + 12\eta(A)\eta(B)\},$

which implies

$\varsigma(A,B) = -8\varpi(A,B) + 6\eta(A)\eta(B).$ (35)

Hence, the result is proved.

V. CODAZZI TYPE

Here, we analyze η-Ricci soliton within Cotton para-Kenmotsu manifold which feature a Ricci tensor of Codazzi type.

Theorem 5.1 *Cotton tensor in Codazzi type of η-Ricci soliton in para-Kenmotsu manifold satisfies following relation in three dimensions.*

$\mathfrak{C}(A,B) = -\frac{1}{4}\{(Ar)B - (Br)A\}$.

Proof: A para-Kenmotsu manifold can be defined as having a Codazzi kind of Ricci tensor, if its Ricci tensor ς of type(0, 2) is not equal to zero that adheres to the following stipulation:

$$(\nabla_C\varsigma)(A,B) = (\nabla_A\varsigma)(B,C), \tag{36}$$

For vector fields A, B, C in M.

Computing covariant derivative of (29), we get

$$(\nabla_C\varsigma)(A,B) = (1-\mu)[(\nabla_C\eta)A\eta(B) - \eta(A)(\nabla_C\eta)B]. \tag{37}$$

using (8) in (37), we get

$$(\nabla_C\varsigma)(A,B) = (1-\mu)\{(\varpi(A,C) - \eta(A)\eta(C))\eta(B) - \eta(A)\big(\varpi(B,C) - \eta(B)\eta(C)\big)\}. \tag{38}$$

which on simplifying implies

$$(\nabla_C\varsigma)(A,B) = (1-\mu)\{(\varpi(A,C)\eta(B) - \eta(A)\varpi(B,C)\}. \tag{39}$$

In view of (36), (39) reduces to

$$(\nabla_C\varsigma)(A,B) - (\nabla_A\varsigma)(B,C) = (1-\mu)\{2\varpi(A,C)\eta(B) - \eta(A)\varpi(B,C) - \eta(C)\varpi(A,B)\}. \tag{40}$$

Taking $C = \xi$ in (40) and using (36), we get

$$(1-\mu)\{2\varpi(A,\xi)\eta(B) - \varpi(B,\xi)\eta(A) - \varpi(A,B)\eta(\xi)\} = 0. \tag{41}$$

$$(1-\mu)\{-\varpi(A,B) + \eta(A)\eta(B)\} = 0.$$

using (4) in (41), we get

$$(1-\mu)\varpi(\phi A, \phi B) = 0, \text{ which implies } \mu = 1. \tag{42}$$

In view of (41) and (16), Cotton tensor reduces to

$$\mathfrak{C}(A,B) = -\frac{1}{4}\{(Ar)B - (Br)A\}. \tag{43}$$

Corollary 5.1*: Codazzi Tensor in Cotton flat η-Ricci soliton in para-Kenmotsu manifold if provided that it meets the following criteria:*

$(Ar)B = (Br)A$.

equation (28) and (42) also implies the following preposition:

Proposition 5.1 A para-Kenmotsu manifold admitting an η-Ricci soliton has a Cotton tensor that vanishes iff it is of Codazzi type.

Definition 5.2 A para-Kenmotsu manifold is considered to exhibit a Ricci tensor if its Ricci tensor ς which is cyclic parallel, categorized as (0, 2) type, is not zero and adheres to the below stipulation:

$$(\nabla_A\varsigma)(B,C) + (\nabla_B\varsigma)(C,A) + (\nabla_C\varsigma)(A,B) = 0,$$

for all vector fields A, B, C on M.

Theorem 5.2 The condition of cyclic parallel Ricci tensor is satisfied by Cotton tensor in η-Ricci soliton admitting para Kenmotsu manifold.

Proof: Assume $(\varpi, \xi, \lambda, \mu)$ is an η-Ricci soliton structure situated in an n-dimensional para Kenmotsu manifold with cyclic parallel Ricci tensor, consequently, definition (5.2) is satisfied.

Calculating the covariant derivative of (29) and using (4), we get

$$(\nabla_C\varsigma)(A,B) = (1-\mu)\{\eta(B)\varpi(A,C) - \eta(A)\varpi(B,C)\}. \tag{44}$$

Similarly, by taking covariant derivative of (29) w.r.t A and B and using (4), we get

$$(\nabla_A\varsigma)(A,B) = (1-\mu)\{\eta(C)\varpi(A,B) - \eta(B)\varpi(A,C)\} \tag{45}$$

$$(\nabla_B\varsigma)(A,B) = (1-\mu)\{\eta(A)\varpi(B,C) - \eta(C)\varpi(A,B)\}. \tag{46}$$

adding (44), (46) and (49), we get

$$(\nabla_A\varsigma)(B,C) + (\nabla_B\varsigma)(C,A) + (\nabla_C\varsigma)(A,B) = 0.$$

Thus, a para-Kenmotsu manifold admitting Cotton tensor in η-Ricci soliton is cyclic parallel Ricci tensor.

Example: 5.1 Considering a manifold with dimension three $M = \{(\mathfrak{A}, \mathfrak{Q}, \mathfrak{T}) \in \mathbb{R}^3, \mathfrak{T} \neq 0\}$, where$(\mathfrak{A}, \mathfrak{Q}, \mathfrak{T})$ is the coordinates in $\mathbb{R}^3$, which is standard Let $(\mathfrak{h}, \mathfrak{s}, \mathfrak{K})$ is the linearly independent vector fields at each point on M, given by $\mathfrak{h} = \mathfrak{T}\frac{\partial}{\partial\mathfrak{A}}$, $\mathfrak{s} = \mathfrak{T}\frac{\partial}{\partial\mathfrak{Q}}$, $\mathfrak{K} = -\mathfrak{T}\frac{\partial}{\partial\mathfrak{T}}$ and hence, it forms a basis for tangent space $T(M^3)$. We also define the Riemannian metric ϖ of the manifold M^3as

$$\varpi = \frac{1}{\mathfrak{T}^2}[d\mathfrak{A} \otimes d\mathfrak{A} + d\mathfrak{Q} \otimes d\mathfrak{Q} + d\mathfrak{T} \otimes d\mathfrak{T}].$$

Assuming η as a 1-form which can be defined as

$$\eta(A') = \varpi(A', \mathfrak{K})$$

For vector field $A' \in \Gamma(TM)$ and consider a tensor field of (1, 1) type, ϕ, defined by

$\phi\mathfrak{h} = -\mathfrak{s}$

$\phi\mathfrak{s} = -\mathfrak{h}$,

$\phi\mathfrak{K} = 0$.

By linear property satisfied by ϕ along with ϖ, following relationships can be easily verified:

$\eta(\mathfrak{K}) = 1$

$$\phi^2(A') = -A' + \eta(A')\mathfrak{K}.$$

$\varpi(\phi A', \phi B') = \varpi(A', B') - \eta(A')\eta(B')$.
For vector fields $A', B' \in T(M^3)$.
When Riemannian metric ϖ is associated with the Levi-Civita connection ∇, we can easily calculate that the following formulations:
$[\mathfrak{h}, \mathfrak{s}] = 0, \quad [\mathfrak{h}, \mathfrak{K}] = \mathfrak{h}, \quad [\mathfrak{s}, \mathfrak{K}] = \mathfrak{s}$.
Now we can recall Koszul's formula as

$$2\varpi(\nabla_{A'}B', C') = A'(\varpi(B', C')) + B'(\varpi(C', D')) - C'(\varpi(A', B')) - \varpi(A', [B', C']) - \varpi(B', [A', C']) + \varpi(C', (A', B')).$$

for arbitrary vector fields $A', B', C' \in T(M^3)$.
Using formula developed by Koszul, we obtained the following:
$\nabla_{\mathfrak{s}}\mathfrak{K} = \mathfrak{s}, \quad \nabla_{\mathfrak{s}}\mathfrak{s} = -\mathfrak{K}, \quad \nabla_{\mathfrak{s}}\mathfrak{h} = 0,$
$\nabla_{\mathfrak{K}}\mathfrak{K} = 0, \quad \nabla_{\mathfrak{K}}\mathfrak{s} = 0, \quad \nabla_{\mathfrak{K}}\mathfrak{h} = 0,$
$\nabla_{\mathfrak{h}}\mathfrak{K} = \mathfrak{h}, \quad \nabla_{\mathfrak{h}}\mathfrak{s} = 0, \quad \nabla_{e_1}\mathfrak{h} = -\mathfrak{K},$
Thus above calculation shows that (M, ϖ) is a para Kenmotsu manifold with cotton tensor admitting η-Ricci soliton.

VI. APPLICATIONS

The Cotton tensor is used in the study of perfect fluid space-times in a number of ways, including:
(i) The Cotton tensor is used in Cotton gravity to specify the energy- momentum tensor. The Bertotti- Robinson, Nariai and Stephani, and space-times are the examples of space-times that involve cotton gravity.
(ii) A perfect fluid solution which is conformally flat, of Einstein's field equations can also be derived by using Cotton tensor in three dimensions.
(iii) In Einstein's theory, the Cotton tensor is used to analyze the relationship between the Cotton tensor and the energy momentum
(iv) Cotton tensor is used to modify Einstein's equation resulting in the equation of Cotton gravity.

VII. SIGNIFICANCE OF THE WORK

We can illustrate some significances of our research work in following points:

- Obtaining the condition of Cotton flat para-Kenmotsu manifold that supports η-Ricci soliton, where r symbolizes Ricci tensor, becomes constant.
- Obtaining some properties of Cotton tensor that has relation with η-Ricci soliton in para Kenmotsu manifold in section (iii)
- Relation between Cotton flat para-Kenmotsu manifold associated with η-Ricci soliton and an η-Einstein manifold in three dimensions is derived in the paper.
- The relation between the Ricci tensor and energy momentum tensor of matter can be restricted by Cotton tensor in the Einstein equations and it also plays a significant role in the Hamiltonian formalism in general relativity.

VIII. CONCLUSION

Cotton tensors have a peculiar importance in geometry. It can be extensively applicable in Cotton gravity recently proposed by Harada. We also conclude that a cotton tensor is a third-order tensor in differential geometry that appears in the identities given by Bianchi and can be used to study fluid space time. The necessary and sufficient condition we investigated for a manifold to be locally conformal flat in three dimensions, is a vanishing cotton tensor. But in higher dimensions, vanishing of the Cotton tensor is not sufficient condition, only necessary.

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