Bifurcations of a Delayed Fractional-order Simplified Five-neuron BAM Neural Network via Parameter Perturbations

Libo Wang, Guigui Xu and Pan Wang

Abstract—**This paper investigates the bifurcation behavior of a delayed fractional-order simplified five-neuron BAM neural network (FOFBAMNN). The corresponding bifurcation results are derived based on the self-regulating parameter, thereby significantly enhancing the bifurcation results of neural networks. The results demonstrate that a larger self-regulating parameter is more beneficial for the stability of the system. Furthermore, the calculation method can be used as a general approach to solve bifurcation points, including the treatment of bifurcation critical point associated with time delay. Finally, numerical experiments are conducted to validate the theoretical findings of this paper.**

Index Terms—**Fractional-order, BAM neural network, Hopf bifurcation, Self-regulating parameter, Time delay.**

I. Introduction

NEURAL networks are widely concerned for their extensive applicability in diverse fields, such as intelligent control systems, optimization techniques, associative memories, and pattern recognition, among others [1]-[3]. Consequently, delving into the dynamic behaviors of these networks has become one of the central topic in the current research. A substantial body of research has been dedicated to explore the diverse dynamic properties of neural networks, with numerous valuable insights and results shared in the literature. For instance, Tian et al. [4] discussed the bifurcation behavior of delayed BAM neural networks with diffusion. Ncube [5] analyzed the stability switching and Hopf bifurcation for multiple-delayed neural networks with distributed delay. For more details of studies on dynamics, including bifurcations and chaos, one can see [6]-[14].

In general, neural networks are described in a large-scale form with highly complex dynamic behavior. To better understand the dynamics of these neural networks, various simplified forms have been considered by numerous researchers. By examining these simplified models, we can gain insights into the essential laws underlying large-scale neural networks. For instance, Jiang et al. [15] obtained some sufficient criteria to satisfy the existence, uniqueness, and global exponential stability of periodic solutions of delayed neural networks concerning three neuron activations. Hu and Huang [16] investigated the stability and Hopf bifurcation for delayed neural networks involving four neurons.

In [17] and [18], the authors discussed the simplified five-neuron BAM neural network with delays in the following form:

$$\begin{cases} \dot{x}_1(t) = -\mu x_1(t) + c_{11}f(y_1(t-\tau_3)) + c_{12}f(y_2(t-\tau_3)) \\ +c_{13}f(y_3(t-\tau_3)), \\ \dot{x}_2(t) = -\mu x_2(t) + c_{21}f(y_1(t-\tau_4)) + c_{22}f(y_2(t-\tau_4)) \\ +c_{23}f(y_3(t-\tau_4)), \\ \dot{y}_1(t) = -\mu y_1(t) + d_{11}f(x_1(t-\tau_1)) + d_{12}f(x_2(t-\tau_2)), \\ \dot{y}_2(t) = -\mu y_2(t) + d_{21}f(x_1(t-\tau_1)) + d_{22}f(x_2(t-\tau_2)), \\ \dot{y}_3(t) = -\mu y_3(t) + d_{31}f(x_1(t-\tau_1)) + d_{32}f(x_2(t-\tau_2)), \end{cases} \quad (1)$$

where c_{ij} and $d_{ji}(i=1,2; j=1,2,3)$ represents connection weights between the I-layer and J -layer neurons. On the I-layer, x_i stand for the states of neurons and the inputs output by those neurons in the J -layer neurons via activation function f, while on the J -layer the neurons whose associated states denoted by y_j and the inputs output by those neurons in the I-layer via activation function f, $\mu(>0)$ describes the stability of internal neuron processing between I and J layers.

Since fractional calculus possesses the ability to capture memory and hereditary properties in various types of dynamical processes, it is a more effective tool for describing the objective world compared to integer-order counterparts([19], [20]). In recent years, a large number of outstanding works on fractional-order differential models have been continuously reported. Hence, the study of fractional-order delay neural networks has attracted a great deal of attention in research, such that it has become one of the most pressing and intriguing topics in the field.

Generally, the outcome of the bifurcation process is influenced to a greater or lesser extent by the parameters within the system. For instance, in [2], [4]-[7], [9]-[12], [18], [21]-[26], the issue of bifurcation caused by time delays has been extensively discussed. Moreover, as noted in [23], the effect of self-regulating parameters on the bifurcation results of the system are also regular. The concept of self-regulating parameters in neural networks pertains to the rate at which individual neurons revert to a static state in the absence of any synaptic connections. This rate essentially represents the decay rate of neural activity in [24]. Based on [25], we understand that the higher the value chosen, the greater the system's stability. Meanwhile, it is clear that the impact of self-regulating parameters on the system's bifurcation results is consistent. This consistency makes it

Manuscript received March 4, 2025; revised June 17, 2025.

This work was supported by the Foundation Research Project of Kaili University (grant No. 2024YB008), Sub-project of the Action Plan for Improving the Quality of Basic Education in Minority Areas of Guizhou Province(grant No. MJ202521).

Libo Wang is an associate professor of School of Science, Kaili University, Guizhou 556011, China (corresponding author to provide e-mail: wanglibo@kluniv.edu.cn).

Guigui Xu is an associate professor of School of Science, Kaili University, Guizhou 556011, China (e-mail: xuguigui586@163.com).

Pan Wang is a lecturer of School of Science, Xuchang University, Henan 461000, China (e-mail: wp521009@126.com).

valuable to investigate the bifurcation problem with self-regulating parameters as the primary bifurcation variables in the current study.

Inspired by previous studies, this paper investigates the fractional-order case of system (1):

$$\begin{cases} D^{\alpha_1}x_1(t) = -\mu x_1(t) + c_{11}f_{11}(y_1(t-\tau_3)) \\ +c_{12}f_{12}(y_2(t-\tau_3)) + c_{13}f_{13}(y_3(t-\tau_3)), \\ D^{\alpha_2}x_2(t) = -\mu x_2(t) + c_{21}f_{21}(y_1(t-\tau_4)) \\ +c_{22}f_{22}(y_2(t-\tau_4)) + c_{23}f_{23}(y_3(t-\tau_4)), \\ D^{\alpha_3}y_1(t) = -\mu y_1(t) + d_{11}g_{11}(x_1(t-\tau_1)) \\ +d_{12}g_{12}(x_2(t-\tau_2)), \\ D^{\alpha_4}y_2(t) = -\mu y_2(t) + d_{21}g_{21}(x_1(t-\tau_1)) \\ +d_{22}g_{22}(x_2(t-\tau_2)), \\ D^{\alpha_5}y_3(t) = -\mu y_3(t) + d_{31}g_{31}(x_1(t-\tau_1)) \\ +d_{32}g_{32}(x_2(t-\tau_2)), \end{cases} \tag{2}$$

where $\mu > 0$ represents self-regulating parameter of the neurons. $\alpha_k \in (0,1]$ $(k = 1,2,3,4,5)$ is the fractional order, $x_i(t), y_j(t)$ denote state variables. c_{ij} and $d_{ji}(i = 1,2, j = 1,2,3)$ are connection weights, $f_{ij}(\cdot), g_{ji}(\cdot)$ denote activation functions, τ_k stands for time delays.

The novel contributions and advancements presented in this paper are summarized as follows:

(1) For System (2), this paper carries out a thorough theoretical analysis of the bifurcation issue arising from self-regulating parameters, and determines the bifurcation critical point. The paper reveals that selecting substantial self-regulating parameter values is beneficial for preserving neural network stability.

(2) Using the Ferrari method traditionally, the computation of the bifurcation critical value yields a fifth-degree polynomial equation involving a self-regulating parameter. However, this conventional approach is both computationally intensive and susceptible to errors. To improve the efficiency and accuracy of the calculation, we have refined the method of employing implicit functions to determine the bifurcation point.

(3) The implicit array method employed in this paper is capable of addressing the bifurcation critical point issues in other high-dimensional systems, which is a novel contribution.

This paper is organized as follows. In Section II, some preliminaries and equivalent model of FOFBAMNN are presented. The self-regulating parameter induced-bifurcation outcomes are derived in Section III. In Section IV, the effectiveness of the theoretical results is verified through a numerical example. Section V, conclusions are drawn.

II. Preliminaries and Equivalent Model of FOFBAMNN

A. Preliminaries

In this section, several essential fundamental definitions on fractional calculus are prepared.

Definition 1[27] For a function $\phi(t) : [t_0, +\infty) \to R$, the Caputo fractional-order integral of $\phi(t)$ can be defined as

$${}_{t_0}I_t^{\alpha}\phi(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(s)\mathrm{d}s, \alpha > 0$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2[27] For a function $\phi(t) \in C^n([t_0, +\infty), R)$, the Caputo fractional-order derivative of $\phi(t)$ is given by

$${}_{t_0}^{C}D_t^{\alpha}\phi(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{\phi^n(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s, n-1 < \alpha < n$$

where $t \geq t_0$ and n is a positive integer.

The Laplace transform of the Caputo fractional-order derivatives can be deduced that

$$\mathcal{L}\{D^{\alpha}\phi(t);s\} = s^{\alpha}\Phi(s) - \textstyle\sum_{k=0}^{n-1} s^{\alpha-k-1}\phi^{(k)}(0), \quad n-1 < \alpha \leq n \in Z^+.$$

If $\phi^{(k)}(0) = 0, k = 0,1,2,\ldots,n$, then it follows that $\mathcal{L}\{D^{\alpha}\phi(t);s\} = s^{\alpha}\Phi(s)$.

B. Equivalent model of the FOFBAMNN

In order to obtain the main results in this paper, we make the following assumptions:

(A1) $\tau_1 + \tau_3 = \tau_2 + \tau_4 = 2\tau$,

(A2) $f_{ij}(\cdot), g_{ji}(\cdot) \in C^1(R,R), f_{ij}(0) = g_{ji}(0) = 0, (i = 1,2, j = 1,2,3)$.

In this paper, we adopt the same approach as in [17], [18]. To establish Hopf bifurcation conditions for system (2) with unequal delays, we utilize the delays as the bifurcation parameter. However, unlike the literature [17], [18], our paper deal with the bifurcation phenomenon induced by the self-regulating parameter.

In order to reduce the computational problems caused by multiple delays, an equivalent model of FOFBAMNN (2) is obtained.

Let

$$\begin{cases} u_1(t) = x_1\left(t + \frac{\tau_3-\tau_1}{2}\right), \\ u_2(t) = x_2\left(t + \frac{\tau_4-\tau_2}{2}\right), \\ u_3(t) = y_1(t), \\ u_4(t) = y_2(t), \\ u_5(t) = y_3(t). \end{cases}$$

Then, FOFBAMNN (2) can be directly transformed to

$$\begin{cases} D^{\alpha_1}u_1(t) = -\mu u_1(t) + c_{11}f_{11}(u_3(t-\tau)) \\ +c_{12}f_{12}(u_4(t-\tau)) + c_{13}f_{13}(u_5(t-\tau)), \\ D^{\alpha_2}u_2(t) = -\mu u_2(t) + c_{21}f_{21}(u_3(t-\tau)) \\ +c_{22}f_{22}(u_4(t-\tau)) + c_{23}f_{23}(u_5(t-\tau)), \\ D^{\alpha_3}u_3(t) = -\mu u_3(t) + d_{11}g_{11}(u_1(t-\tau)) \\ +d_{12}g_{12}(u_2(t-\tau)), \\ D^{\alpha_4}u_4(t) = -\mu u_4(t) + d_{21}g_{21}(u_1(t-\tau)) \\ +d_{22}g_{22}(u_2(t-\tau)), \\ D^{\alpha_5}u_5(t) = -\mu u_5(t) + d_{31}g_{31}(u_1(t-\tau)) \\ +d_{32}g_{32}(u_2(t-\tau)). \end{cases} \tag{3}$$

III. Main Results

The self-regulating parameter of μ is used as the bifurcation control parameter to detect the bifurcation condition in the module. Using (A2), the following linearized form of FOFBAMNN (2) can be derived.

$$\begin{cases} D^{\alpha_1}u_1(t) = -\mu u_1(t) + a_{11}u_3(t-\tau) + a_{12}u_4(t-\tau) \\ +a_{13}u_5(t-\tau), \\ D^{\alpha_2}u_2(t) = -\mu u_2(t) + a_{21}u_3(t-\tau) + a_{22}u_4(t-\tau) \\ +a_{23}u_5(t-\tau), \\ D^{\alpha_3}u_3(t) = -\mu u_3(t) + a_{31}u_1(t-\tau) + a_{32}u_2(t-\tau), \\ D^{\alpha_4}u_4(t) = -\mu u_4(t) + a_{41}u_1(t-\tau) + a_{42}u_2(t-\tau), \\ D^{\alpha_5}u_5(t) = -\mu u_5(t) + a_{51}u_1(t-\tau) + a_{52}u_2(t-\tau), \end{cases}$$

where $a_{ij} = c_{ij} f'_{ij}(0), a_{(j+2)i} = d_{ji} g'_{ji}(0), (i = 1,2, j = 1,2,3)$.

Under the Laplace transform method, the characteristic equation can be obtained as

$$Q_1(s) + Q_2(s)e^{-2s\tau} + Q_3(s)e^{-4s\tau} = 0, \tag{4}$$

where

$$\begin{aligned}
&Q_1(s) = s^{\sum_{i=1}^{5}\alpha_i} + \mu\sum_{i=1}^{5} s^{\sum_{j=1,j\neq i}^{5}\alpha_j} \\
&+\mu^2\sum_{i=1}^{5}\sum_{j=1,j>i}^{5}\sum_{k=1,k>j}^{5} s^{\alpha_i+\alpha_j+\alpha_k} \\
&+\mu^3\sum_{i=1}^{5}\sum_{j=1,j>i}^{5} s^{\alpha_i+\alpha_j} + \mu^4\sum_{i=1}^{5} s^{\alpha_i} + \mu^5, \\
&Q_2(s) = -\Big\{a_{21}a_{32}s^{\alpha_1+\alpha_4+\alpha_5} + a_{22}a_{42}s^{\alpha_1+\alpha_3+\alpha_5} \\
&+a_{11}a_{32}s^{\alpha_2+\alpha_4+\alpha_5} + a_{12}a_{41}s^{\alpha_2+\alpha_3+\alpha_5} + a_{13}a_{51}\times \\
&s^{\alpha_2+\alpha_3+\alpha_4} + a_{23}a_{52}s^{\alpha_1+\alpha_3+\alpha_4} \\
&+\mu[(a_{21}a_{32} + a_{22}a_{42})s^{\alpha_1+\alpha_5} + (a_{21}a_{32} + a_{23}a_{52})s^{\alpha_1+\alpha_4} \\
&+(a_{22}a_{42} + a_{23}a_{52})s^{\alpha_1+\alpha_3} + (a_{12}a_{41} + a_{13}a_{51})s^{\alpha_2+\alpha_3} \\
&+(a_{11}a_{32} + a_{13}a_{51})s^{\alpha_2+\alpha_4} + (a_{11}a_{32} + a_{12}a_{41})s^{\alpha_2+\alpha_5} \\
&+(a_{13}a_{51} + a_{23}a_{52})s^{\alpha_3+\alpha_4} + (a_{22}a_{42} + a_{12}a_{41})s^{\alpha_3+\alpha_5} \\
&+(a_{21}a_{32} + a_{11}a_{32})s^{\alpha_4+\alpha_5}] \\
&+\mu^2[(a_{21}a_{32} + a_{22}a_{42} + a_{23}a_{52})s^{\alpha_1} + (a_{11}a_{32} + a_{12}a_{41} \\
&+a_{13}a_{51})s^{\alpha_2} + (a_{22}a_{42} + a_{12}a_{41} + a_{13}a_{51} + a_{23}a_{52})s^{\alpha_3} \\
&+(a_{21}a_{32} + a_{11}a_{32} + a_{13}a_{51} + a_{23}a_{52})s^{\alpha_4} + (a_{21}a_{32} \\
&+a_{22}a_{42} + a_{11}a_{32} + a_{12}a_{41})s^{\alpha_5} + \mu^3(a_{21}a_{32} + a_{22}a_{42} \\
&+a_{11}a_{32} + a_{12}a_{41} + a_{13}a_{51} + a_{23}a_{52})]\Big\}, \\
&Q_3(s) = \Big\{[a_{31}a_{42}(a_{11}a_{22} - a_{12}a_{21}) + a_{32}a_{41}\times \\
&(a_{12}a_{21} - a_{11}a_{22})]s^{\alpha_5} + [a_{52}a_{31}(a_{11}a_{23} - a_{13}a_{21}) \\
&+a_{32}a_{51}(a_{13}a_{21} - a_{11}a_{23})]s^{\alpha_4} + [a_{52}a_{41}(a_{12}a_{23} \\
&-a_{13}a_{22}) + a_{42}a_{51}(a_{22}a_{13} - a_{12}a_{23})]s^{\alpha_3} \\
&+\mu[a_{31}a_{42}(a_{11}a_{22} - a_{12}a_{21}) + a_{32}a_{41}(a_{12}a_{21} - a_{11}a_{22}) \\
&+a_{32}a_{51}(a_{13}a_{21} - a_{11}a_{23}) + a_{42}a_{51}(a_{22}a_{13} - a_{12}a_{23}) \\
&+a_{52}a_{31}(a_{11}a_{23} - a_{13}a_{21}) + a_{52}a_{41}(a_{12}a_{23} - a_{13}a_{22})]\Big\}.
\end{aligned}$$

For simplicity, we denote the real and imaginary parts of $Q_p(s)(p = 1,2,3)$ as $Q_p^R(s)$, $Q_p^I(s)$, respectively. This implies that $s = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$, with $\omega > 0$, is a purely imaginary root of Eq. (4) if and only if the following equalities are satisfied.

$$\begin{aligned}
&Q_3^R(s)\cos(4\omega\tau) + Q_3^I(s)\sin(4\omega\tau) + Q_1^R(s) \\
&+Q_2^R(s)\cos(2\omega\tau) + Q_2^I(s)\sin(2\omega\tau) = 0, \\
&Q_3^I(s)\cos(4\omega\tau) - Q_3^R(s)\sin(4\omega\tau) + Q_1^I(s) \\
&+Q_2^I(s)\cos(2\omega\tau) - Q_2^R(s)\sin(2\omega\tau) = 0,
\end{aligned} \tag{5}$$

where $Q_p^R(s), Q_p^I(s)$ are defined by Appendix A.

To determine the bifurcation point, two binary functions $H_1(\mu,\omega)$ and $H_2(\mu,\omega)$, which are related to ω and μ, are described.

$$\begin{aligned}
H_1(\mu,\omega) = {}& Q_3^R\cos(4\omega\tau) + Q_3^I\sin(4\omega\tau) + Q_1^R \\
&+Q_2^R\cos(2\omega\tau) + Q_2^I\sin(2\omega\tau), \\
H_2(\mu,\omega) = {}& Q_3^I\cos(4\omega\tau) - Q_3^R\sin(4\omega\tau) + Q_1^I \\
&+Q_2^I\cos(2\omega\tau) - Q_2^R\sin(2\omega\tau).
\end{aligned}$$

Assume that there exists at least one point $\theta_k(\mu_k,\omega_k)$ where the curves $H_1(\mu,\omega)$ and $H_2(\mu,\omega)$ intersect. This implies that Eq. (5) has at least one positive real root. Based on the above assumption, the domain of the intersection $\theta_k(\mu_k,\omega_k)$ can be investigated, and subsequently utilize the command for implicit function arrays to address it. The solutions are contained within the specified common domain. This gives the critical value of the bifurcation

$$\mu_0 = \min\{\mu_k\}.$$

Hence, the bifurcation frequency ω_0 can be obtained by evaluating ω_k corresponding to μ_0 in $\theta_k(\mu_k,\omega_k)$.

To obtain conditions on the bifurcation, we give the following assumptions.

(A3) $\frac{M_1N_1+M_2N_2}{N_1^2+N_2^2} \neq 0$, where M_1, M_2, N_1, N_2 are represented by Eq. (9).

Lemma 1 Let $s(\mu) = \psi(\mu) + i\omega(\mu)$ be the root of Eq. (5) near $\mu = \mu_0$ obeying $\psi(\mu_0) = 0, \omega(\mu_0) = \omega_0$, then the following transversality condition holds

$$\mathrm{Re}\left[\frac{ds}{d\mu}\right]\bigg|_{(\mu=\mu_0,\omega=\omega_0)} \neq 0.$$

Proof Firstly, in order to simplify the process, we can define the variable components in Eq. (5).

$$\begin{aligned}
&Q_1(s) = h_0(s) + \mu h_1(s) + \mu^2 h_2(s) + \mu^3 h_3(s) + \mu^4 h_4(s) \\
&+\mu^5, \\
&Q_2(s) = -\left(g_0(s) + \mu g_1(s) + \mu^2 g_2(s) + \mu^3 g_3\right) \\
&Q_3(s) = k_0(s) + \mu k_1.
\end{aligned}$$

For computational purposes, we define the above symbols and their meanings

$$\begin{aligned}
&h_0(s) = s^{\sum_{i=1}^{5}\alpha_i}; h_1(s) = \sum_{i=1}^{5} s^{\sum_{j=1,j\neq i}^{5}\alpha_j}; \\
&h_2(s) = \sum_{i=1}^{5}\sum_{j=1,j>i}^{5}\sum_{k=1,k>j}^{5} s^{\alpha_i+\alpha_j+\alpha_k}; \\
&h_3(s) = \sum_{i=1}^{5}\sum_{j=1,j>i}^{5} s^{\alpha_i+\alpha_j}; h_4(s) = \sum_{i=1}^{5} s^{\alpha_i}; \\
&g_0(s) = a_{21}a_{32}s^{\alpha_1+\alpha_4+\alpha_5} + a_{22}a_{42}s^{\alpha_1+\alpha_3+\alpha_5} \\
&+a_{11}a_{32}s^{\alpha_2+\alpha_4+\alpha_5} + a_{12}a_{41}s^{\alpha_2+\alpha_3+\alpha_5} \\
&+a_{13}a_{51}s^{\alpha_2+\alpha_3+\alpha_4} + a_{23}a_{52}s^{\alpha_1+\alpha_3+\alpha_4}; \\
&g_1(s) = (a_{21}a_{32} + a_{22}a_{42})s^{\alpha_1+\alpha_5} \\
&+(a_{21}a_{32} + a_{23}a_{52})s^{\alpha_1+\alpha_4} + (a_{22}a_{42} + a_{23}a_{52})s^{\alpha_1+\alpha_3} \\
&+(a_{12}a_{41} + a_{13}a_{51})s^{\alpha_2+\alpha_3} + (a_{11}a_{32} + a_{13}a_{51})s^{\alpha_2+\alpha_4} \\
&+(a_{11}a_{32} + a_{12}a_{41})s^{\alpha_2+\alpha_5} + (a_{13}a_{51} + a_{23}a_{52})s^{\alpha_3+\alpha_4} \\
&+(a_{22}a_{42} + a_{12}a_{41})s^{\alpha_3+\alpha_5} + (a_{21}a_{32} + a_{11}a_{32})s^{\alpha_4+\alpha_5}; \\
&g_2(s) = (a_{21}a_{32} + a_{22}a_{42} + a_{23}a_{52})s^{\alpha_1} \\
&+(a_{11}a_{32} + a_{12}a_{41} + a_{13}a_{51})s^{\alpha_2} \\
&+(a_{22}a_{42} + a_{12}a_{41} + a_{13}a_{51} + a_{23}a_{52})s^{\alpha_3} \\
&+(a_{21}a_{32} + a_{11}a_{32} + a_{13}a_{51} + a_{23}a_{52})s^{\alpha_4} \\
&+(a_{21}a_{32} + a_{22}a_{42} + a_{11}a_{32} + a_{12}a_{41})s^{\alpha_5}; \\
&g_3 = a_{21}a_{32} + a_{22}a_{42} + a_{11}a_{32} \\
&+a_{12}a_{41} + a_{13}a_{51} + a_{23}a_{52}; \\
&k_0(s) = \\
&[a_{31}a_{42}(a_{11}a_{22} - a_{12}a_{21}) + a_{32}a_{41}(a_{12}a_{21} - a_{11}a_{22})]\, s^{\alpha_5} \\
&+[a_{52}a_{31}(a_{11}a_{23} - a_{13}a_{21}) + a_{32}a_{51}(a_{13}a_{21} - a_{11}a_{23})]s^{\alpha_4} \\
&+[a_{52}a_{41}(a_{12}a_{23} - a_{13}a_{22}) + a_{42}a_{51}(a_{22}a_{13} - a_{12}a_{23})]s^{\alpha_3}; \\
&k_1 = a_{31}a_{42}(a_{11}a_{22} - a_{12}a_{21}) + a_{32}a_{41}(a_{12}a_{21} - a_{11}a_{22}) \\
&+a_{32}a_{51}(a_{13}a_{21} - a_{11}a_{23}) + a_{42}a_{51}(a_{22}a_{13} - a_{12}a_{23}) \\
&+a_{52}a_{31}(a_{11}a_{23} - a_{13}a_{21}) + a_{52}a_{41}(a_{12}a_{23} - a_{13}a_{22}).
\end{aligned}$$

Then, Eq. (4) develops into

$$\begin{aligned}
&h_0(s) + \mu h_1(s) + \mu^2 h_2(s) + \mu^3 h_3(s) + \mu^4 h_4(s) + \mu^5 \\
&-\left(g_0(s) + \mu g_1(s) + \mu^2 g_2(s) + \mu^3 g_3\right)e^{-2s\tau} \\
&+\left(k_0(s) + \mu k_1\right)e^{-4s\tau} = 0,
\end{aligned} \tag{6}$$

For simplicity, the real and imaginary parts of $h_p(s), h'_p(s), g_q(s), g'_q(s), k_l(s),\ k'_l(s) (p = 0,1,2,3,4, q = 0,1,2,3, l = 0,1)$ can be marked with $h_p^R(s)$ and $h_p^I(s)$, $h_p'^R(s)$ and $h_p'^I(s)$, $g_q^R(s)$ and $g_q^I(s)$, $g_q'^R(s)$ and $g_q'^I(s)$,

$k_l^R(s)$ and $k_l^I(s)$, $k_l'^R(s)$ and $k_l'^I(s)$, respectively. By employing the implicit function theorem, we proceed to differentiate μ with respect to the variables in Eq. (6), then

$$\begin{aligned} & h'_0(s)\tfrac{\mathrm{d}s}{\mathrm{d}\mu} + h_1(s) + \mu h'_1(s)\tfrac{\mathrm{d}s}{\mathrm{d}\mu} + 2\mu h_2(s) \\ & +\mu^2 h'_2(s)\tfrac{\mathrm{d}s}{\mathrm{d}\mu} + 3\mu^2 h_3(s) + \mu^3 h'_3(s)\tfrac{\mathrm{d}s}{\mathrm{d}\mu} + 4\mu^3 h_4(s) \\ & +\mu^4 h'_4(s)\tfrac{\mathrm{d}s}{\mathrm{d}\mu} + 5\mu^4 + (g_0(s) + \mu g_1(s) + \mu^2 g_2(s) \\ & +\mu^3 g_3) 2\tau e^{-2s\tau}\tfrac{\mathrm{d}s}{\mathrm{d}\mu} - \Big(g'_0(s)\tfrac{\mathrm{d}s}{\mathrm{d}\mu} + g_1(s) + \mu g'_1(s)\tfrac{\mathrm{d}s}{\mathrm{d}\mu} \\ & +2\mu g_2(s) + \mu^2 g'_2(s)\tfrac{\mathrm{d}s}{\mathrm{d}\mu} + 3\mu^2 g_3\Big) e^{-2s\tau} \\ & + \Big(k'_0(s)\tfrac{\mathrm{d}s}{\mathrm{d}\mu} + k_1\Big) e^{-4s\tau} \\ & - (k_0(s) + \mu k_1)\, 4\tau e^{-4s\tau}\tfrac{\mathrm{d}s}{\mathrm{d}\mu} = 0, \end{aligned} \tag{7}$$

It follows from Eq. (7) that

$$\frac{\mathrm{d}s}{\mathrm{d}\mu} = \frac{M(s)}{N(s)}, \tag{8}$$

where

$$\begin{aligned} & M(s) = \left[g_1(s) + 2\mu g_2(s) + 3\mu^2 g_3\right] e^{-2s\tau} - k_1 e^{-4s\tau} \\ & -h_1(s) - 2\mu h_2(s) - 5\mu^4 - 3\mu^2 h_3(s) - 4\mu^3 h_4(s); \\ & N(s) = h'_0(s) + \mu h'_1(s) + \mu^2 h'_2(s) + \mu^3 h'_3(s) \\ & +\mu^4 h'_4(s) + \big(g_0(s) + \mu g_1(s) + \mu^2 g_2(s + \mu^3 g_3\big) 2\tau e^{-2s\tau} \\ & - \big(g'_0(s) + \mu g'_1(s) + \mu^2 g'_2(s)\big) e^{-2s\tau} + k'_0(s) e^{-4s\tau} \\ & - (k_0(s) + \mu k_1)\, 4\tau e^{-4s\tau}; \\ & h'_0(s) = (\textstyle\sum_{i=1}^5 \alpha_i) s^{\sum_{i=1}^5 \alpha_i - 1}; \\ & h'_1(s) = \textstyle\sum_{i=1}^5 (\sum_{j=1, j\neq i}^5 \alpha_j) s^{\sum_{j=1, j\neq i}^5 \alpha_j - 1}; \\ & h'_2(s) = \textstyle\sum_{i=1}^5 \sum_{j=1, j>i}^5 \sum_{k=1, k>j}^5 (\alpha_i + \alpha_j + \alpha_k) \times \\ & s^{\alpha_i + \alpha_j + \alpha_k - 1}; \\ & h'_3(s) = \textstyle\sum_{i=1}^5 \sum_{j=1, j>i}^5 (\alpha_i + \alpha_j) s^{\alpha_i + \alpha_j - 1}; \\ & h'_4(s) = \textstyle\sum_{i=1}^5 \alpha_i s^{\alpha_i - 1}; \\ & g'_0(s) = a_{21}a_{32}(\alpha_1 + \alpha_4 + \alpha_5) s^{\alpha_1 + \alpha_4 + \alpha_5 - 1} \\ & +a_{22}a_{42}(\alpha_1 + \alpha_3 + \alpha_5) s^{\alpha_1 + \alpha_3 + \alpha_5 - 1} \\ & +a_{11}a_{32}(\alpha_2 + \alpha_4 + \alpha_5) s^{\alpha_2 + \alpha_4 + \alpha_5 - 1} \\ & +a_{12}a_{41}(\alpha_2 + \alpha_3 + \alpha_5) s^{\alpha_2 + \alpha_3 + \alpha_5 - 1} \\ & +a_{13}a_{51}(\alpha_2 + \alpha_3 + \alpha_4) s^{\alpha_2 + \alpha_3 + \alpha_4 - 1} \\ & +a_{23}a_{52}(\alpha_1 + \alpha_3 + \alpha_4) s^{\alpha_1 + \alpha_3 + \alpha_4 - 1}; \\ & g'_1(s) = (a_{21}a_{32} + a_{22}a_{42})(\alpha_1 + \alpha_5) s^{\alpha_1 + \alpha_5 - 1} \\ & +(a_{21}a_{32} + a_{23}a_{52})(\alpha_1 + \alpha_4) s^{\alpha_1 + \alpha_4 - 1} \\ & +(a_{22}a_{42} + a_{23}a_{52})(\alpha_1 + \alpha_3) s^{\alpha_1 + \alpha_3 - 1} \\ & +(a_{12}a_{41} + a_{13}a_{51})(\alpha_2 + \alpha_3) s^{\alpha_2 + \alpha_3 - 1} \\ & +(a_{11}a_{32} + a_{13}a_{51})(\alpha_2 + \alpha_4) s^{\alpha_2 + \alpha_4 - 1} \\ & +(a_{11}a_{32} + a_{12}a_{41})(\alpha_2 + \alpha_5) s^{\alpha_2 + \alpha_5 - 1} \\ & +(a_{13}a_{51} + a_{23}a_{52})(\alpha_3 + \alpha_4) s^{\alpha_3 + \alpha_4 - 1} \\ & +(a_{22}a_{42} + a_{12}a_{41})(\alpha_3 + \alpha_5) s^{\alpha_3 + \alpha_5 - 1} \\ & +(a_{21}a_{32} + a_{11}a_{32})(\alpha_4 + \alpha_5) s^{\alpha_4 + \alpha_5 - 1}; \\ & g'_2(s) = (a_{21}a_{32} + a_{22}a_{42} + a_{23}a_{52})\, \alpha_1 s^{\alpha_1 - 1} \\ & +(a_{11}a_{32} + a_{12}a_{41} + a_{13}a_{51})\alpha_2 s^{\alpha_2 - 1} \\ & +(a_{22}a_{42} + a_{12}a_{41} + a_{13}a_{51} + a_{23}a_{52})\alpha_3 s^{\alpha_3 - 1} \\ & +(a_{21}a_{32} + a_{11}a_{32} + a_{13}a_{51} + a_{23}a_{52})\alpha_4 s^{\alpha_4 - 1} \\ & +(a_{21}a_{32} + a_{22}a_{42} + a_{11}a_{32} + a_{12}a_{41})\alpha_5 s^{\alpha_5 - 1}; \\ & g'_3 = 0; \\ & k'_0(s) = [a_{31}a_{42}(a_{11}a_{22} - a_{12}a_{21}) + a_{32}a_{41} \times \\ & (a_{12}a_{21} - a_{11}a_{22})]\alpha_5 s^{\alpha_5 - 1} + [a_{52}a_{31}(a_{11}a_{23} - a_{13}a_{21}) \\ & +a_{32}a_{51}(a_{13}a_{21} - a_{11}a_{23})]\alpha_4 s^{\alpha_4 - 1} + [a_{52}a_{41} \times \\ & (a_{12}a_{23} - a_{13}a_{22}) + a_{42}a_{51}(a_{22}a_{13} - a_{12}a_{23})]\alpha_3 s^{\alpha_3 - 1}; \\ & k'_1 = 0. \end{aligned}$$

From the calculations derived from Eq. (8), we conclude that:

$$\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\mu}\right]\Bigg|_{(\mu=\mu_0, \omega=\omega_0)} = \frac{M_1 N_1 + M_2 N_2}{N_1^2 + N_2^2}, \tag{9}$$

where M_1, M_2, N_1, N_2 are represented by Appendix B.

(A3) indicates that the transversality condition is viable.

According to the definition of the self-regulating parameter, μ cannot be set to zero, such that the difficulty lies in estimating the stability of the initial state. Consequently, we adopt the limit $\mu \to 0^+$ instead of directly discussing the transversality condition for $\mu_0 = 0$. Under this assumption, we consider the critical value associated with ω to be $\omega_0^\star$.

To make the transversal conditions of this with meaningful content, the following assumption is essential.

(A4) $\frac{\overline{M}_1\overline{N}_1 + \overline{M}_2\overline{N}_2}{\overline{N}_1^2 + \overline{N}_2^2} \neq 0$, where $\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2$ are represented by Eq. (10).

Lemma 2 Let $s(\mu) = \psi(\mu) + i\omega(\mu)$ be the root of Eq. (3) near $\mu \to 0^+$ obeying $\psi(\mu_0) = 0, \omega(\mu_0) = \omega_0^\star$, if (A4) is satisfied, the system (2) is locally unstable at $\mu \to 0^+$.

Proof The real and imaginary parts of $\overline{h}_p(s), \overline{h}_p^{'}(s)$, $\overline{g}_q(s), \overline{g}_q'(s), \overline{k}_0(s), \overline{k}_0^{'}(s) (p = 0,1,2,3,4, q = 0,1,2)$ can be marked with $\overline{h}_p^R(s)$ and $\overline{h}_p^I(s)$, $\overline{h}_p'^R(s)$ and $\overline{h}_p'^I(s)$, $\overline{g}_q^R(s)$ and $\overline{g}_q^I(s)$, $\overline{g}_q'^R(s)$ and $\overline{g}_q'^I(s)$, $\overline{k}_0^R(s)$ and $\overline{k}_0^I(s)$, $\overline{k}_0'^R(s)$ and $\overline{k}_0'^I(s)$, respectively.

By the analogous treatment, in accordance with the condition of Lemma 1, Eq. (9) becomes

$$\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\mu}\right]\Bigg|_{(\mu\to 0^+, \omega=\omega_0^\star)} = \frac{\overline{M}_1\overline{N}_1 + \overline{M}_2\overline{N}_2}{\overline{N}_1^2 + \overline{N}_2^2}, \tag{10}$$

where $\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2$ are represented by Appendix C.

According to (10) and (A4), as the transversal condition approaches $\mu \to 0^+$, the following assertion can be derived:

$$\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\mu}\right]\Bigg|_{(\mu\to 0^+, \omega=\omega_0^\star)} = \frac{\overline{M}_1\overline{N}_1 + \overline{M}_2\overline{N}_2}{\overline{N}_1^2 + \overline{N}_2^2} \neq 0,$$

Therefore, the system (2) is locally unstable as $\mu \to 0^+$.

Theorem 1 Assuming that (A1)–(A4) are satisfied, the following results hold.

(i) The zero equilibrium of the FOFBAMNN (2) is locally unstable for $\mu \in (0, \mu_0)$.

(ii) The zero equilibrium of FOFBAMNN (2) undergoes Hopf bifurcation at $\mu = \mu_0$, meaning that the zero equilibrium of FOFBAMNN (2) is asymptotically stable from the zero equilibrium near $\mu = \mu_0$.

Remark 1 It is crucial to invert the FOFBAMNN (2) directly into the FOFBAMNN (3) through assumption and translation knowledge. The rationale behind this hypothesis is to mitigate the influence of time delays on the computational process. Meanwhile, the primary objective of this paper is to delve into the study of self-regulating parameters, which will not impart a substantial impact on the theoretical analysis or the results presented. Especially, it is worth noting that state translation will not alter any of the outcomes.

Remark 2 The critical value of bifurcation in this paper can be derived through an explicit expression in accordance with Eq. (4) and the root-finding formula of the univariate quintic equation, i.e. Ferrari method ([28]), which is a widely employed computational technique. To enhance the computational method, this paper incorporates a combination of numbers and equations, as well as an implicit array command to determine the bifurcation point. The outcomes are both intuitive and precise.

Remark 3 We discuss the bifurcation of fractional-order systems with time delay in this paper. It is obvious that when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5$, the system (2) reduces to (1) in [29]. The bifurcation results, obtained in virtue of the self-regulating parameter as the bifurcation variable, reveal an exact converse to those induced by time-delay-induced bifurcations (see [29]). The behavior of the system in this paper initially exhibits instability, followed by stability. Conversely, time delay induced bifurcations lead to a system that tends to stabilize before the bifurcation critical point, then it bifurcates.

Remark 4 This paper explores the phenomenon of bifurcation triggered by self-regulating parameters within the FOFBAMNN model, simultaneously, a series of noteworthy findings have been derived. When examining the impact of these self-regulating parameters on system stability, it becomes apparent that their effects differ from those of order ([26]) and time delay ([18]). Large self-regulating parameters are more conducive to the stability of the system, whereas small order and time delay are more conducive to its stability. This is evident when the problem from the perspective of resolving bifurcation points is considered. The method used in this paper is not confined by the structural and dimensional constraints of the model.

IV. Numerical Simulations

In this section, we give a numerical example to show the effectiveness of our main results. Thanks to the equivalence between FOFBAMNN (2) and FOFBAMNN (3), we utilize FOFBAMNN (3) as a foundation to support our theoretical findings through our simulations.

We consider the following example.

$$\begin{cases} D^{\alpha_1}u_1(t) = -\mu u_1(t) - 2\tanh(u_3(t-\tau)) \\ +0.8\tanh(u_4(t-\tau)) + 2\tanh(u_5(t-\tau)), \\ D^{\alpha_2}u_2(t) = -\mu u_2(t) - \tanh(u_3(t-\tau)) \\ -\tanh(u_4(t-\tau)) + 2\tanh(u_5(t-\tau)), \\ D^{\alpha_3}u_3(t) = -\mu u_3(t) - 2\tanh(u_1(t-\tau)) \\ +\tanh(u_2(t-\tau)), \\ D^{\alpha_4}u_4(t) = -\mu u_4(t) - 2\tanh(u_1(t-\tau)) \\ +2\tanh(u_2(t-\tau)), \\ D^{\alpha_5}u_5(t) = -\mu u_5(t) - \tanh(u_1(t-\tau)) \\ -2\tanh(u_2(t-\tau)). \end{cases} \tag{11}$$

The initial conditions are given as $(u_1(0), u_2(0), u_3(0), u_4(0), u_5(0)) = (0.02, 0.06, 0.06, 0.06, 0.05)$. When $\alpha_1 = 0.92, \alpha_2 = 0.93, \alpha_3 = 0.95, \alpha_4 = 0.90, \alpha_5 = 0.92, \tau = 0.25$, we can apply Matlab to calculate the intersection point of $H_1(\mu,\omega)$ and $H_2(\mu,\omega)$ is $(1.9209, 0.0834)$ i.e. $\mu_0 = 1.9209, \omega_0 = 0.0843$, and the corresponding transversality condition is $\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\mu}\right]\Big|_{(\mu=\mu_0,\omega=\omega_0)} \neq 0$. Consequently, verifying the accuracy of Theorem 1 is a straightforward task. On the basis of Theorem 1, the zero equilibrium point of FOFBAMNN (11) is locally unstable when $\mu = 1.86 < \mu_0$, which is demonstrated in Figs. 1-2. The zero equilibrium point of FOFBAMNN (11) is locally asymptotically stable when $\mu = 2.02 > \mu_0$, and Hopf bifurcation occurs, which is manifested in Figs. 3-4. It is observed that the system's five neurons undergo a sudden transformation in state as they approach the bifurcation threshold at $\mu_0 = 1.9209$. This transition marks a shift from a state of bifurcation to one of stability.

V. Conclusion

This paper has delved into the bifurcation issue with self-regulating parameters that arises within the FOFBAMNN model. Our findings indicate that such parameters can lead to the instability of BAM neural networks. This insight is crucial for enhancing the control of systems in practical applications, and ensure that they operate in a stable and reliable state. We have achieved the following accomplishments in this paper: 1) We have confirmed that self-regulating parameters significantly influence the stability of the system (2), and this impact is governed by specific rules. Before reaching the bifurcation critical point, the system is unstable, whereas it becomes stable after passing this critical point. 2) The paper presents an accurate and efficient solution for the bifurcation point, by the means of implicit arrays. This innovative method serves as a remarkable alternative when the explicit representation of the solution is intricate or inherently non-expressible in a simple form.

Appendix A

$$\begin{aligned}
&Q_1^R(s) = \omega^{\sum_{i=1}^5 \alpha_i}\cos\left(\frac{\pi\sum_{i=1}^5 \alpha_i}{2}\right) \\
&+\mu\sum_{i=1}^5 \omega^{\sum_{j=1,j\neq i}^5 \alpha_j}\cos\left(\frac{\pi\sum_{j=1,j\neq i}^5 \alpha_j}{2}\right) + \mu^2\times \\
&\sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5 \omega^{\alpha_i+\alpha_j+\alpha_k}\cos\left(\frac{(\alpha_i+\alpha_j+\alpha_k)\pi}{2}\right) \\
&+\mu^3\sum_{i=1}^5\sum_{j=1,j>i}^5 \omega^{\alpha_i+\alpha_j}\cos\left(\frac{(\alpha_i+\alpha_j)\pi}{2}\right) \\
&+\mu^4\sum_{i=1}^5 \omega^{\alpha_i}\cos\frac{\alpha_i\pi}{2} + \mu^5, \\
&Q_1^I(s) = \omega^{\sum_{i=1}^5 \alpha_i}\sin\left(\frac{\pi\sum_{i=1}^5 \alpha_i}{2}\right) \\
&+\mu\sum_{i=1}^5 \omega^{\sum_{j=1,j\neq i}^5 \alpha_j}\sin\left(\frac{\pi\sum_{j=1,j\neq i}^5 \alpha_j}{2}\right) + \mu^2\times \\
&\sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5 \omega^{\alpha_i+\alpha_j+\alpha_k}\sin\left(\frac{(\alpha_i+\alpha_j+\alpha_k)\pi}{2}\right) \\
&+\mu^3\sum_{i=1}^5\sum_{j=1,j>i}^5 \omega^{\alpha_i+\alpha_j}\sin\left(\frac{(\alpha_i+\alpha_j)\pi}{2}\right) \\
&+\mu^4\sum_{i=1}^5 \omega^{\alpha_i}\sin\frac{\alpha_i\pi}{2} + \mu^5, \\
&Q_2^R(s) = -\Bigg\{a_{21}a_{32}\omega^{\alpha_1+\alpha_4+\alpha_5}\cos\left(\frac{(\alpha_1+\alpha_4+\alpha_5)\pi}{2}\right) \\
&+a_{22}a_{42}\omega^{\alpha_1+\alpha_3+\alpha_5}\cos\left(\frac{(\alpha_1+\alpha_3+\alpha_5)\pi}{2}\right) \\
&+a_{11}a_{32}\omega^{\alpha_2+\alpha_4+\alpha_5}\cos\left(\frac{(\alpha_2+\alpha_4+\alpha_5)\pi}{2}\right) \\
&+a_{12}a_{41}\omega^{\alpha_2+\alpha_3+\alpha_5}\cos\left(\frac{(\alpha_2+\alpha_3+\alpha_5)\pi}{2}\right) \\
&+a_{13}a_{51}\omega^{\alpha_2+\alpha_3+\alpha_4}\cos\left(\frac{(\alpha_2+\alpha_3+\alpha_5)\pi}{2}\right) \\
&+a_{23}a_{52}\omega^{\alpha_1+\alpha_3+\alpha_4}\cos\left(\frac{(\alpha_1+\alpha_3+\alpha_4)\pi}{2}\right) \\
&+\mu\Bigg[(a_{21}a_{32}+a_{22}a_{42})\omega^{\alpha_1+\alpha_5}\cos\left(\frac{(\alpha_1+\alpha_5)\pi}{2}\right) \\
&+(a_{21}a_{32}+a_{23}a_{52})\omega^{\alpha_1+\alpha_4}\cos\left(\frac{(\alpha_1+\alpha_4)\pi}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+(a_{22}a_{42}+a_{23}a_{52})\omega^{\alpha_1+\alpha_3}\cos\left(\tfrac{(\alpha_1+\alpha_3)\pi}{2}\right)\\
&+(a_{12}a_{41}+a_{13}a_{51})\omega^{\alpha_2+\alpha_3}\cos\left(\tfrac{(\alpha_2+\alpha_3)\pi}{2}\right)\\
&+(a_{11}a_{32}+a_{13}a_{51})\omega^{\alpha_2+\alpha_4}\cos\left(\tfrac{(\alpha_2+\alpha_4)\pi}{2}\right)\\
&+(a_{11}a_{32}+a_{12}a_{41})\omega^{\alpha_2+\alpha_5}\cos\left(\tfrac{(\alpha_2+\alpha_5)\pi}{2}\right)\\
&+(a_{13}a_{51}+a_{23}a_{52})\omega^{\alpha_3+\alpha_4}\cos\left(\tfrac{(\alpha_3+\alpha_4)\pi}{2}\right)\\
&+(a_{22}a_{42}+a_{12}a_{41})\omega^{\alpha_3+\alpha_5}\cos\left(\tfrac{(\alpha_3+\alpha_5)\pi}{2}\right)\\
&+(a_{21}a_{32}+a_{11}a_{32})\omega^{\alpha_4+\alpha_5}\cos\left(\tfrac{(\alpha_4+\alpha_5)\pi}{2}\right)\Bigg]\\
&+\mu^2\Bigg[(a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\omega^{\alpha_1}\cos\tfrac{\alpha_1\pi}{2}\\
&+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\omega^{\alpha_2}\cos\tfrac{\alpha_2\pi}{2}\\
&+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}+a_{23}a_{52})\omega^{\alpha_3}\cos\tfrac{\alpha_3\pi}{2}\\
&+(a_{21}a_{32}+a_{11}a_{32}+a_{13}a_{51}+a_{23}a_{52})\omega^{\alpha_4}\cos\tfrac{\alpha_4\pi}{2}\\
&+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\omega^{\alpha_5}\cos\tfrac{\alpha_5\pi}{2}\Bigg]\\
&+\mu^3(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51}\\
&+a_{23}a_{52})\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
&Q_2^I(s)=-\Bigg\{a_{21}a_{32}\omega^{\alpha_1+\alpha_4+\alpha_5}\sin\left(\tfrac{(\alpha_1+\alpha_4+\alpha_5)\pi}{2}\right)\\
&+a_{22}a_{42}\omega^{\alpha_1+\alpha_3+\alpha_5}\sin\left(\tfrac{(\alpha_1+\alpha_3+\alpha_5)\pi}{2}\right)\\
&+a_{11}a_{32}\omega^{\alpha_2+\alpha_4+\alpha_5}\sin\left(\tfrac{(\alpha_2+\alpha_4+\alpha_5)\pi}{2}\right)\\
&+a_{12}a_{41}\omega^{\alpha_2+\alpha_3+\alpha_5}\sin\left(\tfrac{(\alpha_2+\alpha_3+\alpha_5)\pi}{2}\right)\\
&+a_{13}a_{51}\omega^{\alpha_2+\alpha_3+\alpha_4}\sin\left(\tfrac{(\alpha_2+\alpha_3+\alpha_5)\pi}{2}\right)\\
&+a_{23}a_{52}\omega^{\alpha_1+\alpha_3+\alpha_4}\sin\left(\tfrac{(\alpha_1+\alpha_3+\alpha_4)\pi}{2}\right)\\
&+\mu\Bigg[(a_{21}a_{32}+a_{22}a_{42})\omega^{\alpha_1+\alpha_5}\sin\left(\tfrac{(\alpha_1+\alpha_5)\pi}{2}\right)\\
&+(a_{21}a_{32}+a_{23}a_{52})\omega^{\alpha_1+\alpha_4}\sin\left(\tfrac{(\alpha_1+\alpha_4)\pi}{2}\right)\\
&+(a_{22}a_{42}+a_{23}a_{52})\omega^{\alpha_1+\alpha_3}\sin\left(\tfrac{(\alpha_1+\alpha_3)\pi}{2}\right)\\
&+(a_{12}a_{41}+a_{13}a_{51})\omega^{\alpha_2+\alpha_3}\sin\left(\tfrac{(\alpha_2+\alpha_3)\pi}{2}\right)\\
&+(a_{11}a_{32}+a_{13}a_{51})\omega^{\alpha_2+\alpha_4}\sin\left(\tfrac{(\alpha_2+\alpha_4)\pi}{2}\right)\\
&+(a_{11}a_{32}+a_{12}a_{41})\omega^{\alpha_2+\alpha_5}\sin\left(\tfrac{(\alpha_2+\alpha_5)\pi}{2}\right)\\
&+(a_{13}a_{51}+a_{23}a_{52})\omega^{\alpha_3+\alpha_4}\sin\left(\tfrac{(\alpha_3+\alpha_4)\pi}{2}\right)\\
&+(a_{22}a_{42}+a_{12}a_{41})\omega^{\alpha_3+\alpha_5}\sin\left(\tfrac{(\alpha_3+\alpha_5)\pi}{2}\right)\\
&+(a_{21}a_{32}+a_{11}a_{32})\omega^{\alpha_4+\alpha_5}\sin\left(\tfrac{(\alpha_4+\alpha_5)\pi}{2}\right)\Bigg]\\
&+\mu^2\Big[(a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\omega^{\alpha_1}\sin\tfrac{\alpha_1\pi}{2}\\
&+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\omega^{\alpha_2}\sin\tfrac{\alpha_2\pi}{2}\\
&+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}+a_{23}a_{52})\omega^{\alpha_3}\sin\tfrac{\alpha_3\pi}{2}+\\
&(a_{21}a_{32}+a_{11}a_{32}+a_{13}a_{51}+a_{23}a_{52})\omega^{\alpha_4}\sin\tfrac{\alpha_4\pi}{2}\\
&+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\omega^{\alpha_5}\sin\tfrac{\alpha_5\pi}{2}\Big]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
&Q_3^R(s)=\Big\{\big[a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}\times\\
&(a_{12}a_{21}-a_{11}a_{22})\big]\omega^{\alpha_5}\cos\tfrac{\alpha_5\pi}{2}+\big[a_{52}a_{31}(a_{11}a_{23}\\
&-a_{13}a_{21})+a_{32}a_{51}(a_{13}a_{21}-a_{11}a_{23})\big]\omega^{\alpha_4}\cos\tfrac{\alpha_4\pi}{2}\\
&+\big[a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})+a_{42}a_{51}(a_{22}a_{13}-a_{12}a_{23})\big]\\
&\omega^{\alpha_3}\cos\tfrac{\alpha_3\pi}{2}+\mu\big[a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})\\
&+a_{32}a_{41}(a_{12}a_{21}-a_{11}a_{22})+a_{32}a_{51}(a_{13}a_{21}-a_{11}a_{23})\\
&+a_{42}a_{51}(a_{22}a_{13}-a_{12}a_{23})+a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})\\
&+a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})\big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&Q_3^I(s)=\Big\{\big[a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}(a_{12}a_{21}\\
&-a_{11}a_{22})\big]\omega^{\alpha_5}\sin\tfrac{\alpha_5\pi}{2}+\big[a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})\\
&+a_{32}a_{51}(a_{13}a_{21}-a_{11}a_{23})\big]\omega^{\alpha_4}\sin\tfrac{\alpha_4\pi}{2}+\big[a_{52}a_{41}(a_{12}a_{23}\\
&-a_{13}a_{22})+a_{42}a_{51}(a_{22}a_{13}-a_{12}a_{23})\big]\omega^{\alpha_3}\sin\tfrac{\alpha_3\pi}{2}\Big\}.
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
&M_1=\left[g_1^R(s)+2\mu_0g_2^R(s)+3\mu_0^2g_3\right]\cos(2\omega_0\tau)\\
&+\left[g_1^I(s)+2\mu_0g_2^I(s)\right]\sin(2\omega_0\tau)-k_1\cos(4\omega_0\tau)-h_1^R(s)\\
&-2\mu_0h_2^R(s)-3\mu_0^2h_3^R(s)-4\mu_0^3h_4^R(s)-5\mu_0^4,\\
&M_2=-\left[g_1^R(s)+2\mu_0g_2^R(s)+3\mu_0^2g_3\right]\sin(2\omega_0\tau)\\
&+\left[g_1^I(s)+2\mu_0g_2^I(s)\right]\cos(2\omega_0\tau)-k_1\sin(4\omega_0\tau)-h_1^I(s)\\
&-2\mu_0h_2^I(s)-3\mu_0^2h_3^I(s)-4\mu_0^3h_4^I(s),\\
&N_1=h''^R_0(s)+\mu_0h'^R_1(s)+\mu_0^2h'^R_2(s)+\mu_0^3h'^R_3(s)\\
&+\mu_0^4h'^R_4(s)+\left(g_0^R(s)+\mu_0g_1^R(s)+\mu_0^2g_2^R(s)+\mu_0^3g_3\right)2\tau\times\\
&\cos(2\omega_0\tau)+\left(g_0^I(s)+\mu_0g_1^I(s)+\mu_0^2g_2^I(s)\right)2\tau\sin(2\omega_0\tau)\\
&-\left(g'^R_0(s)+\mu_0g'^R_1(s)+\mu_0^2g'^R_2(s)\right)\cos(2\omega_0\tau)\\
&-\left(g'^I_0(s)+\mu_0g'^I_1(s)+\mu_0^2g'^I_2(s)\right)\sin(2\omega_0\tau)\\
&+k'^R_0(s)\cos(4\omega_0\tau)+k'^I_0(s)\sin(4\omega_0\tau)\\
&-\left(k_0^R(s)+\mu_0k_1\right)4\tau\cos(4\omega_0\tau)-k_0^I(s)4\tau\sin(4\omega_0\tau),\\
&N_2=h'^I_0(s)+\mu_0h'^I_1(s)+\mu_0^2h'^I_2(s)+\mu_0^3h'^I_3(s)\\
&+\mu_0^4h'^I_4(s)-\left(g_0^R(s)+\mu_0g_1^R(s)+\mu_0^2g_2^R(s)+\mu_0^3g_3\right)2\tau\times\\
&\sin(2\omega_0\tau)+\left(g_0^I(s)+\mu_0g_1^I(s)+\mu_0^2g_2^I(s)\right)2\tau\cos(2\omega_0\tau)\\
&+\left(g'^R_0(s)+\mu_0g'^R_1(s)+\mu_0^2g'^R_2(s)\right)\sin(2\omega_0\tau)\\
&-\left(g'^I_0(s)+\mu_0g'^I_1(s)+\mu_0^2g'^I_2(s)\right)\cos(2\omega_0\tau)\\
&-k'^R_0(s)\sin(4\omega_0\tau)+k'^I_0(s)\cos(4\omega_0\tau)\\
&+\left(k_0^R(s)+\mu_0k_1\right)4\tau\sin(4\omega_0\tau)-k_0^I(s)4\tau\cos(4\omega_0\tau),\\
&h_0^R(s)=\omega_0^{\sum_{i=1}^5\alpha_i}\cos\left(\frac{\left(\sum_{i=1}^5\alpha_i\right)\pi}{2}\right),\\
&h_0^I(s)=\omega_0^{\sum_{i=1}^5\alpha_i}\sin\left(\frac{\left(\sum_{i=1}^5\alpha_i\right)\pi}{2}\right),\\
&h_1^R(s)=\textstyle\sum_{i=1}^5\omega_0^{\sum_{j=1,j\neq i}^5\alpha_j}\cos\left(\frac{\pi\sum_{j=1,j\neq i}^5\alpha_j}{2}\right),\\
&h_1^I(s)=\textstyle\sum_{i=1}^5\omega_0^{\sum_{j=1,j\neq i}^5\alpha_j}\sin\left(\frac{\pi\sum_{j=1,j\neq i}^5\alpha_j}{2}\right),\\
&h_2^R(s)=\textstyle\sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5\omega_0^{\alpha_i+\alpha_j+\alpha_k}\times\\
&\cos\left(\tfrac{(\alpha_i+\alpha_j+\alpha_k)\pi}{2}\right),\\
&h_2^I(s)=\textstyle\sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5\omega_0^{\alpha_i+\alpha_j+\alpha_k}\times\\
&\sin\left(\tfrac{(\alpha_i+\alpha_j+\alpha_k)\pi}{2}\right),\\
&h_3^R(s)=\textstyle\sum_{i=1}^5\sum_{j=1,j>i}^5\omega_0^{\alpha_i+\alpha_j}\cos\left(\tfrac{(\alpha_i+\alpha_j)\pi}{2}\right),\\
&h_3^I(s)=\textstyle\sum_{i=1}^5\sum_{j=1,j>i}^5\omega_0^{\alpha_i+\alpha_j}\sin\left(\tfrac{(\alpha_i+\alpha_j)\pi}{2}\right),\\
&h_4^R(s)=\textstyle\sum_{i=1}^5\omega_0^{\alpha_i}\cos\left(\tfrac{\alpha_i\pi}{2}\right),\\
&h_4^I(s)=\textstyle\sum_{i=1}^5\omega_0^{\alpha_i}\sin\left(\tfrac{\alpha_i\pi}{2}\right),\\
&g_0^R(s)=a_{21}a_{32}\omega_0^{\alpha_1+\alpha_4+\alpha_5}\cos\left(\tfrac{(\alpha_1+\alpha_4+\alpha_5)\pi}{2}\right)\\
&+a_{22}a_{42}\omega_0^{\alpha_1+\alpha_3+\alpha_5}\cos\left(\tfrac{(\alpha_1+\alpha_3+\alpha_5)\pi}{2}\right)\\
&+a_{11}a_{32}\omega_0^{\alpha_2+\alpha_4+\alpha_5}\cos\left(\tfrac{(\alpha_2+\alpha_4+\alpha_5)\pi}{2}\right)\\
&+a_{12}a_{41}\omega_0^{\alpha_2+\alpha_3+\alpha_5}\cos\left(\tfrac{(\alpha_2+\alpha_3+\alpha_5)\pi}{2}\right)\\
&+a_{13}a_{51}\omega_0^{\alpha_2+\alpha_3+\alpha_4}\cos\left(\tfrac{(\alpha_2+\alpha_3+\alpha_4)\pi}{2}\right)\\
&+a_{23}a_{52}\omega_0^{\alpha_1+\alpha_3+\alpha_4}\cos\left(\tfrac{(\alpha_1+\alpha_3+\alpha_4)\pi}{2}\right),
\end{aligned}
$$

$$g_0^I(s) = a_{21}a_{32}\omega_0^{\alpha_1+\alpha_4+\alpha_5}\sin\left(\frac{(\alpha_1+\alpha_4+\alpha_5)\pi}{2}\right)$$
$$+a_{22}a_{42}\omega_0^{\alpha_1+\alpha_3+\alpha_5}\sin\left(\frac{(\alpha_1+\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+a_{11}a_{32}\omega_0^{\alpha_2+\alpha_4+\alpha_5}\sin\left(\frac{(\alpha_2+\alpha_4+\alpha_5)\pi}{2}\right)$$
$$+a_{12}a_{41}\omega_0^{\alpha_2+\alpha_3+\alpha_5}\sin\left(\frac{(\alpha_2+\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+a_{13}a_{51}\omega_0^{\alpha_2+\alpha_3+\alpha_4}\sin\left(\frac{(\alpha_2+\alpha_3+\alpha_4)\pi}{2}\right)$$
$$+a_{23}a_{52}\omega_0^{\alpha_1+\alpha_3+\alpha_4}\sin\left(\frac{(\alpha_1+\alpha_3+\alpha_4)\pi}{2}\right),$$

$$g_1^R(s) = (a_{21}a_{32}+a_{22}a_{42})\omega_0^{\alpha_1+\alpha_5}\cos\left(\frac{(\alpha_1+\alpha_5)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{23}a_{52})\omega_0^{\alpha_1+\alpha_4}\cos\left(\frac{(\alpha_1+\alpha_4)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{23}a_{52})\omega_0^{\alpha_1+\alpha_3}\cos\left(\frac{(\alpha_1+\alpha_3)\pi}{2}\right)$$
$$+(a_{12}a_{41}+a_{13}a_{51})\omega_0^{\alpha_2+\alpha_3}\cos\left(\frac{(\alpha_2+\alpha_3)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{13}a_{51})\omega_0^{\alpha_2+\alpha_4}\cos\left(\frac{(\alpha_2+\alpha_4)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41})\omega_0^{\alpha_2+\alpha_5}\cos\left(\frac{(\alpha_2+\alpha_5)\pi}{2}\right)$$
$$+(a_{13}a_{51}+a_{23}a_{52})\omega_0^{\alpha_3+\alpha_4}\cos\left(\frac{(\alpha_3+\alpha_4)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41})\omega_0^{\alpha_3+\alpha_5}\cos\left(\frac{(\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32})\omega_0^{\alpha_4+\alpha_5}\cos\left(\frac{(\alpha_4+\alpha_5)\pi}{2}\right),$$

$$g_1^I(s) = (a_{21}a_{32}+a_{22}a_{42})\omega_0^{\alpha_1+\alpha_5}\sin\left(\frac{(\alpha_1+\alpha_5)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{23}a_{52})\omega_0^{\alpha_1+\alpha_4}\sin\left(\frac{(\alpha_1+\alpha_4)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{23}a_{52})\omega_0^{\alpha_1+\alpha_3}\sin\left(\frac{(\alpha_1+\alpha_3)\pi}{2}\right)$$
$$+(a_{12}a_{41}+a_{13}a_{51})\omega_0^{\alpha_2+\alpha_3}\sin\left(\frac{(\alpha_2+\alpha_3)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{13}a_{51})\omega_0^{\alpha_2+\alpha_4}\sin\left(\frac{(\alpha_2+\alpha_4)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41})\omega_0^{\alpha_2+\alpha_5}\sin\left(\frac{(\alpha_2+\alpha_5)\pi}{2}\right)$$
$$+(a_{13}a_{51}+a_{23}a_{52})\omega_0^{\alpha_3+\alpha_4}\sin\left(\frac{(\alpha_3+\alpha_4)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41})\omega_0^{\alpha_3+\alpha_5}\sin\left(\frac{(\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32})\omega_0^{\alpha_4+\alpha_5}\sin\left(\frac{(\alpha_4+\alpha_5)\pi}{2}\right),$$

$$g_2^R(s) = (a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\omega_0^{\alpha_1}\cos\left(\frac{\alpha_1\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\omega_0^{\alpha_2}\cos\left(\frac{\alpha_2\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}+a_{23}a_{52})\omega_0^{\alpha_3}\cos\left(\frac{\alpha_3\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32}+a_{13}a_{51}+a_{23}a_{52})\omega_0^{\alpha_4}\cos\left(\frac{\alpha_4\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\omega_0^{\alpha_5}\cos\left(\frac{\alpha_5\pi}{2}\right),$$

$$g_2^I(s) = (a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\omega_0^{\alpha_1}\sin\left(\frac{\alpha_1\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\omega_0^{\alpha_2}\sin\left(\frac{\alpha_2\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}+a_{23}a_{52})\omega_0^{\alpha_3}\sin\left(\frac{\alpha_3\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32}+a_{13}a_{51}+a_{23}a_{52})\omega_0^{\alpha_4}\sin\left(\frac{\alpha_4\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\omega_0^{\alpha_5}\sin\left(\frac{\alpha_5\pi}{2}\right),$$

$$k_0^R(s) = [a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}(a_{12}a_{21}$$
$$-a_{11}a_{22})]\omega_0^{\alpha_5}\cos\left(\frac{\alpha_5\pi}{2}\right)+[a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})$$
$$+a_{32}a_{51}(a_{13}a_{21}-a_{11}a_{23})]\omega_0^{\alpha_4}\cos\left(\frac{\alpha_4\pi}{2}\right)$$
$$+[a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})$$
$$+a_{42}a_{51}(a_{22}a_{13}-a_{12}a_{23})]\omega_0^{\alpha_3}\cos\left(\frac{\alpha_3\pi}{2}\right),$$

$$k_0^I(s) = [a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}(a_{12}a_{21}$$
$$-a_{11}a_{22})]\omega_0^{\alpha_5}\sin\left(\frac{\alpha_5\pi}{2}\right)+[a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})$$
$$+a_{32}a_{51}(a_{13}a_{21}-a_{11}a_{23})]\omega_0^{\alpha_4}\sin\left(\frac{\alpha_4\pi}{2}\right)$$
$$+[a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})$$
$$+a_{42}a_{51}(a_{22}a_{13}-a_{12}a_{23})]\omega_0^{\alpha_3}\sin\left(\frac{\alpha_3\pi}{2}\right),$$

$$h'^R_0(s) = \left(\sum_{i=1}^5\alpha_i\right)\omega_0^{\sum_{i=1}^5\alpha_i-1}\cos\left(\frac{(\sum_{i=1}^5\alpha_i-1)\pi}{2}\right),$$

$$h'^I_0(s) = \left(\sum_{i=1}^5\alpha_i\right)\omega_0^{\sum_{i=1}^5\alpha_i-1}\sin\left(\frac{(\sum_{i=1}^5\alpha_i-1)\pi}{2}\right),$$

$$h'^R_1(s) = \sum_{i=1}^5\left(\sum_{j=1,j\neq i}^5\alpha_j\right)\omega_0^{\sum_{j=1,j\neq i}^5\alpha_j-1}\times$$
$$\cos\left(\frac{(\sum_{j=1,j\neq i}^5\alpha_j-1)\pi}{2}\right),$$

$$h'^I_1(s) = \sum_{i=1}^5\left(\sum_{j=1,j\neq i}^5\alpha_j\right)\omega_0^{\sum_{j=1,j\neq i}^5\alpha_j-1}\times$$
$$\sin\left(\frac{(\sum_{j=1,j\neq i}^5\alpha_j-1)\pi}{2}\right),$$

$$h'^R_2(s) = \sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5(\alpha_i+\alpha_j+\alpha_k)\times$$
$$\omega_0^{\alpha_i+\alpha_j+\alpha_k-1}\cos\left(\frac{(\alpha_i+\alpha_j+\alpha_k-1)\pi}{2}\right),$$

$$h'^I_2(s) = \sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5(\alpha_i+\alpha_j+\alpha_k)\times$$
$$\omega_0^{\alpha_i+\alpha_j+\alpha_k-1}\sin\left(\frac{(\alpha_i+\alpha_j+\alpha_k-1)\pi}{2}\right),$$

$$h'^R_3(s) = \sum_{i=1}^5\sum_{j=1,j>i}^5(\alpha_i+\alpha_j)\omega_0^{\alpha_i+\alpha_j-1}\times$$
$$\cos\left(\frac{(\alpha_i+\alpha_j-1)\pi}{2}\right),$$

$$h'^I_3(s) = \sum_{i=1}^5\sum_{j=1,j>i}^5(\alpha_i+\alpha_j)\omega_0^{\alpha_i+\alpha_j-1}\times$$
$$\sin\left(\frac{(\alpha_i+\alpha_j-1)\pi}{2}\right),$$

$$h'^R_4(s) = \sum_{i=1}^5\alpha_i\omega_0^{\alpha_i-1}\cos\left(\frac{(\alpha_i-1)\pi}{2}\right),$$

$$h'^I_4(s) = \sum_{i=1}^5\alpha_i\omega_0^{\alpha_i-1}\sin\left(\frac{(\alpha_i-1)\pi}{2}\right),$$

$$g'^R_0(s) =$$
$$a_{21}a_{32}(\alpha_1+\alpha_4+\alpha_5)\omega_0^{\alpha_1+\alpha_4+\alpha_5-1}\cos\left(\frac{(\alpha_1+\alpha_4+\alpha_5-1)\pi}{2}\right)$$
$$+a_{22}a_{42}(\alpha_1+\alpha_3+\alpha_5)\omega_0^{\alpha_1+\alpha_3+\alpha_5-1}\cos\left(\frac{(\alpha_1+\alpha_3+\alpha_5-1)\pi}{2}\right)$$
$$+a_{11}a_{32}(\alpha_2+\alpha_4+\alpha_5)\omega_0^{\alpha_2+\alpha_4+\alpha_5-1}\cos\left(\frac{(\alpha_2+\alpha_4+\alpha_5-1)\pi}{2}\right)$$
$$+a_{12}a_{41}(\alpha_2+\alpha_3+\alpha_5)\omega_0^{\alpha_2+\alpha_3+\alpha_5-1}\cos\left(\frac{(\alpha_2+\alpha_3+\alpha_5-1)\pi}{2}\right)$$
$$+a_{13}a_{51}(\alpha_2+\alpha_3+\alpha_4)\omega_0^{\alpha_2+\alpha_3+\alpha_4-1}\cos\left(\frac{(\alpha_2+\alpha_3+\alpha_4-1)\pi}{2}\right)$$
$$+a_{23}a_{52}(\alpha_1+\alpha_3+\alpha_4)\omega_0^{\alpha_1+\alpha_3+\alpha_4-1}\cos\left(\frac{(\alpha_1+\alpha_3+\alpha_4-1)\pi}{2}\right),$$

$$g'^I_0(s) =$$
$$a_{21}a_{32}(\alpha_1+\alpha_4+\alpha_5)\omega_0^{\alpha_1+\alpha_4+\alpha_5-1}\sin\left(\frac{(\alpha_1+\alpha_4+\alpha_5-1)\pi}{2}\right)$$
$$+a_{22}a_{42}(\alpha_1+\alpha_3+\alpha_5)\omega_0^{\alpha_1+\alpha_3+\alpha_5-1}\sin\left(\frac{(\alpha_1+\alpha_3+\alpha_5-1)\pi}{2}\right)$$
$$+a_{11}a_{32}(\alpha_2+\alpha_4+\alpha_5)\omega_0^{\alpha_2+\alpha_4+\alpha_5-1}\sin\left(\frac{(\alpha_2+\alpha_4+\alpha_5-1)\pi}{2}\right)$$
$$+a_{12}a_{41}(\alpha_2+\alpha_3+\alpha_5)\omega_0^{\alpha_2+\alpha_3+\alpha_5-1}\sin\left(\frac{(\alpha_2+\alpha_3+\alpha_5-1)\pi}{2}\right)$$
$$+a_{13}a_{51}(\alpha_2+\alpha_3+\alpha_4)\omega_0^{\alpha_2+\alpha_3+\alpha_4-1}\sin\left(\frac{(\alpha_2+\alpha_3+\alpha_4-1)\pi}{2}\right)$$
$$+a_{23}a_{52}(\alpha_1+\alpha_3+\alpha_4)\omega_0^{\alpha_1+\alpha_3+\alpha_4-1}\sin\left(\frac{(\alpha_1+\alpha_3+\alpha_4-1)\pi}{2}\right),$$

$$g'^R_1(s) =$$
$$(a_{21}a_{32}+a_{22}a_{42})(\alpha_1+\alpha_5)\omega_0^{\alpha_1+\alpha_5-1}\cos\left(\frac{(\alpha_1+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{23}a_{52})(\alpha_1+\alpha_4)\omega_0^{\alpha_1+\alpha_4-1}\cos\left(\frac{(\alpha_1+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{23}a_{52})(\alpha_1+\alpha_3)\omega_0^{\alpha_1+\alpha_3-1}\cos\left(\frac{(\alpha_1+\alpha_3-1)\pi}{2}\right)$$
$$+(a_{12}a_{41}+a_{13}a_{51})(\alpha_2+\alpha_3)\omega_0^{\alpha_2+\alpha_3-1}\cos\left(\frac{(\alpha_2+\alpha_3-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{13}a_{51})(\alpha_2+\alpha_4)\omega_0^{\alpha_2+\alpha_4-1}\cos\left(\frac{(\alpha_2+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41})(\alpha_2+\alpha_5)\omega_0^{\alpha_2+\alpha_5-1}\cos\left(\frac{(\alpha_2+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{13}a_{51}+a_{23}a_{52})(\alpha_3+\alpha_4)\omega_0^{\alpha_3+\alpha_4-1}\cos\left(\frac{(\alpha_3+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41})(\alpha_3+\alpha_5)\omega_0^{\alpha_3+\alpha_5-1}\cos\left(\frac{(\alpha_3+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32})(\alpha_4+\alpha_5)\omega_0^{\alpha_4+\alpha_5-1}\cos\left(\frac{(\alpha_4+\alpha_5-1)\pi}{2}\right),$$

$$g'^{I}_{1}(s)=$$
$$(a_{21}a_{32}+a_{22}a_{42})(\alpha_1+\alpha_5)\omega_0^{\alpha_1+\alpha_5-1}\sin\left(\frac{(\alpha_1+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{23}a_{52})(\alpha_1+\alpha_4)\omega_0^{\alpha_1+\alpha_4-1}\sin\left(\frac{(\alpha_1+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{23}a_{52})(\alpha_1+\alpha_3)\omega_0^{\alpha_1+\alpha_3-1}\sin\left(\frac{(\alpha_1+\alpha_3-1)\pi}{2}\right)$$
$$+(a_{12}a_{41}+a_{13}a_{51})(\alpha_2+\alpha_3)\omega_0^{\alpha_2+\alpha_3-1}\sin\left(\frac{(\alpha_2+\alpha_3-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{13}a_{51})(\alpha_2+\alpha_4)\omega_0^{\alpha_2+\alpha_4-1}\sin\left(\frac{(\alpha_2+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41})(\alpha_2+\alpha_5)\omega_0^{\alpha_2+\alpha_5-1}\sin\left(\frac{(\alpha_2+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{13}a_{51}+a_{23}a_{52})(\alpha_3+\alpha_4)\omega_0^{\alpha_3+\alpha_4-1}\sin\left(\frac{(\alpha_3+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41})(\alpha_3+\alpha_5)\omega_0^{\alpha_3+\alpha_5-1}\sin\left(\frac{(\alpha_3+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32})(\alpha_4+\alpha_5)\omega_0^{\alpha_4+\alpha_5-1}\sin\left(\frac{(\alpha_4+\alpha_5-1)\pi}{2}\right),$$
$$g'^{R}_{2}(s)=$$
$$(a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\alpha_1\omega_0^{\alpha_1-1}\cos\left(\frac{(\alpha_1-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\alpha_2\omega_0^{\alpha_2-1}\cos\left(\frac{(\alpha_2-1)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}+a_{23}a_{52})\alpha_3\times$$
$$\omega_0^{\alpha_3-1}\cos\left(\frac{(\alpha_3-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32}+a_{13}a_{51}+a_{23}a_{52})\alpha_4\times$$
$$\omega_0^{\alpha_4-1}\cos\left(\frac{(\alpha_4-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\alpha_5\times$$
$$\omega_0^{\alpha_5-1}\cos\left(\frac{(\alpha_5-1)\pi}{2}\right),$$
$$g'^{I}_{2}(s)=(a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\alpha_1\omega_0^{\alpha_1-1}\times$$
$$\sin\left(\frac{(\alpha_1-1)\pi}{2}\right)+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\alpha_2\times$$
$$\omega_0^{\alpha_2-1}\sin\left(\frac{(\alpha_2-1)\pi}{2}\right)+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}$$
$$+a_{23}a_{52})\alpha_3\omega_0^{\alpha_3-1}\sin\left(\frac{(\alpha_3-1)\pi}{2}\right)+(a_{21}a_{32}+a_{11}a_{32}$$
$$+a_{13}a_{51}+a_{23}a_{52})\alpha_4\omega_0^{\alpha_4-1}\sin\left(\frac{(\alpha_4-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\times$$
$$\alpha_5\omega_0^{\alpha_5-1}\sin\left(\frac{(\alpha_5-1)\pi}{2}\right),$$
$$k'^{R}_{0}(s)=\big[a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}\times$$
$$(a_{12}a_{21}-a_{11}a_{22})\big]\alpha_5\omega_0^{\alpha_5-1}\cos\left(\frac{(\alpha_5-1)\pi}{2}\right)$$
$$+\big[a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})+a_{32}a_{51}\times$$
$$(a_{13}a_{21}-a_{11}a_{23})\big]\alpha_4\omega_0^{\alpha_4-1}\cos\left(\frac{(\alpha_4-1)\pi}{2}\right)$$
$$+\big[a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})+a_{42}a_{51}\times$$
$$(a_{22}a_{13}-a_{12}a_{23})\big]\alpha_3\omega_0^{\alpha_3-1}\cos\left(\frac{(\alpha_3-1)\pi}{2}\right),$$
$$k'^{I}_{0}(s)=\big[a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}\times$$
$$(a_{12}a_{21}-a_{11}a_{22})\big]\alpha_5\omega_0^{\alpha_5-1}\sin\left(\frac{(\alpha_5-1)\pi}{2}\right)$$
$$+\big[a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})+a_{32}a_{51}\times$$
$$(a_{13}a_{21}-a_{11}a_{23})\big]\alpha_4\omega_0^{\alpha_4-1}\sin\left(\frac{(\alpha_4-1)\pi}{2}\right)$$
$$+\big[a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})+a_{42}a_{51}\times$$
$$(a_{22}a_{13}-a_{12}a_{23})\big]\alpha_3\omega_0^{\alpha_3-1}\sin\left(\frac{(\alpha_3-1)\pi}{2}\right).$$

Appendix C

$$\overline{M}_1=\overline{g}_1^R(s)\cos(2(\omega_0^\star)\tau)+\overline{g}_1^I(s)\sin(2(\omega_0^\star)\tau)$$
$$-k_1\cos(4(\omega_0^\star)\tau)-\overline{h}_1^R(s),$$
$$\overline{M}_2=-\overline{g}_1^R(s)\sin(2(\omega_0^\star)\tau)+g_1^I(s)\cos(2(\omega_0^\star)\tau)$$
$$-k_1\sin(4(\omega_0^\star)\tau)-\overline{h}_1^I(s),$$
$$\overline{N}_1=$$
$$\overline{h}'^{R}_{0}(s)+\overline{g}_0^R(s)2\tau\cos(2(\omega_0^\star)\tau)+\overline{g}_0^I(s)2\tau\sin(2(\omega_0^\star)\tau)$$
$$-\overline{g}'^{R}_{0}(s)\cos(2(\omega_0^\star)\tau)-\overline{g}'^{I}_{0}(s)\sin(2(\omega_0^\star)\tau)$$
$$+\overline{k}'^{R}_{0}(s)\cos(4(\omega_0^\star)\tau)+\overline{k}'^{I}_{0}(s)\sin(4(\omega_0^\star)\tau)$$
$$-\overline{k}_0^R(s)4\tau\cos(4(\omega_0^\star)\tau)-\overline{k}_0^I(s)4\tau\sin(4(\omega_0^\star)\tau),$$
$$\overline{N}_2=$$
$$\overline{h}'^{I}_{0}(s)-\overline{g}_0^R(s)2\tau\sin(2(\omega_0^\star)\tau)+g_0^I(s)2\tau\cos(2(\omega_0^\star)\tau)$$
$$+\overline{g}'^{R}_{0}(s)\sin(2(\omega_0^\star)\tau)-\overline{g}'^{I}_{0}(s)\cos(2(\omega_0^\star)\tau)$$
$$-\overline{k}'^{R}_{0}(s)\sin(4(\omega_0^\star)\tau)+\overline{k}'^{I}_{0}(s)\cos(4(\omega_0^\star)\tau)$$
$$+\overline{k}_0^R(s)4\tau\sin(4(\omega_0^\star)\tau)-\overline{k}_0^I(s)4\tau\cos(4(\omega_0^\star)\tau),$$
$$\overline{h}_0^R(s)=(\omega_0^\star)^{\sum_{i=1}^5\alpha_i}\cos\left(\frac{(\sum_{i=1}^5\alpha_i)\pi}{2}\right),$$
$$\overline{h}_0^I(s)=(\omega_0^\star)^{\sum_{i=1}^5\alpha_i}\sin\left(\frac{(\sum_{i=1}^5\alpha_i)\pi}{2}\right),$$
$$\overline{h}_1^R(s)=\textstyle\sum_{i=1}^5(\omega_0^\star)^{\sum_{j=1,j\neq i}^5\alpha_j}\cos\left(\frac{(\sum_{j=1,j\neq i}^5\alpha_j)\pi}{2}\right),$$
$$\overline{h}_1^I(s)=\textstyle\sum_{i=1}^5(\omega_0^\star)^{\sum_{j=1,j\neq i}^5\alpha_j}\sin\left(\frac{(\sum_{j=1,j\neq i}^5\alpha_j)\pi}{2}\right),$$
$$\overline{h}_2^R(s)=\textstyle\sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5(\omega_0^\star)^{\alpha_i+\alpha_j+\alpha_k}\times$$
$$\cos\left(\frac{(\alpha_i+\alpha_j+\alpha_k)\pi}{2}\right),$$
$$\overline{h}_2^I(s)=\textstyle\sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5(\omega_0^\star)^{\alpha_i+\alpha_j+\alpha_k}\times$$
$$\sin\left(\frac{(\alpha_i+\alpha_j+\alpha_k)\pi}{2}\right),$$
$$\overline{h}_3^R(s)=\textstyle\sum_{i=1}^5\sum_{j=1,j>i}^5(\omega_0^\star)^{\alpha_i+\alpha_j}\cos\left(\frac{(\alpha_i+\alpha_j)\pi}{2}\right),$$
$$\overline{h}_3^I(s)=\textstyle\sum_{i=1}^5\sum_{j=1,j>i}^5(\omega_0^\star)^{\alpha_i+\alpha_j}\sin\left(\frac{(\alpha_i+\alpha_j)\pi}{2}\right),$$
$$\overline{h}_4^R(s)=\textstyle\sum_{i=1}^5(\omega_0^\star)^{\alpha_i}\cos\left(\frac{\alpha_i\pi}{2}\right),$$
$$\overline{h}_4^I(s)=\textstyle\sum_{i=1}^5(\omega_0^\star)^{\alpha_i}\sin\left(\frac{\alpha_i\pi}{2}\right),$$
$$\overline{g}_0^R(s)=a_{21}a_{32}\omega_0^{\star\alpha_1+\alpha_4+\alpha_5}\cos\left(\frac{(\alpha_1+\alpha_4+\alpha_5)\pi}{2}\right)$$
$$+a_{22}a_{42}\omega_0^{\star\alpha_1+\alpha_3+\alpha_5}\cos\left(\frac{(\alpha_1+\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+a_{11}a_{32}\omega_0^{\star\alpha_2+\alpha_4+\alpha_5}\cos\left(\frac{(\alpha_2+\alpha_4+\alpha_5)\pi}{2}\right)$$
$$+a_{12}a_{41}\omega_0^{\star\alpha_2+\alpha_3+\alpha_5}\cos\left(\frac{(\alpha_2+\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+a_{13}a_{51}\omega_0^{\star\alpha_2+\alpha_3+\alpha_4}\cos\left(\frac{(\alpha_2+\alpha_3+\alpha_4)\pi}{2}\right)$$
$$+a_{23}a_{52}\omega_0^{\star\alpha_1+\alpha_3+\alpha_4}\cos\left(\frac{(\alpha_1+\alpha_3+\alpha_4)\pi}{2}\right),$$
$$\overline{g}_0^I(s)=a_{21}a_{32}\omega_0^{\star\alpha_1+\alpha_4+\alpha_5}\sin\left(\frac{(\alpha_1+\alpha_4+\alpha_5)\pi}{2}\right)$$
$$+a_{22}a_{42}\omega_0^{\star\alpha_1+\alpha_3+\alpha_5}\sin\left(\frac{(\alpha_1+\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+a_{11}a_{32}\omega_0^{\star\alpha_2+\alpha_4+\alpha_5}\sin\left(\frac{(\alpha_2+\alpha_4+\alpha_5)\pi}{2}\right)$$
$$+a_{12}a_{41}\omega_0^{\star\alpha_2+\alpha_3+\alpha_5}\sin\left(\frac{(\alpha_2+\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+a_{13}a_{51}\omega_0^{\star\alpha_2+\alpha_3+\alpha_4}\sin\left(\frac{(\alpha_2+\alpha_3+\alpha_4)\pi}{2}\right)$$
$$+a_{23}a_{52}\omega_0^{\star\alpha_1+\alpha_3+\alpha_4}\sin\left(\frac{(\alpha_1+\alpha_3+\alpha_4)\pi}{2}\right),$$
$$\overline{g}_1^R(s)=(a_{21}a_{32}+a_{22}a_{42})\omega_0^{\star\alpha_1+\alpha_5}\cos\left(\frac{(\alpha_1+\alpha_5)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{23}a_{52})\omega_0^{\star\alpha_1+\alpha_4}\cos\left(\frac{(\alpha_1+\alpha_4)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{23}a_{52})\omega_0^{\star\alpha_1+\alpha_3}\cos\left(\frac{(\alpha_1+\alpha_3)\pi}{2}\right)$$
$$+(a_{12}a_{41}+a_{13}a_{51})\omega_0^{\star\alpha_2+\alpha_3}\cos\left(\frac{(\alpha_2+\alpha_3)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{13}a_{51})\omega_0^{\star\alpha_2+\alpha_4}\cos\left(\frac{(\alpha_2+\alpha_4)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41})\omega_0^{\star\alpha_2+\alpha_5}\cos\left(\frac{(\alpha_2+\alpha_5)\pi}{2}\right)$$
$$+(a_{13}a_{51}+a_{23}a_{52})\omega_0^{\star\alpha_3+\alpha_4}\cos\left(\frac{(\alpha_3+\alpha_4)\pi}{2}\right)$$

$$+(a_{22}a_{42}+a_{12}a_{41})\omega_0^{\star\alpha_3+\alpha_5}\cos\left(\frac{(\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32})\omega_0^{\star\alpha_4+\alpha_5}\cos\left(\frac{(\alpha_4+\alpha_5)\pi}{2}\right),$$
$$\overline{g}_1^I(s)=(a_{21}a_{32}+a_{22}a_{42})\omega_0^{\star\alpha_1+\alpha_5}\sin\left(\frac{(\alpha_1+\alpha_5)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{23}a_{52})\omega_0^{\star\alpha_1+\alpha_4}\sin\left(\frac{(\alpha_1+\alpha_4)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{23}a_{52})\omega_0^{\star\alpha_1+\alpha_3}\sin\left(\frac{(\alpha_1+\alpha_3)\pi}{2}\right)$$
$$+(a_{12}a_{41}+a_{13}a_{51})\omega_0^{\star\alpha_2+\alpha_3}\sin\left(\frac{(\alpha_2+\alpha_3)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{13}a_{51})\omega_0^{\star\alpha_2+\alpha_4}\sin\left(\frac{(\alpha_2+\alpha_4)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41})\omega_0^{\star\alpha_2+\alpha_5}\sin\left(\frac{(\alpha_2+\alpha_5)\pi}{2}\right)$$
$$+(a_{13}a_{51}+a_{23}a_{52})\omega_0^{\star\alpha_3+\alpha_4}\sin\left(\frac{(\alpha_3+\alpha_4)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41})\omega_0^{\star\alpha_3+\alpha_5}\sin\left(\frac{(\alpha_3+\alpha_5)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32})\omega_0^{\star\alpha_4+\alpha_5}\sin\left(\frac{(\alpha_4+\alpha_5)\pi}{2}\right),$$
$$\overline{g}_2^R(s)=(a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\omega_0^{\star\alpha_1}\cos\left(\frac{\alpha_1\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\omega_0^{\star\alpha_2}\cos\left(\frac{\alpha_2\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}+a_{23}a_{52})\omega_0^{\star\alpha_3}\cos\left(\frac{\alpha_3\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32}+a_{13}a_{51}+a_{23}a_{52})\omega_0^{\star\alpha_4}\cos\left(\frac{\alpha_4\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\omega_0^{\star\alpha_5}\cos\left(\frac{\alpha_5\pi}{2}\right);$$
$$\overline{g}_2^I(s)=(a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\omega_0^{\star\alpha_1}\sin\left(\frac{\alpha_1\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\omega_0^{\star\alpha_2}\sin\left(\frac{\alpha_2\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}+a_{23}a_{52})\omega_0^{\star\alpha_3}\sin\left(\frac{\alpha_3\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32}+a_{13}a_{51}+a_{23}a_{52})\omega_0^{\star\alpha_4}\sin\left(\frac{\alpha_4\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\omega_0^{\star\alpha_5}\sin\left(\frac{\alpha_5\pi}{2}\right),$$
$$\overline{k}_0^R(s)=[a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}(a_{12}a_{21}$$
$$-a_{11}a_{22})]\omega_0^{\star\alpha_5}\cos\left(\frac{\alpha_5\pi}{2}\right)+[a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})$$
$$+a_{32}a_{51}(a_{13}a_{21}-a_{11}a_{23})]\omega_0^{\star\alpha_4}\cos\left(\frac{\alpha_4\pi}{2}\right)$$
$$+[a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})$$
$$+a_{42}a_{51}(a_{22}a_{13}-a_{12}a_{23})]\omega_0^{\star\alpha_3}\cos\left(\frac{\alpha_3\pi}{2}\right),$$
$$\overline{k}_0^I(s)=[a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}(a_{12}a_{21}$$
$$-a_{11}a_{22})]\omega_0^{\star\alpha_5}\sin\left(\frac{\alpha_5\pi}{2}\right)+[a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})$$
$$+a_{32}a_{51}(a_{13}a_{21}-a_{11}a_{23})]\omega_0^{\star\alpha_4}\sin\left(\frac{\alpha_4\pi}{2}\right)$$
$$+[a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})$$
$$+a_{42}a_{51}(a_{22}a_{13}-a_{12}a_{23})]\omega_0^{\star\alpha_3}\sin\left(\frac{\alpha_3\pi}{2}\right),$$
$$\overline{h}'^R_0(s)=\left(\sum_{i=1}^5\alpha_i\right)(\omega_0^\star)^{\sum_{i=1}^5\alpha_i-1}\cos\left(\frac{(\sum_{i=1}^5\alpha_i-1)\pi}{2}\right),$$
$$\overline{h}'^I_0(s)=\left(\sum_{i=1}^5\alpha_i\right)(\omega_0^\star)^{\sum_{i=1}^5\alpha_i-1}\sin\left(\frac{(\sum_{i=1}^5\alpha_i-1)\pi}{2}\right),$$
$$\overline{h}'^R_1(s)=\sum_{i=1}^5\left(\sum_{j=1,j\neq i}^5\alpha_j\right)(\omega_0^\star)^{\sum_{j=1,j\neq i}^5\alpha_j-1}\times$$
$$\cos\left(\frac{(\sum_{j=1,j\neq i}^5\alpha_j-1)\pi}{2}\right),$$
$$\overline{h}'^I_1(s)=\sum_{i=1}^5\left(\sum_{j=1,j\neq i}^5\alpha_j\right)(\omega_0^\star)^{\sum_{j=1,j\neq i}^5\alpha_j-1}\times$$
$$\sin\left(\frac{(\sum_{j=1,j\neq i}^5\alpha_j-1)\pi}{2}\right),$$
$$\overline{h}'^R_2(s)=\sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5(\alpha_i+\alpha_j+\alpha_k)\times$$
$$(\omega_0^\star)^{\alpha_i+\alpha_j+\alpha_k-1}\cos\left(\frac{(\alpha_i+\alpha_j+\alpha_k-1)\pi}{2}\right),$$
$$\overline{h}'^I_2(s)=\sum_{i=1}^5\sum_{j=1,j>i}^5\sum_{k=1,k>j}^5(\alpha_i+\alpha_j+\alpha_k)\times$$
$$(\omega_0^\star)^{\alpha_i+\alpha_j+\alpha_k-1}\sin\left(\frac{(\alpha_i+\alpha_j+\alpha_k-1)\pi}{2}\right),$$
$$\overline{h}'^R_3(s)=\sum_{i=1}^5\sum_{j=1,j>i}^5(\alpha_i+\alpha_j)(\omega_0^\star)^{\alpha_i+\alpha_j-1}\times$$
$$\cos\left(\frac{(\alpha_i+\alpha_j-1)\pi}{2}\right),$$
$$\overline{h}'^I_3(s)=\sum_{i=1}^5\sum_{j=1,j>i}^5(\alpha_i+\alpha_j)(\omega_0^\star)^{\alpha_i+\alpha_j-1}\times$$
$$\sin\left(\frac{(\alpha_i+\alpha_j-1)\pi}{2}\right),$$
$$\overline{h}'^R_4(s)=\sum_{i=1}^5\alpha_i(\omega_0^\star)^{\alpha_i-1}\cos\left(\frac{(\alpha_i-1)\pi}{2}\right),$$
$$\overline{h}'^I_4(s)=\sum_{i=1}^5\alpha_i(\omega_0^\star)^{\alpha_i-1}\sin\left(\frac{(\alpha_i-1)\pi}{2}\right),$$
$$\overline{g}'^R_0(s)=a_{21}a_{32}(\alpha_1+\alpha_4+\alpha_5)\omega_0^{\star\alpha_1+\alpha_4+\alpha_5-1}\times$$
$$\cos\left(\frac{(\alpha_1+\alpha_4+\alpha_5-1)\pi}{2}\right)+a_{22}a_{42}(\alpha_1+\alpha_3+\alpha_5)\times$$
$$\omega_0^{\star\alpha_1+\alpha_3+\alpha_5-1}\cos\left(\frac{(\alpha_1+\alpha_3+\alpha_5-1)\pi}{2}\right)+a_{11}a_{32}\times$$
$$(\alpha_2+\alpha_4+\alpha_5)\omega_0^{\star\alpha_2+\alpha_4+\alpha_5-1}\cos\left(\frac{(\alpha_2+\alpha_4+\alpha_5-1)\pi}{2}\right)$$
$$+a_{12}a_{41}(\alpha_2+\alpha_3+\alpha_5)\omega_0^{\star\alpha_2+\alpha_3+\alpha_5-1}\times$$
$$\cos\left(\frac{(\alpha_2+\alpha_3+\alpha_5-1)\pi}{2}\right)+a_{13}a_{51}(\alpha_2+\alpha_3+\alpha_4)\times$$
$$\omega_0^{\star\alpha_2+\alpha_3+\alpha_4-1}\cos\left(\frac{(\alpha_2+\alpha_3+\alpha_4-1)\pi}{2}\right)+a_{23}a_{52}\times$$
$$(\alpha_1+\alpha_3+\alpha_4)\omega_0^{\star\alpha_1+\alpha_3+\alpha_4-1}\cos\left(\frac{(\alpha_1+\alpha_3+\alpha_4-1)\pi}{2}\right),$$
$$\overline{g}'^I_0(s)=a_{21}a_{32}(\alpha_1+\alpha_4+\alpha_5)\omega_0^{\star\alpha_1+\alpha_4+\alpha_5-1}\times$$
$$\sin\left(\frac{(\alpha_1+\alpha_4+\alpha_5-1)\pi}{2}\right)+a_{22}a_{42}(\alpha_1+\alpha_3+\alpha_5)\times$$
$$\omega_0^{\star\alpha_1+\alpha_3+\alpha_5-1}\sin\left(\frac{(\alpha_1+\alpha_3+\alpha_5-1)\pi}{2}\right)+a_{11}a_{32}\times$$
$$(\alpha_2+\alpha_4+\alpha_5)\omega_0^{\star\alpha_2+\alpha_4+\alpha_5-1}\sin\left(\frac{(\alpha_2+\alpha_4+\alpha_5-1)\pi}{2}\right)$$
$$+a_{12}a_{41}(\alpha_2+\alpha_3+\alpha_5)\omega_0^{\star\alpha_2+\alpha_3+\alpha_5-1}\times$$
$$\sin\left(\frac{(\alpha_2+\alpha_3+\alpha_5-1)\pi}{2}\right)+a_{13}a_{51}(\alpha_2+\alpha_3+\alpha_4)\times$$
$$\omega_0^{\star\alpha_2+\alpha_3+\alpha_4-1}\sin\left(\frac{(\alpha_2+\alpha_3+\alpha_4-1)\pi}{2}\right)+a_{23}a_{52}\times$$
$$(\alpha_1+\alpha_3+\alpha_4)\omega_0^{\star\alpha_1+\alpha_3+\alpha_4-1}\sin\left(\frac{(\alpha_1+\alpha_3+\alpha_4-1)\pi}{2}\right),$$
$$\overline{g}'^R_1(s)=$$
$$(a_{21}a_{32}+a_{22}a_{42})(\alpha_1+\alpha_5)\omega_0^{\star\alpha_1+\alpha_5-1}\cos\left(\frac{(\alpha_1+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{23}a_{52})(\alpha_1+\alpha_4)\omega_0^{\star\alpha_1+\alpha_4-1}\cos\left(\frac{(\alpha_1+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{23}a_{52})(\alpha_1+\alpha_3)\omega_0^{\star\alpha_1+\alpha_3-1}\cos\left(\frac{(\alpha_1+\alpha_3-1)\pi}{2}\right)$$
$$+(a_{12}a_{41}+a_{13}a_{51})(\alpha_2+\alpha_3)\omega_0^{\star\alpha_2+\alpha_3-1}\cos\left(\frac{(\alpha_2+\alpha_3-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{13}a_{51})(\alpha_2+\alpha_4)\omega_0^{\star\alpha_2+\alpha_4-1}\cos\left(\frac{(\alpha_2+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41})(\alpha_2+\alpha_5)\omega_0^{\star\alpha_2+\alpha_5-1}\cos\left(\frac{(\alpha_2+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{13}a_{51}+a_{23}a_{52})(\alpha_3+\alpha_4)\omega_0^{\star\alpha_3+\alpha_4-1}\cos\left(\frac{(\alpha_3+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41})(\alpha_3+\alpha_5)\omega_0^{\star\alpha_3+\alpha_5-1}\cos\left(\frac{(\alpha_3+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32})(\alpha_4+\alpha_5)\omega_0^{\star\alpha_4+\alpha_5-1}\cos\left(\frac{(\alpha_4+\alpha_5-1)\pi}{2}\right),$$
$$\overline{g}'^I_1(s)=$$
$$(a_{21}a_{32}+a_{22}a_{42})(\alpha_1+\alpha_5)\omega_0^{\star\alpha_1+\alpha_5-1}\sin\left(\frac{(\alpha_1+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{23}a_{52})(\alpha_1+\alpha_4)\omega_0^{\star\alpha_1+\alpha_4-1}\sin\left(\frac{(\alpha_1+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{23}a_{52})(\alpha_1+\alpha_3)\omega_0^{\star\alpha_1+\alpha_3-1}\sin\left(\frac{(\alpha_1+\alpha_3-1)\pi}{2}\right)$$
$$+(a_{12}a_{41}+a_{13}a_{51})(\alpha_2+\alpha_3)\omega_0^{\star\alpha_2+\alpha_3-1}\sin\left(\frac{(\alpha_2+\alpha_3-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{13}a_{51})(\alpha_2+\alpha_4)\omega_0^{\star\alpha_2+\alpha_4-1}\sin\left(\frac{(\alpha_2+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41})(\alpha_2+\alpha_5)\omega_0^{\star\alpha_2+\alpha_5-1}\sin\left(\frac{(\alpha_2+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{13}a_{51}+a_{23}a_{52})(\alpha_3+\alpha_4)\omega_0^{\star\alpha_3+\alpha_4-1}\sin\left(\frac{(\alpha_3+\alpha_4-1)\pi}{2}\right)$$
$$+(a_{22}a_{42}+a_{12}a_{41})(\alpha_3+\alpha_5)\omega_0^{\star\alpha_3+\alpha_5-1}\sin\left(\frac{(\alpha_3+\alpha_5-1)\pi}{2}\right)$$
$$+(a_{21}a_{32}+a_{11}a_{32})(\alpha_4+\alpha_5)\omega_0^{\star\alpha_4+\alpha_5-1}\sin\left(\frac{(\alpha_4+\alpha_5-1)\pi}{2}\right),$$
$$\overline{g}'^R_2(s)=$$
$$(a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\alpha_1\omega_0^{\star\alpha_1-1}\cos\left(\frac{(\alpha_1-1)\pi}{2}\right)$$
$$+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\alpha_2\omega_0^{\star\alpha_2-1}\cos\left(\frac{(\alpha_2-1)\pi}{2}\right)$$

$+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}+a_{23}a_{52})\times$
$\alpha_3\omega_0^{\star\alpha_3-1}\cos\left(\frac{(\alpha_3-1)\pi}{2}\right)$
$+(a_{21}a_{32}+a_{11}a_{32}+a_{13}a_{51}+a_{23}a_{52})\times$
$\alpha_4\omega_0^{\star\alpha_4-1}\cos\left(\frac{(\alpha_4-1)\pi}{2}\right)$
$+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\alpha_5\times$
$\omega_0^{\star\alpha_5-1}\cos\left(\frac{(\alpha_5-1)\pi}{2}\right),$
$g'^{I}_{2}(s)=$
$(a_{21}a_{32}+a_{22}a_{42}+a_{23}a_{52})\alpha_1\omega_0^{\star\alpha_1-1}\sin\left(\frac{(\alpha_1-1)\pi}{2}\right)$
$+(a_{11}a_{32}+a_{12}a_{41}+a_{13}a_{51})\alpha_2\omega_0^{\star\alpha_2-1}\sin\left(\frac{(\alpha_2-1)\pi}{2}\right)$
$+(a_{22}a_{42}+a_{12}a_{41}+a_{13}a_{51}+a_{23}a_{52})\times$
$\alpha_3\omega_0^{\star\alpha_3-1}\sin\left(\frac{(\alpha_3-1)\pi}{2}\right)$
$+(a_{21}a_{32}+a_{11}a_{32}+a_{13}a_{51}+a_{23}a_{52})\times$
$\alpha_4\omega_0^{\star\alpha_4-1}\sin\left(\frac{(\alpha_4-1)\pi}{2}\right)$
$+(a_{21}a_{32}+a_{22}a_{42}+a_{11}a_{32}+a_{12}a_{41})\times$
$\alpha_5\omega_0^{\star\alpha_5-1}\sin\left(\frac{(\alpha_5-1)\pi}{2}\right),$
$\overline{k}'^{R}_{0}(s)=\big[a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}\times$
$(a_{12}a_{21}-a_{11}a_{22})\big]\alpha_5\omega_0^{\star\alpha_5-1}\cos\left(\frac{(\alpha_5-1)\pi}{2}\right)+$
$\big[a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})+a_{32}a_{51}(a_{13}a_{21}-a_{11}a_{23})\big]\times$
$\alpha_4\omega_0^{\star\alpha_4-1}\cos\left(\frac{(\alpha_4-1)\pi}{2}\right)+\big[a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})$
$+a_{42}a_{51}(a_{22}a_{13}-a_{12}a_{23})\big]\alpha_3\omega_0^{\star\alpha_3-1}\cos\left(\frac{(\alpha_3-1)\pi}{2}\right),$
$\overline{k}'^{I}_{0}(s)=\big[a_{31}a_{42}(a_{11}a_{22}-a_{12}a_{21})+a_{32}a_{41}\times$
$(a_{12}a_{21}-a_{11}a_{22})\big]\alpha_5\omega_0^{\star\alpha_5-1}\sin\left(\frac{(\alpha_5-1)\pi}{2}\right)+$
$\big[a_{52}a_{31}(a_{11}a_{23}-a_{13}a_{21})+a_{32}a_{51}(a_{13}a_{21}-a_{11}a_{23})\big]\times$
$\alpha_4\omega_0^{\star\alpha_4-1}\sin\left(\frac{(\alpha_4-1)\pi}{2}\right)+\big[a_{52}a_{41}(a_{12}a_{23}-a_{13}a_{22})$
$+a_{42}a_{51}(a_{22}a_{13}-a_{12}a_{23})\big]\alpha_3\omega_0^{\star\alpha_3-1}\sin\left(\frac{(\alpha_3-1)\pi}{2}\right).$

REFERENCES

[1] M. Xiao, W. Zheng, G. Jiang, and J. Cao, "Undamped oscillations generated by Hopf bifurcations in fractional order recurrent neural networks with Caputo derivative," *IEEE Trans Neural Netw Learn Syst.*, vol. 26, pp. 3201–3214, 2015.

[2] M. Xiao, W. Zheng, and J. Cao, "Hopf bifurcation of an $(n+1)$-neuron bidirectional associative memory neural network model with delays," *IEEE Trans Neural Netw Learn Syst.*, vol. 24, pp. 118–132, 2013.

[3] C. Xu, "Local and global Hopf bifurcation analysis on simplified bidirectional associative memory neural networks with multiple delays," *Math Comput Simul.*, vol.149, pp. 69–90, 2018.

[4] X. Tian, R. Xu, and Q. Gan, "Hopf bifurcation analysis of a BAM neural network with multiple time delays and diffusion," *Appl Math Comput.*, vol. 266, pp. 909–926, 2015.

[5] I. Ncube, "Stability switching and Hopf bifurcation in a multiple-delayed neural network with distributed delay," *J Math Anal Appl.*, vol. 407, no. 1, pp. 141–146, 2013.

[6] C. Huang, J. Cao, M. Xiao, A. Alsaedi, and T. Hayat, "Bifurcations in a delayed fractional complex-valued neural network," *Appl Math Comput.*, vol. 292, pp. 210–227, 2017.

[7] C. Xu, X. Tang, and M. Liao, "Stability and bifurcation analysis of a six-neuron BAM neural network model with discrete delays," *Neurocomputing*, vol. 74, pp. 689–707, 2011.

[8] B. Zheng, Y. Zhang, and C. Zhang, "Global existence of periodic solutions on a simplified BAM neural network model with delays," *Chaos Solitons Fractals*, vol. 37, pp. 1397–1408, 2008.

[9] Y. Song, M. Han, and J. Wei, "Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays," *Physica D*, vol. 200, pp. 185–204, 2005.

[10] W. Yu and J. Cao, "Stability and Hopf bifurcation analysis on a four neuron BAM neural network with time delays," *Phys Lett A* , vol. 351, pp. 64–78, 2006.

[11] J. Cao and M. Xiao, "Stability and Hopf bifurcation in a simplified BAM neural network with two time delays," *IEEE Trans Neural Netw.*, vol.18, pp. 416–430, 2007.

[12] W. Yu and J. Cao, "Stability and Hopf bifurcation analysis on a four-neuron BAM neural network with delays," *Phys Lett A.*, vol. 351, pp. 64–78, 2006.

[13] L. Wang and G. Xu, "Quasi-synchronization of fractional-order complex-value BAM neural networks with time delays and discontinuous activations," *Comp. Appl. Math.*, vol. 43, no. 6, pp. 391, 2024.

[14] S. Rao and T. Zhang, "Global asymptotic stability and asymptotically periodic oscillation in fractional-order fuzzy cohen-grossberg neural networks with delays," *Engineering Letters*, vol. 32,no. 1, pp. 12–20, 2024.

[15] F. Jiang, J. Shen, and X. Li, "The LMI method for stationary oscillation of interval neural networks with three neuron activations under impulsive efects," *Nonlinear Anal: Real World Appl.*, vol. 14, no. 3, pp. 1404–1416, 2013.

[16] H. Hu and L. Huang. "Stability and Hopf bifurcation analysis on a ring of four neurons with delays," *Appl Math Comput.*, vol. 213, no. 2, pp. 587–599, 2009.

[17] J. Ge and J. Xu, "Synchronization and synchronized periodic solution in a simplifed five-neuron BAM neural network with delays," *Neurocomputing*, vol. 74, no. 6, pp. 993–999, 2011.

[18] C. Xu, M. Liao, P. Li, and Y. Guo, "Bifurcation analysis for simplified five-neuron Bidirectional Associative Memory neural networks with four delays," *Neural Process. Lett.*, vol. 50, no. 3, pp. 2219–2245, 2019.

[19] J. Zhang, J. Wu, H. Bao, and J. Cao, "Synchronization analysis of fractional-order three-neuron BAM neural networks with multiple time delays," *Appl Math Comput.*, vol. 339, pp. 441–450, 2018.

[20] A Pratap, R Raja, J. Cao, G. Rajchakit, and F. Alsaadi, "Further synchronization in fnite time analysis for time-varying delayed fractional order memristive competitive neural networks with leakage delay," *Neurocomputing*, vol. 317, pp. 110–126, 2018.

[21] C. Huang, J. Wang, X. Chen, and J. Cao, "Bifurcations in a fractional-order BAM neural network with four diferent delays," *Neural Netw.*, vol. 141, pp. 344–354, 2021.

[22] C. Xu, Z. Liu, M. Liao, P. Li, Q. Xiao, and S. Yuan, "Fractional order bidirectional associate memory (BAM) neural networks with multiple delays: The case of Hopf bifurcation," *Math Comput Simul.*, vol. 182, pp. 471–494, 2021.

[23] C. Huang, H. Wang, H. Liu, and J. Cao "Bifurcations of a delayed fractional-order BAM neural network via new parameter perturbations," *Neural Netw.*, vol. 168, pp. 123–142, 2023.

[24] J. Chen, M. Xiao, Y. Wan, C. Huang, and F. Xu, "Dynamical bifurcation for a class of large-scale fractional delayed neural networks with complex ring-hub structure and hybrid coupling," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 34, no. 5, pp. 2659–2669, 2023.

[25] C. Huang, Z. Li, D. Ding, and J. Cao, "Bifurcation analysis in a delayed fractional neural network involving self-connection," *Neurocomputing*, vol. 314, pp. 186–197, 2018.

[26] C. Huang, H. Wang, and J. Cao, "Fractional order-induced bifurcations in a delayed neural network with three neurons," *Chaos*, vol. 33, no. 3, pp. 033143, 2023.

[27] I. Podlubny, *Fractional differential equations,* San Diego California: Academic Press, 1999.

[28] J. Tignol, *Galois′ Theory of Algebraic Equations,* New Jersey: World Scientific Publishing Company, 2016.

[29] C. Xu, C. Aouiti, and Z. Liu, "A further study on bifurcation for fractional order BAM neural networks with multiple delays," *Neurocomputing*, vol. 417, pp. 501–515, 2020.

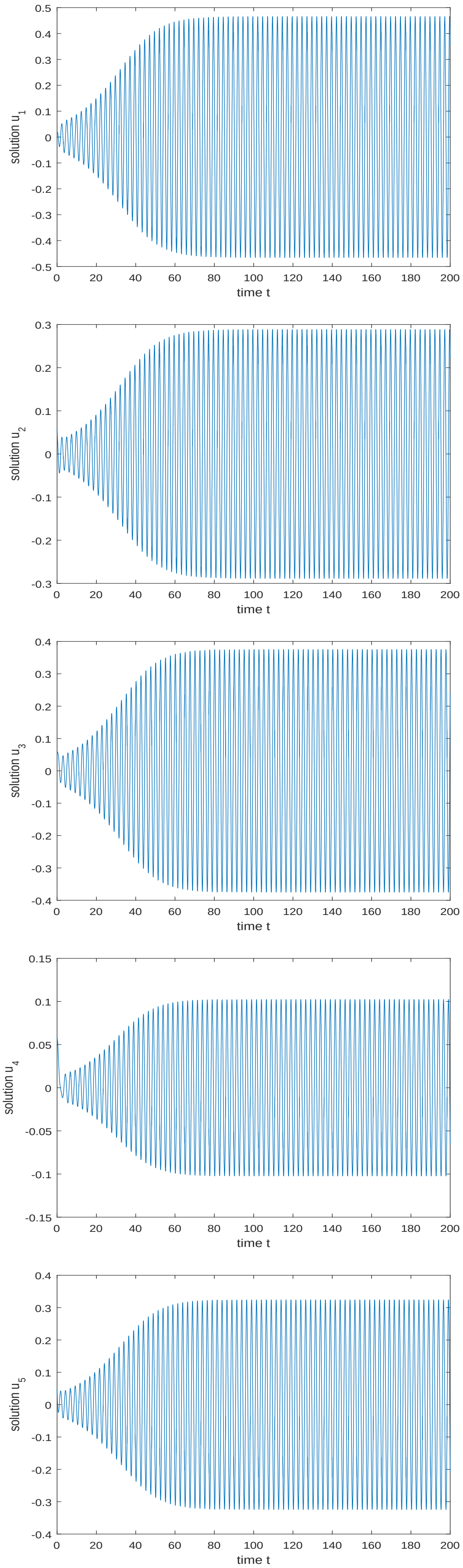

Fig. 1. Evolution of states of FOFBAMNN (11) with $\tau = 0.25, \mu = 1.86 < \mu_0$.

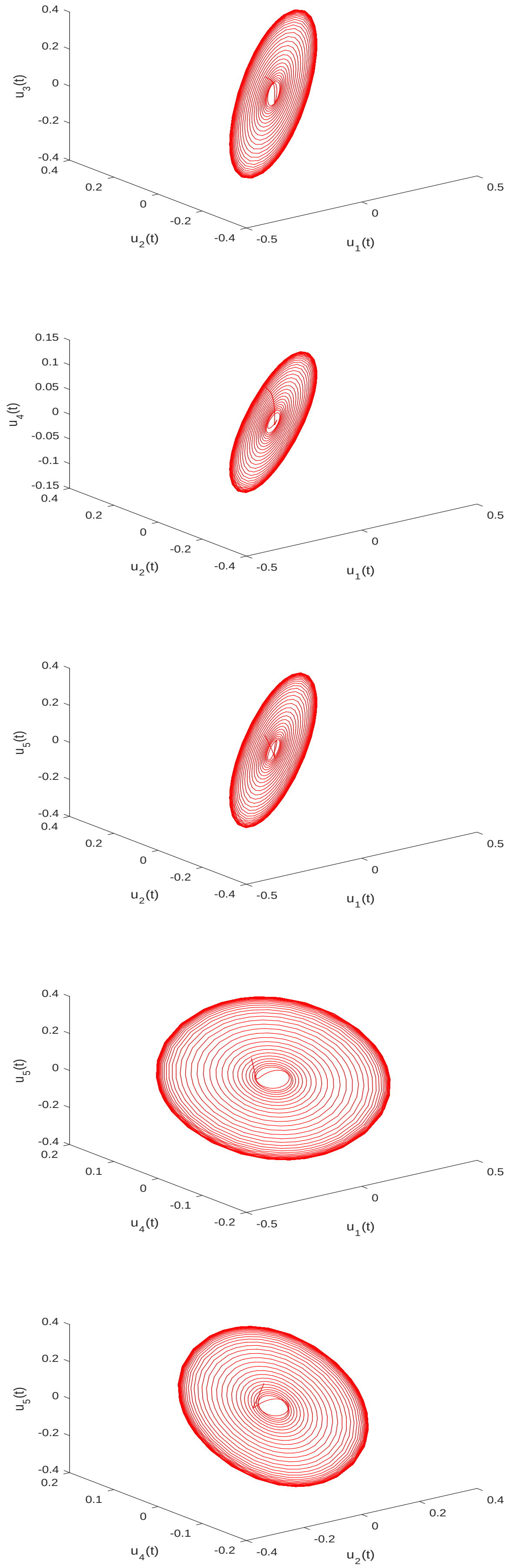

Fig. 2. Phase graphs of FOFBAMNN (11) with $\tau = 0.25, \mu = 1.86 < \mu_0$.

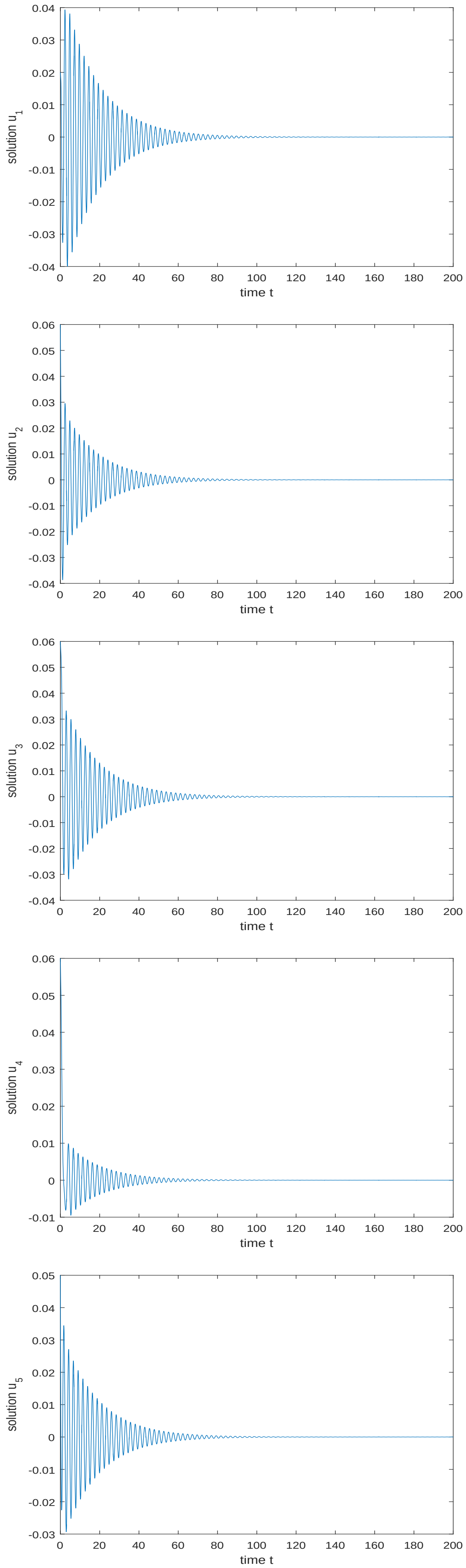

Fig. 3. Evolution of states of FOFBAMNN (11) with $\tau = 0.25, \mu = 2.02 > \mu_0$.

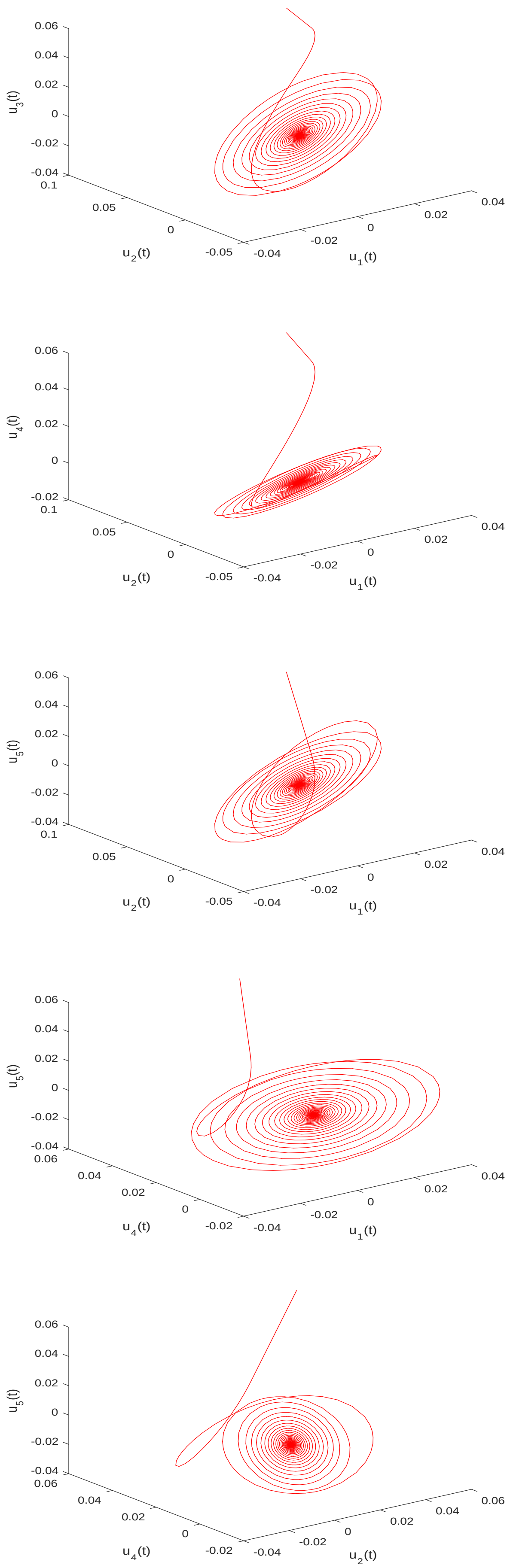

Fig. 4. Phase graphs of FOFBAMNN (11) with $\tau = 0.25, \mu = 2.02 > \mu_0$.