Solvability of a Resonant Hadamard Fractional Boundary Value Problem with Infinite Point Boundary Conditions

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***Abstract*—In this article, conditions for the existence of solution for a Hadamard fractional boundary problem with infinite point boundary value conditions at resonance will be established using the coincidence degree theorem due to Mawhin.**

***Index Terms*—Coincidence degree, Hadamard fractional derivative, infinite point, resonance.**

I. Introduction

In this research article, we study the following Hadamard fractional boundary value problem (HFBV) with infinite point boundary condition

$$ {}^{H}D_{1^+}^{\alpha}u(t) = w\left(t, u(t)\right),\ t \in [1, e], \tag{1}$$

$$\begin{aligned} &u(1) = {}^{H} D_{1^+}^{\alpha-2}u(1) = \cdots = {}^{H} D_{1^+}^{\alpha-(M-1)}u(1) = 0, \\ &{}^{H}I_{1^+}^{2-\alpha}u(e) = \sum_{i=1}^{\infty} \beta_i {}^{H}I_{1^+}^{\delta}u(\zeta_i), \end{aligned} \tag{2}$$

where ${}^{H}D_{1^+}^{\alpha}$, is the Hadamard fractional derivatives of order α while ${}^{H}I_{1^+}^{\alpha}$ is the Hadamard fractional integral, $M-1 < \alpha \le M$, $1 < \zeta_1 < \zeta_2 < \cdots < \zeta_{i-1} < \zeta_i < \cdots < e$, $0 \le \delta \le 1$, $\beta_i \in \mathbb{R}$, $\{\zeta_i\}_{i=1}^{\infty}$ is a monotone increasing sequence with $\lim_{i\to\infty} \zeta_i = v$, $v \in [1, e]$, $\beta_i \in \mathbb{R}$, and $w : [1, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

Recently, there has been increased attention paid to research on fractional differential equations as a result of their wide application in modeling of various processes in different fields such as in chemistry, physics, biology, mechanical engineering, economics, biology, system control, etc [4], [5]. There has also been a significant development of the theory of fractional differential equations. Many researchers have studied the problem of existence of solutions, oscillation properties, stability analysis and other qualitative properties of different types of differential equations. [6]–[13], [15], [17], [18].

Guo et al [13] studied an infinite point fractional boundary value problem

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) + f(t, u(t), u(t)) = 0, \quad t \in (0,1), \\ u(0) = u''(0) = 0,\ u'(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j), \end{cases}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order α, $2 < \alpha \le 3$, $\eta_j \ge 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$. The authors obtained existence results using the Avery-Peterson's fixed point theorem.

In [14], the authors considered the following fractional differential equation with infinite point boundary conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t, u(t)), \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α, $\alpha > 2$, $n-1 < \alpha \le n$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \ge 0, 0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ $(j = 1, 2, \dots)$. They used fixed-point theorem in a cone to obtain existence and uniqueness results.

Bohner et al. [16] considered the following Hadamard fractional differential equation at resonance

$$\begin{cases} (-{}^{H}D^{\gamma}u)(t) = f(t, u(t)), \quad t \in (1, e), \\ u(1) = 0,\ u(e) = \int_1^e u(t)dA, \end{cases}$$

where ${}^{H}D^{\gamma}$ is the Hadamard fractional derivative of order γ, $1 < \gamma \le 2$, $f : [1, e] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Caratheodory conditions and $\int_1^e u(t)dA$ is the Riemann-Stielejes integration. They obtained existence results using the coincidence degree theory.

Although many researchers have considered Hadamard fractional differential equations, the Hadamard fractional differential equations with infinite point boundary conditions have nor received much attention in literature. Motivated by the above results, we study the solvability for a Hadamard fractional differential equation with infinite point boundary value conditions. The rest of this article is organized as follows: in Section 2 of this work, required lemmas, theorem and definitions will be presented, Section 3 contains conditions for existence of solution. An example will be given in Section 4 to demonstrate the results obtained.

The HFBV $(1) - (2)$ is said to be at resonance because its associated homogeneous boundary value problem

$$\begin{gathered} {}^{H}D_{1^+}^{\alpha}u(t) = 0,\ t \in [1, e], \\ u(1) = {}^{H} D_{1^+}^{\alpha-2}u(1) = \cdots = {}^{H} D_{1^+}^{\alpha-(M-1)}u(1) = 0, \\ {}^{H}I_{1^+}^{2-\alpha}u(e) = \sum_{i=1}^{\infty} \beta_i {}^{H}I_{1^+}^{\delta}u(\zeta_i), \end{gathered}$$

has a nontrivial solution $u(t) = c(\log t)^{\alpha-1}$.

We assume the following throughout this paper:

(A_1) $\displaystyle\sum_{i=1}^{\infty} \frac{\beta_i(\log \zeta_i)^{\delta+\alpha-2}}{\Gamma(\delta+\alpha)} = 1;$

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(A_2) there exists functions $\sigma,\ \varphi_i \in G,\ i = 1, \cdots, M-1$ such that $\forall u \in \mathbb{R}^2,\ t \in [1, e]$

$$|w(t, u(t))| \le \sigma + |u| \sum_{i=1}^{M-1} \varphi_i; \tag{3}$$

(A_3) there exists a constant $B > 0$ such that for $u \in \text{dom } L$, if $\left|{}^H D_{1^+}^{\alpha-1} u(t)\right| \ge B,\ \forall\ t \in [1, e]$, then

$$\begin{aligned}&\int_1^e (1-\log s)\, w(s, u(s)) \frac{ds}{s} - \sum_{i=1}^{\infty} \frac{\beta_i}{\Gamma(\alpha+\delta)} \\ &\quad \times \int_1^{\zeta_i} (\log \zeta_i - \log s)^{\alpha+\delta-1} w(s, u(s)) \frac{ds}{s} \ne 0;\end{aligned} \tag{4}$$

(A_4) there exists $M_1 > 0$ such that for any $u(t) = c_0(\log t)^{\alpha-1} \in \ker L$ with $|c_0| > M_1$, either

$$\begin{aligned}&c_0 \left[\int_1^e (1-\log s) w\left(s, c_0(\log t)^{\alpha-1}\right) \frac{ds}{s} - \sum_{i=1}^{\infty} \frac{\beta_i}{\Gamma(\alpha+\delta)} \right. \\ &\left. \times \int_1^{\zeta_i} (\log \zeta_i - \log s)^{\alpha+\delta-1} w\left(s, c_0(\log t)^{\alpha-1},\right) \frac{ds}{s} \right] < 0,\end{aligned} \tag{5}$$

or

$$\begin{aligned}&c_0 \left[\int_1^e (1-\log s) w\left(s, c_0(\log t)^{\alpha-1}\right) \frac{ds}{s} - \sum_{i=1}^{\infty} \frac{\beta_i}{\Gamma(\alpha+\delta)} \right. \\ &\left. \times \int_1^{\zeta_i} (\log \zeta_i - \log s)^{\alpha+\delta-1} w\left(s, c_0(\log t)^{\alpha-1},\right) \frac{ds}{s} \right] > 0,\end{aligned} \tag{6}$$

then the HFBV $(1)-(2)$ has at least one solution in X if

$$1 - 2\left(\sum_{i=1}^{M-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)} \right) \sum_{i=1}^{M-1} \|\varphi_i\|_1 > 0. \tag{7}$$

II. Preliminaries

In this section, we will give definitions, lemmas and theorems that will be used in this work. Let us start with some definitions relating to fractional calculus.

Definition 1. [1] The nth Hadamard fractional order derivative of a function $w : (1, +\infty) \to \mathbb{R}$ is defined by

$$\begin{aligned}&{}^H D_{1^+}^n w(t) \\ &= \frac{1}{\Gamma(a-n)} \left(t \frac{d}{dt}\right)^a \int_1^t (\log t - \log s)^{a-n-1} \frac{w(s)}{s} ds,\end{aligned} \tag{8}$$

where $n > 0$, $n = [a] + 1$, and $[a]$ denotes the largest integer which is less than or equal to a. Similarly, the nth Hadamard fractional order integral of a function $w : (1, +\infty) \to \mathbb{R}$ is defined by

$${}^H I_{1^+}^n w(t) = \frac{1}{\Gamma(n)} \int_1^t (\log t - \log s)^{n-1} \frac{w(s)}{s} ds. \tag{9}$$

Definition 2. [1] Let $n > 0$, $n = [a] + 1$, then

$${}^H I_{1^+}^n {}^H D_{1^+}^n w(t) = w(t) + \sum_{j=1}^{n} c_j (\log t)^{n-j}. \tag{10}$$

Also, if ${}^H D_{1^+}^n w(t) = 0$, then

$$w(t) = \sum_{j=1}^{n} c_j (\log t)^{n-j}, \tag{11}$$

where $c_j = 1, 2, \ldots, n$ are some real numbers and $n - 1 < j < n$.

Definition 3. [2] Given $a, \alpha, \beta > 0$, then

$$\begin{aligned}\text{(i)}\ &\left({}^H I_{1^+}^{\alpha} (\log t - \log a)^{\beta-1}\right)(x) \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\log t - \log a)^{\beta+\alpha-1},\end{aligned} \tag{12}$$

$$\text{(ii)}\ {}^H D_{1^+}^{\alpha} {}^H D_{1^+}^{\beta} w(t) = {}^H D_{1^+}^{\alpha+\beta} w(t) \quad \text{where } {}^H D = \frac{d}{dt}. \tag{13}$$

The following are notations about coincidence degree theorem that will be used throughout this work.

Let X and G be real Banach spaces and $L : \text{dom } L \subset X \to G$ be a Fredholm map of index zero. Let the operators $P : X \to X$ and $Q : G \to G$ be continuous projectors such that Im $L = \ker Q$, $\ker P = \text{Im } P$, $X = \ker L \oplus \ker P$, $G = \text{Im } L \oplus \text{Im } Q$. Then, the operator $L|_{\text{dom } L \cap \ker P} : \text{dom } L \cap \ker P \to \text{Im } L$ is invertible and its inverse is denoted by K_p. Let the set $\Omega \subset X$ be open and bounded, then the map N is *L-compact* on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_{P,Q}N = K_P(I-Q)N : \bar{\Omega} \to X$ is compact.

Theorem 1. [3] Let the set $\Omega \subset X$ be bounded and open, $L : \text{dom } L \subset X \to G$ be a Fredholm operator of index zero and $N : X \to G$ be L-compact on $\bar{\Omega}$. If the following conditions are satisfied:

(i) $Lu \ne \lambda Nu$ for every $(u, \lambda) \in [(\text{dom } L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$;
(ii) $Nu \notin \text{Im } L$ for every $u \in \ker L \cap \partial\Omega$;
(iii) $\deg(QN|_{\ker L}, \ker L \cap \Omega, 0) \ne 0$, where $Q : G \to G$ is a projection such that $\ker Q = \text{Im } L$,

then, the operator equation $Lu = Nu$ has at least one solution in dom $L \cap \bar{\Omega}$.

We define the spaces

$$\begin{aligned}X = \{u(t) : u(t), {}^H D_{1^+}^{\alpha-i} u(t) \in C[1, e], \\ i = 1, 2, \ldots, M-1\},\end{aligned}$$

with norm

$$\|u\|_X = \|u\|_\infty + \left\|{}^H D_{1^+}^{\alpha-1} u\right\|_\infty + \cdots + \left\|{}^H D_{1^+}^{\alpha-(M-1)} u\right\|_\infty,$$

where $\|u\|_\infty = \sup_{t \in [1,e]} |u(t)|$ then $(X, \|\cdot\|_X)$ is a Banach space.

Let $G = L^1[1, e]$, with the norm $\|g\|_G = \|g\|_1 = \int_1^t |g(t)| \frac{dt}{ds}$. Then $(G, \|\cdot\|_1)$ is a Banach space.

We define the operator $L : \text{dom } L \subset X \to G$ as

$$Lu = {}^H D_{1^+}^{\alpha} u, \tag{14}$$

where

$$\begin{aligned}\text{dom } L = \Big\{&u \in X : {}^H D_{1^+}^{\alpha} u(t) \in G, \\ &u(1) = {}^H D_{1^+}^{\alpha-2} u(1) = \cdots {}^H D_{1^+}^{\alpha-(M-1)} u(1) = 0, \\ &{}^H I_{1^+}^{2-\alpha} u(e) = \sum_{i=1}^{\infty} \beta_i^H I_{1^+}^{\delta} u(\zeta_i). \Big\}\end{aligned}$$

We also define the operator $N : X \to G$ by

$$Nu = w(t, u(t), {}^H D_{1^+}^{\alpha-1} u(t), {}^H D_{1^+}^{\alpha-2} u(t), \ldots,$$

$$^{H}D_{1^+}^{\alpha-(M-1)}u(t)).$$

Hence, the HFBV $(1)-(2)$ is written in abstract form as $Lu = Nu$.

For easy of computation, for $g \in G$, we define operator A by

$$Ag(t) = \int_1^e (1-\log s)\, g(s)\frac{ds}{s} - \sum_{i=1}^{\infty} \frac{\beta_i}{\Gamma(\alpha+\delta)} \int_1^{\zeta_i} (\log\zeta_i - \log s)^{\alpha+\delta-1} g(s)\frac{ds}{s}.$$

Lemma 1. Let the operator L be as defined in (14), then

$$\ker L = \{u(t) \in \text{dom } L : u(t) = c(\log t)^{\alpha-1}, c \in \mathbb{R}, \forall t \in [1,e]\},$$

and

$$\text{Im } L = \{g \in G : Ag(t) = 0\}.$$

Proof: By (10) and (14), the kernel can easily be obtained as

$$\ker L = \left\{u(t) \in \text{dom } L : u(t) = c(\log t)^{\alpha-1}, c \in \mathbb{R}\right\}.$$

Next, we find Im L. Let

$$g(t) = w(t, u(t), {}^{H}D_{1^+}^{\alpha-1}u(t), {}^{H}D_{1^+}^{\alpha-2}u(t), \dots, {}^{H}D_{1^+}^{\alpha-(M-1)}u(t)),\ t \in [1,e],$$

then (1) can be written as

$$^{H}D_{1^+}^{\alpha}u = g(t) \tag{15}$$

with $u(t)$ as the solution of (15) subject to (2). By (10), we obtain from (14)

$$u(t) = {}^{H}I_{1^+}^{\alpha}g(t) + c_1(\log t)^{\alpha-1} + \dots + c_N(\log t)^{\alpha-M}.$$

By $u(1) = {}^{H}D_{1^+}^{\alpha-2}u(1) = \dots = {}^{H}D_{1^+}^{\alpha-(N-1)}u(1) = 0$, we obtain $c_2 = \cdots c_M = 0$, which implies

$$u(t) = {}^{H}I_{1^+}^{\alpha}g(t) + c_1(\log t)^{\alpha-1}.$$

Let $B(\delta,\alpha)$ be the beta function, then by ${}^{H}I_{1^+}^{2-\alpha}u(e) = \sum_{i=1}^{\infty}\beta_i^{H}I_{1^+}^{\delta}u(\zeta_i)$, we have

$$\begin{aligned}
&{}^{H}I_{1^+}^{2}g(e) + c_1\Gamma(\alpha)\\
&= \sum_{i=1}^{\infty}\beta_i\left({}^{H}I_{1^+}^{\alpha+\delta}g(\zeta_i) + c_1 I_{1^+}^{\delta}(\log\zeta_i)^{\alpha-1}\right)\\
&= \sum_{i=1}^{\infty}\frac{\beta_i}{\Gamma(\alpha+\delta)}\int_1^{\zeta_i}(\log\zeta_i-\log s)^{\alpha+\delta-1}g(s)\frac{ds}{s}\\
&\quad + \sum_{i=1}^{\infty}\frac{\beta_i c_1}{\Gamma(\delta)}\int_1^{\zeta_i}(\log\zeta_i-\log s)^{\delta-1}(\log s)^{\alpha-1}\frac{ds}{s}\\
&= \sum_{i=1}^{\infty}\frac{\beta_i}{\Gamma(\alpha+\delta)}\int_1^{\zeta_i}(\log\zeta_i-\log s)^{\alpha+\delta-1}g(s)\frac{ds}{s}\\
&\quad + \sum_{i=1}^{\infty}\frac{\beta_i c_1(\log\zeta_i)^{\delta+\alpha-2}}{\Gamma(\delta)}\\
&\quad \times \int_1^{\zeta_i}\left(1-\frac{\log s}{\log\zeta_i}\right)^{\delta-1}\left(\frac{\log s}{\log\zeta_i}\right)^{\alpha-1}\frac{ds}{s}\\
&= \sum_{i=1}^{\infty}\frac{\beta_i}{\Gamma(\alpha+\delta)}\int_1^{\zeta_i}(\log\zeta_i-\log s)^{\alpha+\delta-1}g(s)\frac{ds}{s}\\
&\quad + \sum_{i=1}^{\infty}\frac{\beta_i c_1(\log\zeta_i)^{\delta+\alpha-2}}{\Gamma(\delta)}B(\delta,\alpha)\\
&= \sum_{i=1}^{\infty}\frac{\beta_i}{\Gamma(\alpha+\delta)}\int_1^{\zeta_i}(\log\zeta_i-\log s)^{\alpha+\delta-1}g(s)\frac{ds}{s}\\
&\quad + c_1\Gamma(\alpha).
\end{aligned}$$

Therefore,

$$\int_1^e (1-\log s)g(s)\frac{ds}{s} - \sum_{i=1}^{\infty}\frac{\beta_i}{\Gamma(\alpha+\delta)}\int_1^{\zeta_i}(\log\zeta_i-\log s)^{\alpha+\delta-1}g(s)\frac{ds}{s} = 0. \tag{16}$$

On the other hand, suppose (16) holds, let

$$u(t) = {}^{H}I_{1^+}^{\alpha}g(t) + c(\log t)^{\alpha-1}$$

where c is an arbitrary constant, then $u \in$ dom L and $Lu(t) = {}^{H}D_{1^+}^{\alpha}u(t) = g(t)$. Hence, $g(t) \in$ Im L and

$$\text{Im } L = \{g \in G : Ag(t) = 0\}.$$

The proof is concluded. ■

Lemma 2. The operator L : dom $L \subset X \to G$ defined in (14) is a Fredholm mapping of index zero and the linear continuous projectors $P : X \to X$ and $Q : G \to G$ can be defined as

$$Pu(t) = \frac{1}{\Gamma(\alpha)}{}^{H}D_{1^+}^{\alpha-1}u(1)(\log t)^{\alpha-1}, \quad \forall t \in [1,e],$$

$$Qg(t) = Q = dAg(t)$$

where $d = \left(\frac{1}{2} - \sum_{i=1}^{\infty}\frac{\beta_i(\log\zeta_i)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\right)^{-1}$, $g(t) \in G$.

In addition, the linear operator K_P : Im $L \to$ dom $L \cap \ker P$ can be defined as $K_P g(t) = {}^{H}I_{1^+}^{\alpha}g(t)$.

Proof: For any $g(t) \in G$, we have

$$Q^2g(t) = Qg(t)\cdot dA = Qg(t).$$

Let $g_1 = g - Qg$ then

$$Ag_1(t) = Ag(t) - AQ(t) = d^{-1}(Qg(t) - Q^2g(t)) = 0,$$

implying $g_1 \in$ Im L. Hence, $G =$ Im L + Im Q. Since, Im $L \cap$ Im $Q = \{0\}$, we have $G =$ Im $L \oplus$ Im Q. Therefore,

$$\dim\ker L = \dim\text{Im } Q = \text{codim Im } L = 1.$$

Thus, L is a Fredholm operator of index zero.

For any $u \in X$,

$$P(Pu) = P\left(\frac{1}{\Gamma(\alpha)}{}^{H}D_{1^+}^{\alpha-1}u(1)(\log t)^{\alpha-1}\right) = \frac{1}{\Gamma(\alpha)}{}^{H}D_{1^+}^{\alpha-1}(\log t)^{\alpha-1} = Pu.$$

Thus, $P^2 = P$.

For $u \in X$, we have

$$\begin{aligned}\|Pu\|_X &= \frac{1}{\Gamma(\alpha)}|{}^H D_{1^+}^{\alpha-1}u(1)| \cdot \|(\log t)^{\alpha-1}\|_X \\ &= \frac{1}{\Gamma(\alpha)}\left|{}^H D_{1^+}^{\alpha-1}u(1)\right| \cdot \Big[\left\|(\log t)^{\alpha-1}\right\|_\infty \\ &\quad + \left\|{}^H D_{1^+}^{\alpha-1}(\log t)^{\alpha-1}\right\|_\infty + \cdots \\ &\quad + \left\|{}^H D_{1^+}^{\alpha-1}(\log t)^{\alpha-(M-1)}\right\|_\infty\Big] \\ &= \left(\sum_{i=1}^{M-1}\frac{|(\log t)^{i-1}|}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right)\left|{}^H D_{1^+}^{\alpha-1}u(1)\right| \\ &\leq \left(\sum_{i=1}^{M-1}\frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right)\left|{}^H D_{1^+}^{\alpha-1}u(1)\right| \end{aligned} \tag{17}$$

From $u = u - Pu + Pu$, we can see that $X = \ker P + \ker L$. Let $u \in \ker L \cap \ker P$, then $u = c_1(\log t)^{\alpha-1}$. From ${}^H D_{1^+}^{\alpha-1} c_1(\log t)^{\alpha-1}\big|_{t=e} = 0$, we can obtain $c_1 = 0$, hence,

$$X = \ker L \oplus \ker P.$$

Let us define $K_P : \text{Im } L \to \text{dom } L \cap \ker P$ by $K_P u = {}^H I_{1^+}^{\alpha} u$, then for $g \in \text{Im } L$, we have

$$LK_P g = {}^H D_{1^+}^{\alpha}\, {}^H I_{1^+}^{\alpha} g(t) = g(t). \tag{18}$$

Now, for $u \in \text{dom } L \cap \ker P$, we have $u(1) = {}^H D_{1^+}^{\alpha-1}u(1) = \cdots = {}^H D_{1^+}^{\alpha-(N-2)}u(1) = 0$. By (10), we obtain

$$\begin{aligned}{}^H D_{1^+}^{\alpha}\, {}^H I_{1^+}^{\alpha} Lu(t) &= {}^H D_{1^+}^{\alpha}\, {}^H I_{1^+}^{\alpha}\, {}^H D_{1^+}^{\alpha} u(t) \\ &= {}^H I_{1^+}^{\alpha} g(t) + c_1(\log t)^{\alpha-1} + \cdots + c_N(\log t)^{\alpha-N}\end{aligned}$$

where $c_1, \ldots, c_N \in \mathbb{R}$. For $u \in \text{dom } L$, the constants $c_2, \ldots, c_N$ are all equal to zero, therefore

$$K_P L u = {}^H I_{1^+}^{\alpha}\, {}^H D_{1^+}^{\alpha} u(t) = u. \tag{19}$$

By (18) and (19), we see that $K_P = (L|_{\text{dom } L \cap \ker P})^{-1}$. Furthermore, $\forall\ g \in \text{Im } L$,

$$\begin{aligned}\|K_P g\|_X &= \|{}^H I_{1^+}^{\alpha} g\|_X = \|{}^H D_{1^+}^{\alpha-1}\, {}^H I_{1^+}^{\alpha} g\|_\infty + \cdots \\ &\quad + \|{}^H D_{1^+}^{\alpha-(N-1)}\, {}^H I_{1^+}^{\alpha} g\|_\infty + \|{}^H I_{1^+}^{\alpha} g\|_\infty \\ &= \left\|\int_1^t g(s)\frac{ds}{s}\right\|_\infty + \cdots \\ &\quad + \frac{1}{\Gamma(N-1)}\left\|\int_1^t (\log t - \log s)^{M-2} g(s)\frac{ds}{s}\right\|_\infty \\ &\quad + \frac{1}{\Gamma(\alpha)}\left\|\int_1^t (\log t - \log s)^{\alpha-1} g(s)\frac{ds}{s}\right\|_\infty \\ &\leq \left(\sum_{i=1}^{M-1}\frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right)\|g\|_1.\end{aligned} \tag{20}$$

The proof is concluded. ■

Lemma 3. Suppose $\Omega \subset X$ is an open and bounded subset such that $\text{dom } L \cap \overline{\Omega} \neq \emptyset$, then the operator N is L-compact on $\overline{\Omega}$.

Proof: Since w is continuous, it can be shown that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. Also, there exists $\phi > 0$ such that $|(I-Q)Nu| \leq \phi$, $\forall x \in \overline{\Omega},\ t \in [1, e]$. Thus, by the Arzela-Ascoli theorem, we require only to prove that $K_P(I-Q)N(\Omega) \subset X$ is equicontinuous.

For $1 \leq e^{t_1} \leq e^{t_2} \leq e$ and $u \in \overline{\Omega}$, we have

$$\begin{aligned}&|K_P(I-Q)Nu(e^{t_2}) - K_P(I-Q)Nu(e^{t_1})| \\ &\leq \frac{\phi}{\Gamma(\alpha)}\int_1^{e^{t_1}} [(\log t_1 - \log s)^{\alpha-1} - (\log t_2 - \log s)^{\alpha-1}]\frac{ds}{s} \\ &\quad + \frac{\phi}{\Gamma(\alpha)}\int_{e^{t_1}}^{e^{t_2}} [(\log t_2 - \log s)^{\alpha-1}\frac{ds}{s} \\ &= \frac{\phi}{\Gamma(\alpha+1)}[t_1^\alpha - t_2^\alpha + 2(t_2^\alpha - t_1^\alpha)] \\ &\leq \frac{\phi}{\Gamma(\alpha+1)}[t_2^\alpha - t_1^\alpha + 2(t_2^\alpha - t_1^\alpha)] \\ &\leq \frac{\phi}{\Gamma(\alpha+1)}\left[t_2^\alpha\left(1 - \frac{t_1^\alpha}{t_2^\alpha}\right) + 2t_2^\alpha\left(1 - \frac{t_1}{t_2}\right)^\alpha\right] \\ &\leq \frac{3\phi t_2^\alpha}{\Gamma(\alpha+1)}\end{aligned}$$

Similarly, we have

$$\begin{aligned}&|{}^H D_{0^+}^{\alpha-i} K_P(I-Q)Nu(e^{t_2}) - {}^H D_{0^+}^{\alpha-i} K_P(I-Q)Nu(e^{t_1})| \\ &\leq \frac{\phi}{\Gamma(i)}\int_1^{e^{t_1}} [(\log t_1 - \log s)^{i-1} - (\log t_2 - \log s)^{i-1}]\frac{ds}{s} \\ &\quad + \frac{\phi}{\Gamma(i)}\int_{e^{t_1}}^{e^{t_2}} [(\log t_2 - \log s)^{i-1}\frac{ds}{s} \\ &= \frac{\phi}{\Gamma(i+1)}[t_1^i - t_2^i + 2(t_2^i - t_1^i)] \\ &\leq \frac{\phi}{\Gamma(i+1)}[t_2^i - t_1^i + 2(t_2^i - t_1^i)] \\ &\leq \frac{\phi}{\Gamma(i+1)}\left[t_2^i\left(1 - \frac{t_1^i}{t_2^i}\right) + 2t_2^i\left(1 - \frac{t_1}{t_2}\right)^i\right] \\ &\leq \frac{3\phi t_2^i}{\Gamma(\alpha+1)}\end{aligned}$$

where $i = 1, 2, \ldots, M-1$. From the uniform continuity of t^α and t^i on $[1, e]$, we can obtain $K_P(I-Q)N : \overline{\Omega} \to G$ is compact. The proof is concluded. ■

III. Main Results

In this section, we will state and prove conditions for existence of solution for the HFBV $(1) - (2)$. We start by stating some lemmas that are required to prove existence of solutions.

Lemma 4. Let $E_1 = \{u \in \text{dom } L \setminus \ker L : Lu = \lambda Nu \text{ for some } \lambda \in [0, 1]\}$, then E_1 is bounded.

Proof: Let $\in E_1$, then $\lambda \neq 0$, $Lu = \lambda Nu$, and $Nu \in \text{Im } L$. Therefore, by Lemma 1 we have

$$\int_1^e (1 - \log s)\, g(s)\frac{ds}{s} - \sum_{i=1}^{\infty}\frac{\beta_i}{\Gamma(\alpha+\delta)}\int_1^{\zeta_i}(\log\zeta_i - \log s)^{\alpha+\delta-1} g(s)\frac{ds}{s} = 0.$$

By (A_2), there exists $r \in [1, e]$ such that $|{}^H D_{1^+}^{\alpha-1}u(r)| \leq B$. Since ${}^H D_{1^+}^{\alpha-1}$ is absolutely continuous where $u \in E_1$, by (II) we have ${}^H D\, {}^H D_{1^+}^{\alpha-1}u(t) = {}^H D_{1^+}^{\alpha}u(t)$, then

$${}^H D_{1^+}^{\alpha-1}u(1) = {}^H D_{1^+}^{\alpha-1}u(r) - \int_1^r {}^H D_{1^+}^{\alpha}u(t)dt.$$

That is

$$\left|{}^H D_{1^+}^{\alpha-1}u(1)\right| = B + \|Lu\|_1 \leq B + \|Nu\|_1. \tag{21}$$

Also, for $u \in E_1$ and $u \in \text{dom } L\backslash \ker L$, we have $(I-P)u \in \text{dom } L \cap \ker P$ and $LPu = 0$, therefore from (20)

$$\begin{aligned}\|(I-P)u\|_X &= \|K_P L(I-P)u\|_X \\ &\le \left(\sum_{i=1}^{M-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right) \|L(I-P)u\|_1 \\ &\le \left(\sum_{i=1}^{M-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right) \|Nu\|_1.\end{aligned} \tag{22}$$

In addition, by (A_2)

$$\begin{aligned}\|Nu\|_1 &= \int_1^e \left(\sigma + |u| \sum_{i=1}^{M-1} \varphi_i\right) \frac{ds}{s} \\ &\le \|\sigma\|_1 + \|u\|_X \sum_{i=1}^{M-1} \|\varphi_i\|_1\end{aligned} \tag{23}$$

Then by (17), (21), (22) and (23)

$$\begin{aligned}\|u\|_X &= \|Pu + (I-Pu)\|_X \le \|Pu\|_X + \|(I-Pu)\|_X \\ &\le \left(\sum_{i=1}^{M-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right) \left(\left|{}^H D_{1^+}^{\alpha-1} u(1)\right| + \|Nu\|_1\right) \\ &\le B\left(\sum_{i=1}^{M-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right) + 2\left(\sum_{i=1}^{M-1} \frac{1}{\Gamma(i)} \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)}\right) \left[\|\sigma\|_1 + \|\|u\|_X \sum_{i=1}^{M-1} \|\varphi_i\|_1\right].\end{aligned}$$

Setting $z = \sum_{i=1}^{M-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}$, we have

$$\|u\|_X \le \frac{Bz + 2z\|\sigma\|_1}{1 - 2z\sum_{i=1}^{M-1} \|\varphi_i\|_1},$$

hence, (7) holds and E_1 is bounded. The proof is concluded. ■

Lemma 5. The set $E_2 = \{u \in \ker L : Nu \in \text{Im } L\}$ is bounded if (A_3) holds.

Proof: Let $u \in E_2$, we have $u(t) = c(\log t)^{\alpha-1}$, $c \in \mathbb{R}$ and $Nu \in \text{Im } L$. Then

$$\int_1^e (1-\log s)\, w\left(s, c(\log t)^{\alpha-1}\right) \frac{ds}{s} - \sum_{i=1}^{\infty} \frac{\beta_i}{\Gamma(\alpha+\delta)} \times \int_1^{\zeta_i} (\log \zeta_i - \log s)^{\alpha+\delta-1} w\left(s, c(\log t)^{\alpha-1}\right) \frac{ds}{s} = 0.$$

From (A_3), we obtain $|c| \le \frac{B}{\Gamma(\alpha)}$. Thus, E_2 is bounded. The proof is concluded. ■

Lemma 6. The set $E_3 = \{u \in \ker L : \nu\lambda u + (I-Q)Nu = 0,\ \lambda \in [0,1]\}$ is bounded if (A_4) holds where

$$\nu = \begin{cases} 1, & \text{if (5) holds,} \\ -1, & \text{if (6) holds.} \end{cases} \tag{24}$$

Proof: Let $u \in E_3$, then $u \in \ker L$ with

$$u(t) = c_0(\log t)^{\alpha-1},\ c \in \mathbb{R}$$

and

$$\begin{aligned}&\lambda\nu c_0(\log t)^{\alpha-1} \\ &+ (1-\lambda)d\Bigg(\int_1^e (1-\log s)\, w\left(s, c(\log t)^{\alpha-1}\right) \frac{ds}{s} \\ &- \sum_{i=1}^{\infty} \frac{\beta_i}{\Gamma(\alpha+\delta)} \\ &\int_1^{\zeta_i} (\log \zeta_i - \log s)^{\alpha+\delta-1} w\left(s, c(\log t)^{\alpha-1}\right) \frac{ds}{s}\Bigg) = 0\end{aligned} \tag{25}$$

If $\lambda = 0$, then $|c_0| \le M_1$. If $\lambda = 1$, we have $c_0 = 0$. For $\lambda \in (0,1)$ and $|c_0| > M_1$,

$$\begin{aligned}&\nu\lambda c_0^2(\log t)^{\alpha-1} \\ &+ (1-\lambda)d\Bigg(\int_1^e (1-\log s)\, c_0 w\left(s, c(\log t)^{\alpha-1}\right) \frac{ds}{s} \\ &- \sum_{i=1}^{\infty} \frac{\beta_i}{\Gamma(\alpha+\delta)} \\ &\times \int_1^{\zeta_i} (\log \zeta_i - \log s)^{\alpha+\delta-1} c_0 w\left(s, c(\log t)^{\alpha-1}\right) \frac{ds}{s} \ne 0\Bigg),\end{aligned}$$

which contradicts (25). Therefore, E_3 is bounded. The proof is concluded. ■

Theorem 2. The HFBV $(1)-(2)$ has at least one solution in X if (A_1)-(A_4) hold.

Proof: Let $E \subset X$ be an open and bounded such that $\cup_{n=1}^3 \overline{E_n} \subset E$. It follows from Lemma 2 that the operator L is a Fredholm mapping of index zero while Lemma 3 shows that the operator N is L-compact. By Lemmas 4 and 5, we see that the following conditions of Theorem 1 are satisfied:

(i) $Lu \ne \lambda Nu$ for every $(u, \lambda) \in [(\text{dom } L\backslash \ker L) \cap \partial\Omega] \times (0,1)$;

(ii) $Nu \notin \text{Im } L$ for every $u \in \ker L \cap \partial\Omega$.

Next, we will verify statement (iii) of Theorem 1. Let

$$H(u, \lambda) = \pm(1-\lambda)QNu.$$

By Lemma 6, we see that

$$H(u, \lambda) \ne 0,\ \forall\ u \in \partial E \cap \ker L.$$

Therefore, by the homotopy property of degree, we obtain

$$\begin{aligned}\deg(QN|_{\ker L}, E \cap \ker L, 0) &= \deg(H(\cdot,0), E \cap \ker L, 0) \\ &= \deg(H(\cdot,1), E \cap \ker L, 0) \\ &= \deg(\pm I, E \cap \ker L, 0) \ne 0.\end{aligned}$$

Hence, statement (iii) of Theorem 1 is satisfied. Therefore, since statements $(i)-(iii)$ of Theorem 1 are satisfied, the operator equation $Lu = Nu$ has at lest one solution in $\text{dom } L \cap \overline{E}$, implying that the HFBV $(1)-(2)$ has at least one solution in X. The proof is concluded. ■

IV. Example

Consider the following Hardamard fractional boundary value problem for $t \in [1, e]$

$$\begin{aligned}{}^H D_{1^+}^{3.5} =& 1 - (\log t)^2 + \frac{1}{27} {}^H D_{1^+}^{2.5} u(t) + \frac{\sin\left({}^H D_{1^+}^{1.5} u(t)\right)}{27} \\ &+ \frac{1}{34} {}^H D_{1^+}^{0.5} u(t) + \frac{\cos^2 u(t)}{28},\end{aligned} \tag{26}$$

subject to the boundary conditions

$$u(1) = {}^H D_{1^+}^{1.5}u(1) = {}^H D_{1^+}^{1.5}u(1) = 0,$$
$${}^H I_{1^+}^{-2.5}u(e) = \sum_{i=1}^{\infty} \beta_i {}^H I_{1^+}^{\delta} u(\zeta_i), \tag{27}$$

where, $\beta_i = \frac{543}{2^{i/2}}$, $\zeta_i = \frac{2i+1}{i+2}$ and $\delta = 1$.
Corresponding to HFBV $(26) - (27)$, we have $\alpha = 3.5$, and $w(t,a,b,c,f) = 1 - (\log t)^2 + \frac{\cos^2 a}{28} + \frac{b}{27} + \frac{\sin c}{27} + \frac{f}{34}$ then $|w(t,a,b,c,f)| = \frac{|a|}{18} + \frac{|b|}{27} + \frac{|c|}{27} + \frac{|f|}{34}$.
Taking $\varphi_1 = \frac{1}{18}$, $\varphi_2 = \frac{1}{3}$, $\varphi_3 = \frac{1}{17}$, $\varphi_4 = \frac{1}{14}$, then $1 - 2\left(\sum_{i=1}^{N-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right)\sum_{i=1}^{N-1} \|\varphi_i\|_1 = 1 - 0.8262 = 0.1738 > 0$.

$$\sum_{i=1}^{\infty} \frac{\beta_i (\log \zeta_i)^{\delta+\alpha-2}}{\Gamma(\delta+\alpha)} = \sum_{i=1}^{\infty} \frac{\frac{543}{2^{i/2}}\left(\log \frac{2i+1}{i+2}\right)^{2.5}}{\Gamma(3.5)} = 1.00.$$

If we choose $B = 51$. For ${}^H D_{1^+}^{\alpha-1} > 51$, we have $w(t,a,b,c,f) > \frac{1}{28} + \frac{51}{27} + \frac{1}{27} + \frac{1}{34} > 0$. Also, if ${}^H D_{1^+}^{\alpha-1} < -51$ then $w(t,a,b,c,f) > \frac{1}{28} - \frac{51}{27} + \frac{1}{27} + \frac{1}{34} < 0$. Therefore, if $|{}^H D_{1^+}^{\alpha-1}| > B > 51$, then

$$\int_1^e (1 - \log s)\, w(s, u(s)) \frac{ds}{s} - \sum_{i=1}^{\infty} \frac{\beta_i}{\Gamma(\alpha+\delta)} \int_1^{\zeta_i} (\log \zeta_i - \log s)^{\alpha+\delta-1} w(s, u(s)) \frac{ds}{s} \neq 0.$$

Hence, (A_3) is satisfied.
Let $u \in \ker L$ and $u(t) = c_0\Theta = c_0(\log t)^{\alpha-1}$, we choose $|c_0| > 0$, then

$$c_0\left[\int_1^e (1-\log s) w\left(s, c_0(\log t)^{\alpha-1}\right)\frac{ds}{s} - \sum_{i=1}^{\infty}\frac{\beta_i}{\Gamma(\alpha+\delta)} \times \int_1^{\zeta_i} (\log \zeta_i - \log s)^{\alpha+\delta-1} w\left(s, c_0(\log t)^{\alpha-1},\right)\frac{ds}{s}\right] > 0.$$

since $w\left(s, c_0\Theta, {}^H D_{1^+}^{\alpha-1}c_0\Theta, {}^H D_{1^+}^{\alpha-2}c_0\Theta, {}^H D_{1^+}^{\alpha-3}c_0\Theta\right) = 1 - (\log t)^2 + \frac{c_0\Gamma(\alpha)}{27} + \frac{\sqrt{\sin c_0(\Gamma(\alpha)(\log t))}}{27} + \frac{c_0\Gamma(\alpha)(\log t)^2}{34\Gamma(3)} + \frac{\cos^2 c_0(\log t)^{\alpha-1})}{28} > 1 - 1 + \frac{|c_0|\Gamma(\alpha)}{27} + \frac{|c_0|(\Gamma(\alpha))}{27} + \frac{|c_0|\Gamma(\alpha)}{34\Gamma(3)} + \frac{|c_0|}{28} = \frac{|975811c_0|}{25704} > 0$, implying that $|c| > 0$. Therefore, condition (A_4) is satisfied. Hence by Theorem 3.1, HFBV $(26) - (27)$ has at least one solution.

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