Convergence Theorems of Common Fixed Point by an Iterative Scheme for Nearly Asymptotically Nonexpansive Mappings

Mohd Jamshed Ali, Gaurav Aggarwal and Virendra Singh Chouhan

Abstract— **In this paper, we investigate the convergence analysis of an iterative scheme for approximating common fixed point of nonself mappings in Banach space. We introduce an explicit iteration scheme and prove strong convergence theorems for a countably infinite family of nonself nearly asymptotically nonexpansive mappings. Our results extend and generalize several existing theorems in the literature by considering more general mappings and a broader class of iterative processes. Specifically, we establish conditions under which the proposed iterative sequence converges strongly to a common fixed point of the given family of mappings. The key tools in our analysis include techniques from functional analysis and fixed-point theory, combined with careful estimates of the iterative sequence. Several corollaries and examples are provided to illustrate the applicability and significance of our main results.**

Index Terms— **Banach space, fixed point, nearly asymptotically nonexpansive mappings, convergence.**

I. INTRODUCTION

THE fixed-point theory has many use-full applications in different branches. There are numerous practical applications of fixed-point theory in various fields. Numerous fixed-point convergence findings on nonexpansive mapping defined in accordance with various convex spaces have recently been explored in various sorts. Fixed point theory serves as a cornerstone of nonlinear analysis, particularly in examining the stability and convergence of iterative processes in Banach spaces. Over the years, classical fixed point theorems have been significantly extended to accommodate broader classes of mappings, including nearly asymptotically nonexpansive mappings and non-Lipschitzian mappings, thereby expanding their applicability in various mathematical domains. These generalizations have proven instrumental in solving problems related to differential equations, dynamical systems, and mathematical optimization. Among the foundational results in this field, the Banach contraction principle remains pivotal, offering a basis for more advanced theoretical developments, particularly in fixed point theorem for monotone asymptotically nonexpansive mappings in metric spaces Alfuraidan & Khamsi [2] in 2018.

Manuscript received September 24, 2024; revised April 12, 2025.

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The geometric nature of metric spaces, defined by their curvature-like properties, makes them a natural framework for analyzing convergence behavior, especially for nonexpansive and asymptotically nonexpansive mappings Leustean [13] in 2010.

Recent studies have expanded fixed point theory to include numerous mappings, particularly those that do not adhere to strict Lipschitz conditions. For instance, Bachar et al. [5] in 2018 established the existence of common fixed points for monotone Lipschitzian mappings in metric spaces, emphasizing the significance of contraction-type conditions in ensuring convergence Bachar et al. [5] in 2018. Their results were further refined by Aggarwal et al. [1] in 2019, who examined monotone nearly asymptotically nonexpansive mappings in partially ordered metric spaces, proving key existence and convergence theorems Aggarwal et al. [1] in 2019. This work provided a critical extension of classical contraction principles by permitting small asymptotic deviations from nonexpansiveness, making them highly relevant in dynamic systems where mappings exhibit nonuniform contraction behavior.

A growing trend in fixed point theory involves utilizing graph structures in metric spaces to generalize classical results. Unlike partial ordering, graph-based structures offer a more flexible framework for studying fixed points. Jachymski [10] in 2008 pioneered the contraction principle in metric spaces endowed with graphs, inspiring subsequent research in this direction Jachymski [10] in 2008. Expanding on this, Bin Dehaish & Khamsi [6] in 2015 explored approximate fixed points of metric spaces, demonstrating that fixed points can be obtained under generalized graph-theoretic conditions Bin Dehaish & Khamsi [6] in 2015. The practical implications of these findings are vast, particularly in stability analysis of iterative algorithms in optimization and game theory.

In addition to theoretical advancements, recent research has focused on iterative approximation methods in metric spaces. Song et al. [22] in 2016 and Uddin et al. [23] in 2016 investigated convergence properties of iterative schemes for nonexpansive mappings in ordered Banach spaces, offering critical insights into the computational aspects of fixed point theory Song et al. [22] in 2016, Uddin et al. [23] in 2016. Further developments by Shukla et al. [21] in 2017 extended these findings to α-nonexpansive mappings, proving their stability in partially ordered metric spaces Shukla et al. [21] in 2017. These contributions collectively highlight the growing interplay between fixed point theory, metric geometry, and computational analysis.

Numerous fixed-point convergence findings on nearly asymptotically nonexpansive mapping defined in accordance with various convex spaces have recently been explored in various sorts. For those applied science situations when one is impossible to determine the analytical value of the wanted answer, fixed-point theory proposes very fruitful and alternate strategies for existence as well as an iterative approximation of desired solutions. To start, the required desired answer must be expressed as a fixed point of an operator, may be defined on a relevant subset of a Banach space.

Let E be any Banach space whose convex closed subset of K is not empty. The nonexpansive retraction of E to K is denoted by $R: E \to K$. A point $a \in K$ where $T(a) = a$ is a nonself mapping $T: K \to E$ with a fixed point of T. Furthermore, the $F(T)$ represents the set of all fixed points of T. The existence theorems of fixed points for asymptotically nonexpansive mappings have been studied by a few authors [7, 8, 11, 16, 24].

Given a Banach space E and a nonempty subset C, fix sequence $\{a_n\}$ in $[0, \infty]$ with $a_n \to 0$. If, for every $n\ in\ \mathbb{N}, \exists$ a constant $k_n \geq 0$ such that, for $\{a_n\}$, a mapping $T: C \to C$ is nearly Lipschitzian

$$(1.1) \quad \|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n),$$

$$for\ all\ x, y \in C.$$

The nearly Lipschitz constant, represented as $\eta(T^n)$, is the minimum of constants k_n for which (1.1) is holds. Observe that

$$\eta(T^n) = sup\left\{\frac{\|T^n x - T^n y\|}{\|x - y\| + a_n} : x, y \in C, x \neq y\right\}.$$

A nearly Lipschitzian map T with sequence $\{(a_n, \eta(T^n))\}$ is called nearly asymptotically nonexpansive [17], if

$$\eta(T^n) \geq 1, \forall\ n \in \mathbb{N}\ and\ \lim_{n\to\infty} \eta(T^n) \leq 1.$$

Sahu [17] in 2005, established the class of nearly asymptotically nonexpansive mapping.

Understanding the geometry of spaces, sequence convergence, and the demiclosed ness principle of nonlinear mappings all depend on the Opial's condition. The Opial's condition is satisfied by any space X if a sequence $\{\gamma_n\}$ defined on space X converges weakly to any $\gamma_0 \in X$ then

$$\lim_{n\to\infty} inf\left||\gamma_n - \gamma_0|\right| < \lim_{n\to\infty} inf\left||\gamma_n - \gamma|\right|\ \forall\gamma \in X\ and\ \gamma \neq \gamma_0$$

here, we can establish weak Opial's condition by substituting the inequality $\leq$ for the strict inequality $<$.

Let $T: D \to E$ be a mapping defined from D to Banach space E and let D be any non-empty subset of Banach space E. The mapping T is therefore demiclosed at each $s \in H$ if it satisfies the following criterion for any associated sequence $y_n \in D$:

$$y_n \to b \in D\ and\ Ty_n \to d\ \Rightarrow\ Tb\ = d.$$

If : $D \to E$ is a nonexpansive mapping and D is a nonempty convex closed subset of space E, then $I - T$ is demi closed, where I is the identity mapping. This is an example of an Opial's Condition defined on reflexive Banach space E.

Theorem 1. [18] Assume that B is a Hilbert space and $G \neq \emptyset$ is convex closed & bounded subset of B. Let T be a mapping of nearly asymptotically nonexpansive from $G\ to\ G$ with sequence $\{k_n\} \subset [1, \infty)\ for\ all\ n \geq 1$, $\lim_{n\to\infty} = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in [0,1] that, for some constant a, b, satisfies the constraint $0 < a \leq \alpha_n \leq b < 1, n \geq 1$. Then the $\{x_n\}$ sequence produced from any $x_1 \in K$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, n \geq 1,$$

strongly convergence to any fixed point of T.

Since then, fixed points in Banach space or Hilbert space that are asymptotically nonexpansive mappings have been frequently approximated using Schu's iteration [18] process. Assume that real linear normed space E has a $K \neq \emptyset$ subset. The nonexpansive retraction of E onto K is denoted by $R: E \to K$. $T: K \to E$ is a nonself mapping, with the subsequent iteration procedure:

$$x_1 \in K, \qquad x_{n+1} = R(1 - \alpha_n)x_n + \alpha_n T(RT)^{n-1}x_n, \quad n \geq 1.$$

Several theorems of strong and weak convergence in convex Banach spaces for asymptotically nonexpansive mappings were developed by Chidume, Ofoedu, and Zegeye [8] in 2003.

Theorems of strong and weak convergence on convex uniformly Banach spaces and other convex spaces for nonexpansive mappings have been studied by a few authors [3, 12, 14, 24, 26].

Wang [14] discussed additional fixed-point results using asymptotically nonexpansive mapping. John and Shaini recently explored the fixed-point theorems for Suzuki nonexpansive mapping in Banach space [20]. For fixed points of nonself asymptotically nonexpansive mappings, Wei and Jing [15] introduce a novel iteration technique.

Motivated by these concepts, we present an iterative method for evaluating the common fixed points convergence theorem of an infinite family of mappings $\{T_\lambda: \lambda \in \Lambda\}$ formed on Banach space that are nearly asymptotically nonexpansive. The set of common fixed points of an infinite family of nonself nearly asymptotically nonexpansive mappings $\{T_\lambda: \lambda \in \Lambda\}$ is denoted throughout this paper by $F(T) = \{a \in K: Ta = a, \lambda \in \Lambda\}$. Analysing the convergence of these schemes is a crucial part of the work. This study contributes to the broader field of fixed point theory by extending existing results to a more general class of mappings, offering insights into the behaviour of nearly asymptotically nonexpansive mappings in the context of common fixed point problems.

II. Preliminaries

In this section, we provide certain definitions and lemma's in Banach space that will be useful in our main results.

Definition 2.1. [9] A mapping $T: C \to C$ defined on space C is called nonexpansive mapping if

$$\|T\mu - T\gamma\| \leq \|\mu - \gamma\|$$

, $\forall\ \mu, \gamma$ in space C.

Definition 2.2. [17] Let E be a Banach space with nonempty subset K, fix sequence $\{a_n\}$ in $[0,\infty]$ with $a_n \to 0$. Let nonexpansive retraction R from E to K. A map $T: K \to E$ is called nearly Lipschitzian with $\{a_n\}$ if $\forall\, n \in \mathbb{N}, \exists\ a\ constant\ k_n \geq 0$ such that

$$(1.2)\quad \|T(RT)^{n-1}x - T(RT)^{n-1}y\| \leq k_n(\|x-y\| + a_n),$$

$$for\ all\ x, y \in K.$$

The nearly Lipschitz constant, represented as $\eta(T^n)$, is the infimum of constants k_n for which (1.2) is holds. Observe that

$$\eta(T^n) = sup\left\{\frac{\|T(RT)^{n-1}x - T(RT)^{n-1}y\|}{\|x-y\| + a_n} : x, y \in C, x \neq y\right\}.$$

A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be

a) nearly contraction if $\eta(T^n) < 1, for\ all\ n \in \mathbb{N}$,
b) nearly nonexpansive if $\eta(T^n) \leq 1, for\ all\ n \in \mathbb{N}$,
c) nearly asymptotically nonexpansive, if $\eta(T^n) \geq 1, \forall\, n \in \mathbb{N}\ and\ \lim_{n\to\infty} \eta(T^n) \leq 1$,
d) nearly uniformly k-Lipschitzian if $\eta(T^n) \leq k\ for\ all\ n \in \mathbb{N}$,
e) nearly uniform k-contraction if $\eta(T^n) \leq k < 1\ for\ all\ n \in \mathbb{N}$.

Result 2.3. [25] Let E be a Banach space with K be nonempty convex closed subset of E. K is a nonexpansive retract of Banach space E with nonexpansive retraction R. Let $\{T_{\lambda_1}, T_{\lambda_2}, T_{\lambda_3}, \dots\dots T_{\lambda_n} \dots.\}$ be a countable subset of $\{T_\lambda: \lambda \in \Lambda\}$. Define mapping T_λ from K to Banach space E. Then the explicit iteration scheme is

$$z_n = R\big((1-p_n)z_{n-1} + p_n T_n^*(RT_n^*)^{m_n-1}z_{n-1}\big), for\ all\ n \geq 1$$

where $\{p_n\}$ is a sequence in $[\alpha, 1-\alpha]$ for some $\alpha \in (0,1)$ and $T_n^* = T_{i_n}$ such that i_n and m_n are the solutions of $n = i + \frac{(m-1)m}{2}, m \geq i, n \in \mathbb{N}$.

Lemma 2.4. [4] Let $\{a_n\}, \{\delta_n\}$, and $\{b_n\}$ be sequence of real, nonnegative numbers satisfying

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \forall\, n \geq 1$$

if $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.5. [19] Let E be a real Banach space that is uniformly convex, and let a and b be two constants such that $0 < a < b < 1$. Assume that there are two sequences in E, $\{x_n\}, \{y_n\}$ and that $\{t_n\} \subset [a,b]$ is a real sequence. Then

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d,$$

$$\lim_{n\to\infty} sup\|x_n\| \leq d, \qquad \lim_{n\to\infty} sup\|y_n\| \leq d$$

imply that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0,$$

where $d \geq 0$ is a constant.

Lemma 2.6. [8] Let E be a real uniformly convex Banach space, K a nonempty closed subset of E, and let $T: K \to E$ be a nonself nearly asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)\ and\ k_n \to 1\ as\ n \to \infty$. Then $I - T$ is demiclosed at zero.

Definition 2.7. [8] Let E be a Banach space and K subset of Banach space E, subset K is called retract of E if $\exists$ a continuous $R: E \to K$ such that $Rx = x, \forall x \in K$. Every closed convex set of a uniformly convex Banach space is a retract. A mapping $R: E \to E$ is called retraction if $R^2 = R$. It follows that if a map R is a retraction then $Rz = z, \forall\ z\ in$ the range of R.

III. Main Results

In this theorem, we demonstrate that our iteration process converges to a fixed point.

Theorem 3.1. Let a Banach space E, whose nonempty convex closed subset K is a nonexpansive retract of E with nonexpansive retraction R. Let $\{T_{\lambda_1}, T_{\lambda_2}, T_{\lambda_3}, \dots\dots T_{\lambda_n} \dots.\}$ be a countably finite subset of the nonself nearly asymptotically nonexpansive mapping $\{T_\lambda: \lambda \in \Lambda\}$ defined from $K\ to\ E$ with corresponding sequence $\{k_n^{(i)}\} \subset [1,\infty)\ and\ \{a_n{}^{(i)}\} \subset [0,\infty)\ such\ that\ \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} p_n a_n{}^{(i)} < \infty\ for\ all\ i \geq 1$. Suppose that $\{z_n\}$ be a sequence which is explicit iteration scheme

$$z_n = R\big((1-p_n)z_{n-1} + p_n T_n^*(RT_n^*)^{m_n-1}z_{n-1}\big), for\ all\ n \geq 1$$

where $\{p_n\}$ is a sequence in $[\alpha, 1-\alpha]$ for some $\alpha \in [0,1)$ and $T_n^* = T_{i_n}$ such that i_n and m_n are the solutions of $n = i + \frac{(m-1)m}{2}, m \geq i, n \in \mathbb{N}$. If $F(T) \neq \emptyset$, then $\lim_{n\to\infty} \|z_n - x\|$ exists for each fixed point in $F(T)$.

Proof: Let $\{z_n\}$ be a sequence which satisfy the explicit iteration scheme is

$$z_n = R\big((1-p_n)z_{n-1} + p_n T_n^*(RT_n^*)^{m_n-1}z_{n-1}\big), for\ all\ n \geq 1$$

then, for any fixed-point x, we have

$$\|z_n - x\| = \big\|R\big((1-p_n)z_{n-1} + p_n T_n^*(RT_n^*)^{m_n-1}z_{n-1}\big) - Rx\big\|$$

$$\leq \|(1-p_n)(z_{n-1} - x) + p_n(T_n^*(RT_n^*)^{m_n-1}z_{n-1} - x)\|$$

$$\leq \Big(1 - p_n + p_n K_{m_n}^{(i_n)}\Big)\|z_{n-1} - x\| + p_n a_{m_n}^{(i_n)}$$

$$\leq \left(1+u_{m_n}^{(i_n)}\right)\|z_{n-1}-x\|+p_n a_{m_n}^{(i_n)} \quad \ldots\ldots(1)$$

where

$$K_{m_n}^{(i_n)}=1+u_{m_n}^{(i_n)}, \qquad \forall\, n\in\mathbb{N}$$

and i_n, m_n satisfies the positive integer equation

$$n=i+\frac{(m-1)m}{2}, m\geq i, n\in\mathbb{N}.$$

Now,

$$\sum_{n=1}^{\infty}u_{m_n}^{(i_n)}=\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}(K_n^{(i)}-1)$$
$$\leq\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(K_n^{(i)}-1)<\infty \;\ldots\ldots(2)$$

$$\sum_{n=1}^{\infty}p_n a_{m_n}^{(i_n)}=\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}p_{n-1}\,a_n^{(i)}\leq\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}p_n\,a_n^{(i)} <\infty \ldots\ldots(A)$$

from lemma 2.4, we know that if

$$c_{n+1}\leq(1+\delta_n)c_n+b_n, \forall\, n\in\mathbb{N}$$

such that $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}b_n<\infty$ then $\lim_{n\to\infty}c_n$ exists where

$$\delta_n=u_{m_n}^{(i_n)}, \qquad b_n=p_n a_{m_n}^{(i_n)}\; and\; c_n=\|z_{n-1}-x\|.$$

So, from equation $(1), (2)$ and (A), we get $\lim_{n\to\infty}\|z_n-x\|$ exists.

In this theorem, we demonstrate an approximate fixed point exists for our iteration process.

Theorem 3.2. Let a Banach space E, whose nonempty convex closed subset K is a nonexpansive retract of E with nonexpansive retraction R. Let $\{T_{\lambda_1}, T_{\lambda_2}, T_{\lambda_3}, \ldots\ldots T_{\lambda_n}\ldots.\}$ be a countably finite subset of the nonself nearly asymptotically nonexpansive mapping $\{T_\lambda: \lambda\in\Lambda\}$ defined from $K\; to\; E$ with corresponding sequence $\left\{k_n^{(i)}\right\}\subset[1,\infty)\; and\; \left\{a_n{}^{(i)}\right\}\subset[0,\infty)\; such\; that \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left(k_n^{(i)}-1\right)<\infty$ and $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}p_n a_n{}^{(i)}<\infty\; for\; all\; i\geq 1$. Suppose that $\{z_n\}$ be a sequence which is explicit iteration scheme

$$z_n=R\left((1-p_n)z_{n-1}+p_nT_n^*(RT_n^*)^{m_n-1}z_{n-1}\right), for\; all\; n\geq 1$$

where $\{p_n\}$ is a sequence in $[\alpha, 1-\alpha]$ for some $\alpha\in(0,1)$ and $T_n^*=T_{i_n}$ such that i_n and m_n are the solutions of $n=i+\frac{(m-1)m}{2}, m\geq i, n\in\mathbb{N}$. If $F(T)\neq\emptyset$ then $for\; each\; i\geq 1$, $\exists$ a corresponding $\{z_{n_i}\}_{n_i\in\mathbb{N}}$ subsequence $of\; \{z_n\}\; such\; that$ $\lim_{n\to\infty}\|z_n-T_iz_n\|=0$.

Proof. According to theorem 3.1, we known that for any fixed-point $x\in F$,

$$\lim_{n\to\infty}\|z_n-x\|=C.$$

Let $\{z_n\}$ be a sequence which satisfy the explicit iteration scheme

$$z_n=R\left((1-p_n)z_{n-1}+p_nT_n^*(RT_n^*)^{m_n-1}z_{n-1}\right), for\; all\; n\geq 1$$

then, for any fixed-point x, we have

$$\|z_{n+1}-x\|=\left\|R\left((1-p_{n+1})z_n+p_{n+1}T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n\right)-Rx\right\|$$

$$\leq\; \|(1-p_{n+1})(z_n-x)+p_{n+1}(T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n-x)\|$$

taking both sides lim inf, we get

$$\lim_{n\to\infty}inf\;\|z_{n+1}-x\| \leq\lim_{n\to\infty}\inf\|(1-p_{n+1})(z_n-x)+p_{n+1}(T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n-x)\|$$

$$C\leq\lim_{n\to\infty}inf\|(1-p_{n+1})(z_n-x)+p_{n+1}(T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n-x)\| \quad\ldots\ldots(3)$$

in addition, we have

$$\left(1+u_{m_{n+1}}^{(i_{n+1})}\right)\|z_n-x\|+p_{n+1}a_{m_{n+1}}^{(i_{n+1})}\geq\|(1-p_{n+1})(z_n-x)+p_{n+1}(T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n-x)\|$$

taking both sides lim sup, we get

$$\lim_{n\to\infty}sup\;\left(1+u_{m_{n+1}}^{(i_{n+1})}\right)\|z_n-x\|+p_{n+1}a_{m_{n+1}}^{(i_{n+1})}\geq\lim_{n\to\infty}sup\;\|(1-p_{n+1})(z_n-x)+p_{n+1}(T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n-x)\|$$

$$C\geq\lim_{n\to\infty}sup\;\|(1-p_{n+1})(z_n-x)+p_{n+1}(T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n-x)\| \quad\ldots\ldots(4)$$

hence, from equation (3) and (4) we get

$$\lim_{n\to\infty}\|(1-p_{n+1})(z_n-x)+p_{n+1}(T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n-x)\|=C \quad\ldots(5)$$

since

$$\lim_{n\to\infty}\|z_n-x\|=C\; and$$

$$\lim_{n\to\infty}sup\|T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n-x\|\leq C,$$

from Lemma 2.5, we get

$$\lim_{n\to\infty}\|z_n-T_{n+1}^*(RT_{n+1}^*)^{m_{n+1}-1}z_n\|=0 \qquad\ldots\ldots(6)$$

hence,

$$\|z_{n+1} - z_n\| \leq p_{n+1}\|z_n - T^*_{n+1}(RT^*_{n+1})^{m_{n+1}-1}z_n\|$$

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| \leq \lim_{n\to\infty} p_{n+1}\|z_n - T^*_{n+1}(RT^*_{n+1})^{m_{n+1}-1}z_n\|$$

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| \leq 0 \qquad \dots\dots(7)$$

$\lim_{n\to\infty}\|z_{n+1} - z_n\|$ tends to zero as n tends to infinity

now,

$$\|z_n - T_i z_n\| \leq \|z_n - T_i(RT_i)^{m_{n+1}-1}z_n\| + \|T_i(RT_i)^{m_{n+1}-1}z_n - T_i z_n\|$$

therefore, from equation (5), (6) and (7) we get

$$\|z_n - T_i z_n\| \leq 0$$

$$\lim_{n\to\infty}\|z_n - T_i z_n\| \leq 0$$

$$\lim_{n\to\infty}\|z_n - T_i z_n\| = 0$$

hence, $for\ each\ i \geq 1$, ∃ a corresponding $\{z_{n_i}\}_{n_i\in\mathbb{N}}$ subsequence such that $\|z_n - T_i z_n\| = 0$.

In this theorem, we demonstrated strong converges of common fixed point for infinitely family of nearly asymptotically nonexpansive mappings.

Theorem 3.3. Let a Banach space E, whose nonempty closed convex subset K is a nonexpansive retract of E with nonexpansive retraction R. Let $\{T_{\lambda_1}, T_{\lambda_2}, T_{\lambda_3}, \dots\dots T_{\lambda_n} \dots.\}$ be a countably finite subset of the nonself nearly asymptotically nonexpansive mapping $\{T_\lambda : \lambda \in \Lambda\}$ defined from $K\ to\ E$ with corresponding sequence $\{k_n^{(i)}\} \subset [1,\infty)\ and\ \{a_n{}^{(i)}\} \subset [0,\infty)\ such\ that \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} p_n a_n{}^{(i)} < \infty\ for\ all\ i \geq 1$. Suppose that $\{z_n\}$ be a sequence which is explicit iteration scheme

$$z_n = R\big((1-p_n)z_{n-1} + p_n T^*_n(RT^*_n)^{m_n-1}z_{n-1}\big), for\ all\ n \geq 1$$

where $\{p_n\}$ is a sequence in $[\alpha, 1-\alpha]$ for some $\alpha \in (0,1)$ and $T^*_n = T_{i_n}$ such that i_n and m_n are the solutions of $n = i + \frac{(m-1)m}{2}, m \geq i, n \in \mathbb{N}$. If $F(T) = \{\alpha \in k \mid T(\alpha) = \alpha\} \neq \emptyset$ and ∃ a mapping $T_i \epsilon \{T_{\lambda_1}, T_{\lambda_2}, T_{\lambda_3}, \dots\dots T_{\lambda_n} \dots.\}$ and the function defined $f: [0,\infty) \to [0,\infty)$ is a nondecreasing such that $f(0) = 0\ and\ f(r) > 0, \forall\ r\epsilon(0,\infty)$ with $f\big(d(z_n, F)\big) \leq \|z_n - T_i z_n\|, \forall\ n\epsilon N$ then the $\{z_n\}$ sequence strongly convergent to some fixed point of the mapping $\{T_{\lambda_1}, T_{\lambda_2}, T_{\lambda_3}, \dots\dots T_{\lambda_n} \dots.\}$.

Proof: We know that for any subsequence $\{z_{n_i}\}$ of $\{z_n\}$

we have

$$\lim_{n\to\infty}\|z_n - T_i z_n\| = 0$$

and $f(x)$ is a non-decreasing function satisfying the condition

$$f(d(z_n, F)) \leq \|z_n - T_i Z_n\|$$

$$\lim_{n\to\infty} f(d(z_n, F)) \leq \lim_{n\to\infty}\|z_n - T_i z_n\|$$

$$\lim_{n\to\infty} f(d(z_n, F)) \leq 0$$

$$\lim_{n\to\infty} f\big(d(z_n, F)\big) = 0, \qquad Since\ f(r) \geq 0$$

and $f(0) = 0$. Thus, we get

$$d\big(z_{n,} F\big) = 0 \qquad \dots\dots(8)$$

now, we will show that the subsequence $\{z_{n_i}\}$ is a Cauchy sequence by theorem 3.1, ∃ a natural number M such that

$$\|z_n - x\| \leq M\|z_m - x\| \qquad \dots\dots(9)$$

for fixed-point $x \in F$ and for all $n > m$.

For any positive $\epsilon > 0, \exists\ a\ natural\ number\ \mathbb{N}$ such that $d(z_n, F) < \frac{\epsilon}{M}, \forall\ n \geq \mathbb{N}$

then

$$\|z_n - z_m\| \leq \|z_n - x\| + \|z_m - x\|$$

$$\|z_n - z_m\| \leq M\|z_n - x\|$$

$$\|z_n - z_m\| \leq Md(z_N, F)$$

$$\|z_n - z_m\| \leq \epsilon \qquad \dots\dots(10)$$

hence, from equation (10), we will say that the subsequence $\{z_n\}$ is a Cauchy sequence.

Therefore, $\exists\ p \in K\ such\ that\ z_n$ tends to p as n tends to infinity and $\lim_{n\to\infty} d(z_n, F) = 0$ which given that

$$d(p, F) = 0 \qquad \dots\dots(11)$$

so, equation (10) and (11) we will say that $p \in F$ and F is closed set. Which implies that the $\lim_{n\to\infty}\|z_n - p\|$ exists as subsequence z_n tends to p and n tends to infinity.

Hence, the subsequence z_n strongly convergent to fixed-point p of the mapping $\{T_{\lambda_1}, T_{\lambda_2}, T_{\lambda_3}, \dots\dots T_{\lambda_n} \dots.\}$.

IV. CONCLUSION

In this research, we have discussed the iterative scheme for identifying the convergence of common fixed-point of infinite family of nearly asymptotically nonexpansive mapping defined on Banach space. We obtained some convergence results concerning the exhibition of iterative method obtained by satisfy the explicit iteration scheme

$$z_n = R\left((1-p_n)z_{n-1} + p_n T_n^*(RT_n^*)^{m_n-1} z_{n-1}\right),$$

for all $n \geq 1$ on the nearly asymptotically nonexpansive mapping on Banach space.

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