A Numerical Model of Groundwater Quality Assessment Using a Special A-D Cubic Spline Method

Pantira Klankaew, Nopparat Pochai

***Abstract*— Nowadays, when the industrial sector and human habitation have expanded rapidly, causing more and more various pollutants to come out, water is considered one type of pollution that is contaminated by these community and industrial sources. When this toxic water seeps through the soil, it can contaminate groundwater sources. Groundwater contamination can affect the health of humans and other living things, both directly and indirectly. In some areas, groundwater contamination can cause the population to become sick with various diseases. Long-term groundwater quality investigations near landfill sites need the use of mathematical models. A one-dimensional advection-diffusion equation (ADE) was used to analyze the groundwater's quality by describing the amount of contamination present. The objectives are mathematical simulations that can be used to assess the quality of groundwater that becomes contaminated over a long period of time. This study proposes numerical simulations for a one-dimensional mathematical model for long-term measurement of groundwater pollution around landfills. The natural cubic spline method, the Crank-Nicolson method, the upwind explicit method, and the special A-D cubic spline method are approximated in the model solution. The exact and approximate solutions are compared in each case. The proposed upwind explicit method analysis provides close to exact and properly accurate solutions. In five to ten years, the proposed numerical model can simulate several scenarios.**

***Index Terms*—groundwater pollution, contamination, advection-diffusion equation, natural cubic spline method, the special A-D cubic spline method**

I. Introduction

The most current groundwater pollution is caused by industry, agriculture, or improper waste disposal. Contamination of groundwater, which is a source of water for human consumption and consumption, will affect the health of humans and living things. It is the cause of people suffering from various diseases. Various mathematical models are employed to preserve the environment for human use, including [1–5]. To estimate pollutant concentrations in terms of total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate concentrations, we examine nitrogen pollutant models derived from the advection-dispersion-reaction equation in [6]. In [7], two distinct models are used to perform a mathematical simulation of water quality over an extended period of flooding: the one-dimensional shallow water equations, which provide the elevation and velocity of the water, and the one-dimensional advection-dispersion equation, which provides the concentrations of pollutants in the water after the sandbag dike has been destroyed. [8] In order to estimate the salt levels in the Lower Chao Phraya River, Thailand, this study aimed to create a numerical model of the one-dimensional advection-diffusion equation.

Mathematical models for evaluating groundwater quality have been used in recent studies. The salinity in the groundwater at different flow velocities is simulated using the mathematical model [9–11]. Water injection station management is optimized using mathematical simulations to minimize costs associated with groundwater management in drought areas [12]. The explicit method, as explained in [13], is applied in the transient two-dimensional groundwater flow model and the transient two-dimensional advection diffusion equation. A measured nitrogen dispersion on total nitrogen transformation effects models have been described in [14]. Groundwater quality in rural areas is affected by landfill construction, as demonstrated by the mathematical models [15]. The one-dimensional groundwater pollution measurement near landfills through heterogeneous soil is explained in [16]. The groundwater contamination with chloride and related compounds has been explained by means of mathematical models in [17]. Groundwater quality assessment is approximated using two-level explicit approaches and the Lax-Wendroff method [18]. The impact of pumping water to nearby settlements on groundwater flow and water quality has been proposed in [19]. For long-term groundwater quality investigations near waste sites, mathematical models are necessary in this instance. Studies on the environmental impacts of projects, including landfills, are based on the long-term expansion of groundwater quality. The amount of pollution in the groundwater was described using an advection-diffusion equation (ADE) in one dimension the groundwater quality analysis. Using the one-dimensional advection-diffusion equation (ADE), the concentration of pollutants in groundwater is expressed.

Manuscript received August 28, 2024; revised April 28, 2025.

This paper is supported by Centre of Excellence in Mathematics, Ministry of Higher Education, Science, Research and Innovation Bangkok, Thailand.

P. Klankaew is a PhD candidate of Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok, 10520, Thailand (e-mail: k.pantira24@gmail.com).

N. Pochai is an Associate Professor of Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok, 10520, Thailand (corresponding author to provide phone: 662329-8400; fax: 662-329-8400; e-mail: nop_math@yahoo.com).

Natural cubic spline, special A-D cubic spline, FTCS, and Crank-Nicolson are the four numerical methods used to estimate the advection and diffusion components of the one-dimensional advection-diffusion equations with constant coefficients in [20]. In [21], the Forward in Time, Center in Space (FTCS) finite difference approach was used to solve the three-dimensional advection-diffusion problem. In [22], Halley and Householder develop a variant of Newton's method with higher-order convergence for solving nonlinear equations utilizing iterative methods and the predictor-corrector methodology. In this research, we studied the groundwater dispersion flow through an inhomogeneous soil model. The finite difference method, which is the natural cubic spline, is used to obtain the approximated solutions. An analytical solution is provided to test the accuracy of the anticipated numerical approaches.

II. Governing Equation

A. Dispersion flow of groundwater pollution utilizing an inhomogeneous soil model

A groundwater efficiency model is governed by a one-dimensional advection-diffusion partial differential equation, as illustrated in [23];

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C}{\partial x}-u(x,t)C\right), \qquad (1)$$

for all $(x,t)\in[0,L]\times[0,T]$, C is the groundwater pollutant's dispersion concentration at point x along the longitudinal direction at time t. D is the dispersion coefficient for the pollutant technique, u is a constant flow velocity, L is the distance measured from the polluting source to the endpoint of the region under consideration and T is the rate of change simulation time. The groundwater flow velocity varies due to the soil's heterogeneity. An growing nature variation has been proposed by Kumar et al. [23]. Additionally, they thought that functions were assigned to the velocity parameters $g_1(x,t)$ and $g_2(x,t)$. It is possible to rewrite Eq. (1) as [24];

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left(D_0 g_1(x,t)\frac{\partial C}{\partial x}-u_0 g_2(x,t)C\right), \qquad (2)$$

Eq. (2) can be expressed as follows;

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left(D_0\frac{\partial g_1(x,t)}{\partial x}-u_0 g_2(x,t)C\right)+D_0 g_1(x,t)\frac{\partial^2 C}{\partial x^2}$$
$$-u_0\frac{\partial g_2(x,t)}{\partial x}C. \qquad (3)$$

In the equation above, D_0 and u_0 are constants, the dimensions of which depend on the expression $g_1(x,t)$ and $g_2(x,t)$. The inhomogeneity of the soil allows the rate of flow to differ. A difference in the growing dispersion of groundwater contaminants in heterogeneous soil has been considered by Kumar et al. [24]. The dispersion parameter is often believed to be proportional to the velocity square. Consequently, Eq. (2) is becoming;

$$g_1(x,t)=(1+ax)^2, \text{ and } g_2(x,t)=1+ax, \qquad (4)$$

the parameter a is the parameter accounting for the inhomogeneity of the medium $(length)^{-1}$, Eq. (3) is becoming;

$$\frac{\partial C}{\partial t}=\left[(1+ax)(2aD_0-u_0)\right]\frac{\partial C}{\partial x}+D_0(1+ax)^2\frac{\partial^2 C}{\partial x^2}-u_0 aC, \qquad (5)$$

$$\frac{\partial C}{\partial t}=p(x)\frac{\partial C}{\partial x}+q(x)\frac{\partial^2 C}{\partial x^2}-KC. \qquad (6)$$

where

$$p(x)=(1+ax)(2aD_0-u_0), \qquad (7)$$

$$q(x)=D_0(1+ax)^2, \qquad (8)$$

$$K=au_0, \qquad (9)$$

$$-\beta=p(x), \qquad (10)$$

$$\alpha=q(x). \qquad (11)$$

B. Initial and boundary conditions

The initial condition suggested by the soil's free state of concentration, which was initially contaminated by groundwater, is as follows:

$$C(x,0)=s(x),\ 0\le x\le L,\ t=0. \qquad (12)$$

where $s(x)$ is a given initially measured groundwater pollutant function. The concentration gradient at the end point is established by the average chance rate of groundwater pollutant concentration surrounding them, which is established by the following boundary conditions. The concentration of groundwater pollutants is introduced at the origin by a continuous input:

$$C(x,t)=C_0,\ x=0,\ t>0, \qquad (13)$$

$$\frac{\partial C(x,t)}{\partial x}=C_s,\ x=L,\ t\ge 0. \qquad (14)$$

where C_0 is a given average groundwater pollutant concentration at the considered landfill, and C_s is the pollutant concentration change rate in the vicinity of the far field monitoring station.

III. Numerical Techniques

Next, we divide the interval to discretize the domain $[0,L]$ into M subintervals such that $M\Delta x=L$ and the time interval $[0,T]$ into N subintervals such that $N\Delta t=T$. The grid points (x_i,t_n) are defined by $x_i=i\Delta x$ for all $i=0,1,2,...,M$ and $t_n=n\Delta t$ for all $n=0,1,2,...,N$ in which M and N are positive integers. We can then approximate $C(x_i,t_n)$ by C_i^n, value of the difference approximation of $C(x,t)$ at point $x=i\Delta x$ and $t=n\Delta t$, where $0\le i\le M$ and $0\le n\le N$. The forward time central space finite difference scheme (FTCS) will be utilized in the solution of Eq. (2).

A. The Natural Cubic Spline Method

The "natural" cubic spline is defined for this study as follows:

(i) The interpolating spline segments are cubic polynomial functions on each sub-interval $[x_k, x_{k+1}]$, $k=1,2,...,N$, and the segments agree with the function values at the grid-points;

(ii) The cubic spline segments' first and second derivatives are continuous at the internal positions;

(iii) At the first and last grid points, the second derivatives of the cubic spline segments are zero.

The following are satisfied by the cubic spline method's approximation of the governing equation:

$$\frac{\partial C}{\partial t} = p(x)P_k^n + q(x)Q_k^n - KC, \qquad (15)$$

$$\frac{C_k^{n+1}-C_k^n}{\Delta t} = p(x)P_k^n + q(x)Q_k^n - KC_k^n, \qquad (16)$$

for $k=1,2,...,N+1$; $n=0,1,2,...$ where P_k^n is the first derivative and Q_k^n the second derivative of the cubic spline function at the point x_k at time $n\Delta t$. Eq. (16) can be written in the explicit form:

$$C_k^{n+1} = \Delta t \cdot p(x)P_k^n + \Delta t \cdot q(x)Q_k^n + (1-\Delta t \cdot K)C_k^n, \qquad (17)$$

The values of the slopes P_k^n is produced by solving the set of simultaneous equations that follows (which are derived by manipulating the equations that come from the continuity conditions for the spline segments; for further information on algebraic working, see [26]):

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ \alpha_2 & 2 & \mu_2 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & \alpha_3 & 2 & \mu_3 & 0 & . & . & . & 0 & 0 & 0 \\ . & & & & & . & & & & & . \\ . & & & & & . & & & & & . \\ . & & & & & . & & & & & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_{N-1} & 2 & \mu_{N-1} \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} P_1^n \\ P_2^n \\ P_3^n \\ . \\ . \\ . \\ P_N^n \\ P_{N+1}^n \end{bmatrix} = \begin{bmatrix} d_1^n \\ d_2^n \\ d_3^n \\ . \\ . \\ . \\ d_N^n \\ d_{N+1}^n \end{bmatrix}, \qquad (18)$$

where

$$d_1^n = 3\left(\frac{C_2^n - C_1^n}{x_2 - x_1}\right),$$

$$d_k^n = 3\frac{\mu_k}{h_{k+1}}\left(C_{k+1}^n - C_k^n\right) + 3\frac{\alpha_k}{h_k}\left(C_k^n - C_{k-1}^n\right), \text{ for } k=2,3,...,N,$$

$$d_{N+1}^n = 3\left(\frac{C_{N+1}^n - C_N^n}{x_{N+1} - x_N}\right),$$

and where $\alpha_k = \frac{h_{k+1}}{(h_k + h_{k+1})}$, $\mu_k = 1-\alpha_k = \frac{h_k}{(h_k + h_{k+1})}$, $h_{k+1} = x_{k+1} - x_k$ and $h_k = x_k - x_{k-1}$.

The values of Q_k^n are the second derivatives of cubic spline at points x_k for $k=2,3,...,N$, at time $n\Delta t$. For the natural cubic spline it is assumed that $s_1''(x_1) = s_n''(x_{n+1}) = 0$ $(i.e.\ Q_1^n = Q_{N+1}^n = 0)$. Then we have:

$$Q_k^n = 6\frac{C_{k+1}^n - C_k^n}{(x_{k+1}-x_k)^2} - 4\frac{P_k^n}{x_{k+1}-x_k} - 2\frac{P_{k+1}^n}{x_{k+1}-x_k} \qquad (19)$$

for $k=2,3,...,N$.

The finite difference method's stability condition is used in this study.

B. The Crank-Nicolson Method

Analyse the following discretizations of the Crank-Nicolson scheme for the advection-diffusion equation:

$$\frac{\partial C}{\partial t} \approx \frac{\left(C_k^{n+1} - C_k^n\right)}{\Delta t}, \qquad (20)$$

$$\frac{\partial C}{\partial x} \approx \frac{\left(C_k^{n+1} - C_{k-1}^{n+1}\right)}{\Delta x}, \qquad (21)$$

$$\frac{\partial^2 C}{\partial x^2} \approx \frac{\left(C_{k+1}^{n+1} - 2C_k^{n+1} + C_{k-1}^{n+1}\right)}{(\Delta x)^2}, \qquad (22)$$

Using Crank-Nicolson Method approach, the governing equation's approximate solution satisfies:

$$(\beta-\gamma)C_{k-1}^{n+1} + (\lambda-\beta+2\gamma)C_k^{n+1} - \gamma C_{k+1}^{n+1} = (\lambda-\alpha)C_k^n,$$

for $k=1,2,...,N$ and $n=0,1,2,...,M$

C. The Upwind Explicit Method

Analyze the following discretizations to examine the upwind explicit scheme for the advection-diffusion equation:

$$\frac{\partial C}{\partial t} \approx \frac{\left(C_k^{n+1} - C_k^n\right)}{\Delta t}, \qquad (23)$$

$$\frac{\partial C}{\partial x} \approx \frac{\left(C_k^n - C_{k-1}^n\right)}{\Delta x}, \qquad (24)$$

$$\frac{\partial^2 C}{\partial x^2} \approx \frac{\left(C_{k+1}^n - 2C_k^n + C_{k-1}^n\right)}{(\Delta x)^2}, \qquad (25)$$

Using the upwind explicit approach, the governing equation's approximate solution satisfies:

$$C_k^{n+1} = \frac{(\gamma-\beta)}{\lambda}C_{k-1}^n + \frac{(\beta-2\gamma-\alpha+\lambda)}{\lambda}C_k^n + \frac{\gamma}{\lambda}C_{k+1}^n, \qquad (26)$$

for $k=1,2,...,N$ and $n=0,1,2,...,M$

D. The Special A-D Cubic Spline method

From equation (16) $u = g(x_k)$ and $D = h(x_k)$ we have,

$$\frac{C_k^{n+1}-C_k^n}{\Delta t} = uP_k^n + DQ_k^n - KC_k^n, \qquad (27)$$

for $k=1,2,...,N+1$; $n=0,1,2,...$ where P_k^n is the first derivative and Q_k^n the second derivative of the cubic spline function at the point x_k at time $n\Delta t$. Eq. (16) can be written in the explicit form:

$$C_k^{n+1} = \Delta t \cdot uP_k^n + \Delta t \cdot DQ_k^n + (1-\Delta t \cdot K)C_k^n, \qquad (28)$$

In this method we present a cubic spline interpolation scheme that satisfies the condition (19) with the requirement that $uP_1^n = DQ_1^n$ and $uP_{N+1}^n = DQ_{N+1}^n$. These imply that $Q_1^n = \frac{u}{D}P_1^n$ and $Q_{N+1}^n = \frac{u}{D}P_{N+1}^n$ (compare with natural cubic spline where $Q_1^n = 0$; $Q_{N+1}^n = 0$). The values of P_j^n can then be calculated from the following system:

$$A\overline{P} = \overline{d}, \tag{29}$$

where

$$\begin{bmatrix} 2+\frac{u(x_2-x_1)}{2D} & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ \alpha_2 & 2 & \mu_2 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & \alpha_3 & 2 & \mu_3 & 0 & . & . & . & 0 & 0 & 0 \\ . & & & & & & & & & & . \\ . & & & & & & & & & & . \\ . & & & & & & & & & & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_{n-1} & 2 & \mu_{n-1} \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 1 & \frac{4D-u(x_{N+1}-x_N)}{2D} \end{bmatrix}, \tag{30}$$

$\overline{P}$ and $\overline{d}$ are column vectors,

$$\overline{P} = \left[P_1^n,\ P_2^n,\ P_3^n, ..., P_N^n, P_{N+1}^n\right]^T$$

and

$$\overline{d} = \left[d_1^n,\ d_2^n,\ d_3^n, ..., d_N^n, d_{N+1}^n\right]^T,$$

$$d_1^n = 3\left(\frac{C_2^n - C_1^n}{x_2 - x_1}\right),$$

$$d_i^n = 3\frac{\mu_k}{h_{k+1}}\left(C_{k+1}^n - C_k^n\right) + 3\frac{\alpha_k}{h_k}\left(C_k^n - C_{k-1}^n\right), \text{ for } k = 2,3,...,N,$$

$$d_{N+1}^n = 3\left(\frac{C_{N+1}^n - C_N^n}{x_{N+1} - x_N}\right),$$

and where $\alpha_k = \frac{h_{k+1}}{(h_k + h_{k+1})}$, $\mu_k = 1 - \alpha_k = \frac{h_k}{(h_k + h_{k+1})}$. For this case Q_k^n, $k = 2,3,...,N$ can be obtained from

$$Q_k^n = 6\frac{C_{k+1}^n - C_k^n}{(x_{k+1} - x_k)^2} - 4\frac{P_k^n}{x_{k+1} - x_k} - 2\frac{P_{k+1}^n}{x_{k+1} - x_k}, \tag{31}$$

while Q_1^n and Q_{N+1}^n are calculated directly using the formulae already given above. The stability condition of the FTCS approach is used in this study.

IV. Numerical Experiments

The measured concentration of groundwater pollutants beneath a landfill and in the area around it. The examined region is 1.0 km long overall, which is in line with the longitudinal distance. Leachate from a landfill is released into the earth as a source of pollution. The parameters for pollutants at the dump under consideration are $D_0 = 0.71\ km^{-2}/year$, $C_0 = 1.0\ kg/l$, $u_0 = 0.60\ km/year$ and $a = 1\ km^{-1}$. The numerical experiment discretizes time and space by $\Delta x = 0.1\ km$ and $\Delta t = 0.0001$ year, respectively. The Special A-D Cubic Spline approach, the Crank-Nicolson method, the upwind explicit method, and the natural cubic spline method are used to approximate the groundwater concentration. An analytical solution to the ideal advection-diffusion problem, as suggested in [27], is obtained.

$$C(x,t) = \frac{C_0}{2}\left(\begin{array}{l} (1+ax)^{-1} erfc\left(\frac{\ln(1+ax)}{2a\sqrt{D_0 t}} - \beta_0\sqrt{t}\right) \\ +(1+ax)^{\delta} erfc\left(\frac{\ln(1+ax)}{2a\sqrt{D_0 t}} + \beta_0\sqrt{t}\right) \end{array} \right). \tag{33}$$

where

$$\omega_0 = \left(au_0 - a^2 D_0\right), \tag{34}$$

$$\beta_0 = \sqrt{\frac{\omega_0^2}{4a^2 D_0} + au_0} = \frac{u_0 + aD_0}{2\sqrt{D_0}}, \tag{35}$$

$$\delta = \frac{u_0}{aD_0}. \tag{36}$$

If we employ the natural cubic spline method, in Eqs. (15)-(19), we obtain the estimated groundwater pollution along the examined area for a period of one year, as indicated in Table I and Figs. 1 and 2. We use the Crank-Nicolson method, in Eqs. (20)-(22), the estimated concentration of groundwater pollutants along the longitudinal area under consideration is obtained in Table II and Figs. 3 and 4. Utilizing the upwind explicit scheme, we get the estimated concentration of groundwater pollutants in Figs. 5 and 6 as well as Table III using Eqs. (23)-(25). We derive the estimated groundwater pollutant concentration in Figs. 7 and 8 and Table IV using the special A-D cubic spline method, which is expressed in Eqs. (27)–(31). The comparison of analytical solution, the natural cubic spline, the Crank-Nicolson method, the upwind explicit method and the Special A-D Cubic Spline method at 0.5 year, as shown in Figs. 9.

TABLE I
ESTIMATED GROUNDWATER POLLUTANT CONCENTRATION OVER A CONSIDERED AREA BETWEEN 0.1-1.0 YEARS USING THE NATURAL CUBIC SPLINE METHOD

$C(x,t)$						
t \ x	0.0	0.1	0.2	0.3	0.4	0.5
0.1	1.0000	0.7936	0.6175	0.4762	0.3664	0.2841
0.3	1.0000	0.8930	0.8024	0.7297	0.6727	0.6293
0.5	1.0000	0.9310	0.8737	0.8287	0.7939	0.7678
0.7	1.0000	0.9463	0.9025	0.8686	0.8428	0.8236
1.0	1.0000	0.9540	0.9170	0.8887	0.8675	0.8518
t \ x	0.6	0.7	0.8	0.9	1.0	
0.1	0.2252	0.1859	0.1629	0.1535	0.1532	
0.3	0.5978	0.5765	0.5639	0.5587	0.5586	
0.5	0.7489	0.7363	0.7288	0.7258	0.7257	
0.7	0.8099	0.8007	0.7954	0.7932	0.7931	
1.0	0.8406	0.8332	0.8289	0.8271	0.8271	

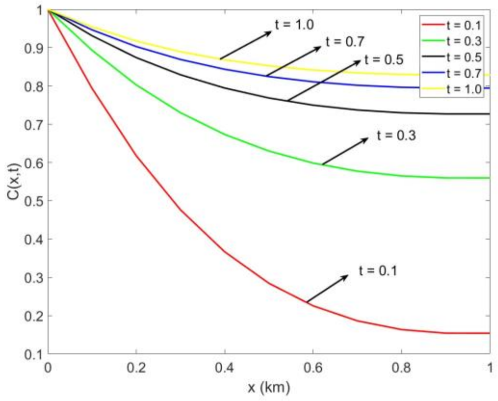

Fig 1. Groundwater pollutant by using the natural cubic spline method at 0.1, 0.3, 0.5, 0.7 and 1.0 years.

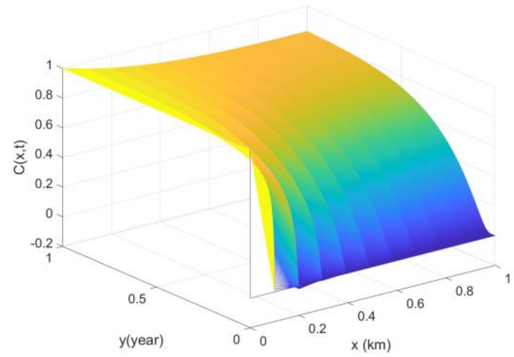

Fig 2. The surface plot of groundwater pollutant by using the natural cubic spline method.

TABLE II
ESTIMATED GROUNDWATER POLLUTANT CONCENTRATION OVER A CONSIDERED AREA BETWEEN 0.1-1.0 YEARS USING THE CRANK-NICOLSON METHOD

$C(x,t)$

t \ x	0.0	0.1	0.2	0.3	0.4	0.5
0.1	1.0000	0.7788	0.5972	0.4526	0.3399	0.2538
0.3	1.0000	0.8569	0.7367	0.6354	0.5498	0.4773
0.5	1.0000	0.8783	0.7746	0.6901	0.6163	0.5529
0.7	1.0000	0.8884	0.7951	0.7161	0.6485	0.5901
1.0	1.0000	0.8960	0.8094	0.7362	0.6736	0.6194

t \ x	0.6	0.7	0.8	0.9	1.0
0.1	0.1888	0.1404	0.1046	0.0786	0.0598
0.3	0.4157	0.3633	0.3186	0.2804	0.2477
0.5	0.4980	0.4502	0.4084	0.3717	0.3394
0.7	0.5393	0.4948	0.4556	0.4208	0.3899
1.0	0.5722	0.5307	0.4940	0.4614	0.4321

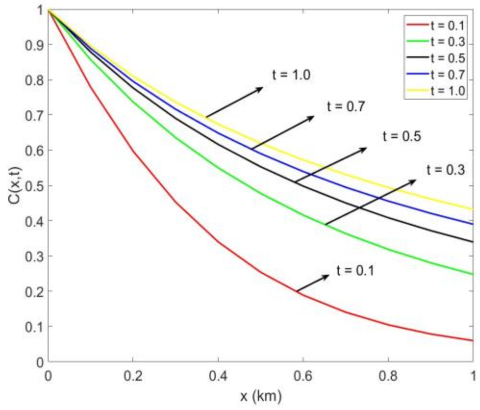

Fig 3. Groundwater pollutant by using the Crank-Nicolson method at 0.1, 0.3, 0.5, 0.7 and 1.0 years.

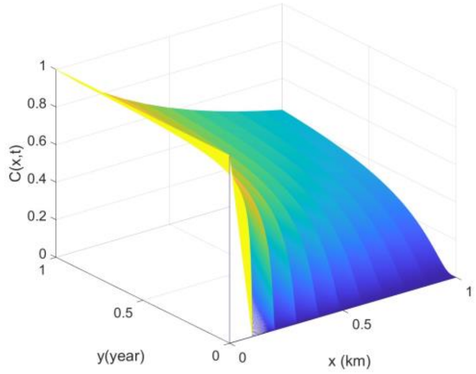

Fig 4. The surface plot of groundwater pollutant by using Crank-Nicolson method.

TABLE III
ESTIMATED GROUNDWATER POLLUTANT CONCENTRATION OVER A CONSIDERED AREA BETWEEN 0.1-1.0 YEARS USING THE UPWIND EXPLICIT METHOD

$C(x,t)$

t \ x	0.0	0.1	0.2	0.3	0.4	0.5
0.1	1.0000	0.7857	0.6081	0.4651	0.3524	0.2651
0.3	1.0000	0.8607	0.7427	0.6424	0.5570	0.4840
0.5	1.0000	0.8812	0.7808	0.6952	0.6215	0.5577
0.7	1.0000	0.8909	0.7989	0.7205	0.6530	0.5943
1.0	1.0000	0.8983	0.8129	0.7402	0.6776	0.6232

t \ x	0.6	0.7	0.8	0.9	1.0
0.1	0.1983	0.1477	0.1095	0.0810	0.0598
0.3	0.4215	0.3679	0.3218	0.2820	0.2477
0.5	0.5021	0.4535	0.4107	0.3729	0.3394
0.7	0.5429	0.4979	0.4575	0.4218	0.3899
1.0	0.5755	0.5333	0.4958	0.4623	0.4321

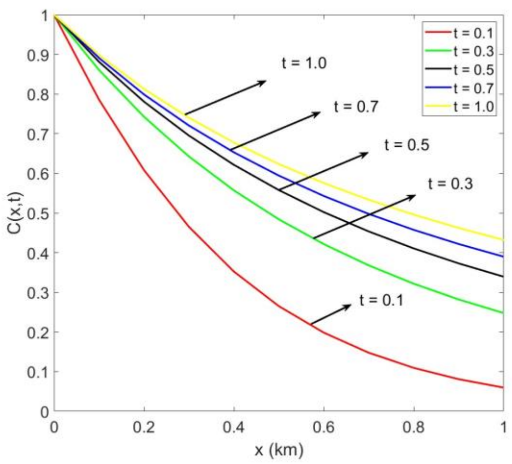

Fig 5. Groundwater pollutant by using the upwind explicit method at 0.1, 0.3, 0.5, 0.7 and 1.0 years.

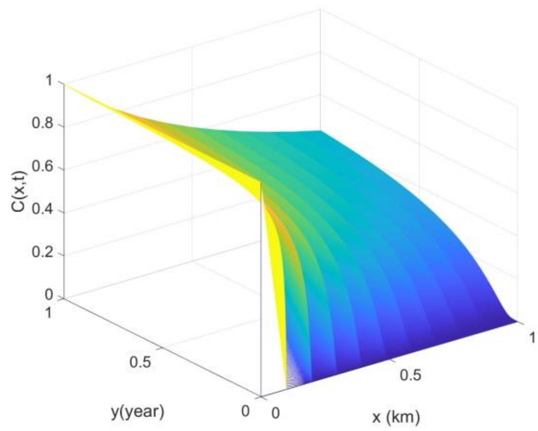

Fig 6. The surface plot of groundwater pollutant by using the upwind explicit method.

TABLE IV
ESTIMATED GROUNDWATER POLLUTANT CONCENTRATION OVER A CONSIDERED AREA BETWEEN 0.1-1.0 YEARS USING THE SPECIAL A-D CUBIC SPLINE METHOD

$C(x,t)$						
t \ x	0.0	0.1	0.2	0.3	0.4	0.5
0.1	1.0000	0.7936	0.6176	0.4763	0.3666	0.2844
0.3	1.0000	0.8931	0.8027	0.7300	0.6731	0.6298
0.5	1.0000	0.9311	0.8740	0.8291	0.7944	0.7684
0.7	1.0000	0.9465	0.9028	0.8691	0.8434	0.8243
1.0	1.0000	0.9542	0.9173	0.8892	0.8681	0.8525
t \ x	0.6	0.7	0.8	0.9	1.0	
0.1	0.2255	0.1862	0.1634	0.1541	0.1539	
0.3	0.5984	0.5772	0.5647	0.5597	0.5596	
0.5	0.7497	0.7371	0.7298	0.7268	0.7268	
0.7	0.8107	0.8017	0.7964	0.7943	0.7943	
1.0	0.8415	0.8342	0.8299	0.8283	0.8283	

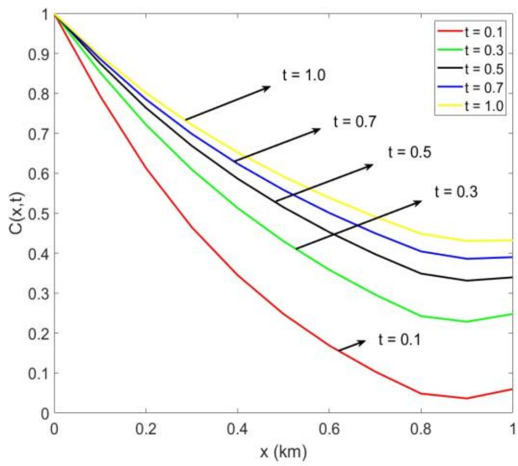

Fig 7. Groundwater pollutant by using the special A-D cubic spline method at 0.1, 0.3, 0.5, 0.7 and 1.0 years.

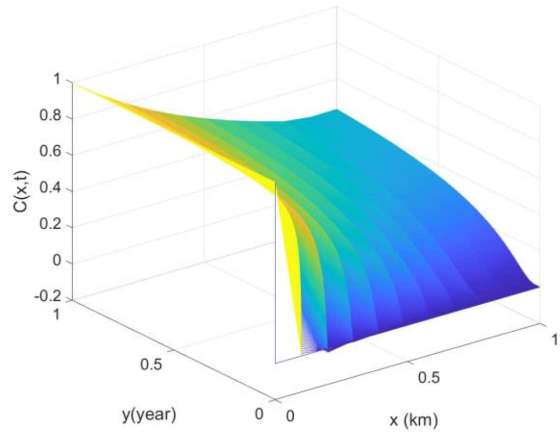

Fig 8. The surface plot of groundwater pollutant by using the special A-D cubic spline method.

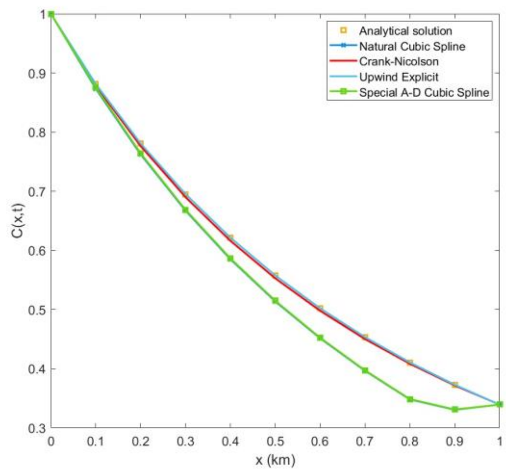

Fig 9. The comparison of analytical solution, the natural cubic spline, the Crank-Nicolson method, the upwind explicit method and the Special A-D cubic spline method at 0.5 year.

V. DISCUSSION

The natural cubic spline method, the Crank-Nicolson method, the upwind explicit method, and the special A-D cubic spline method give good agreement for approximated groundwater pollutant concentration in an ideal case, as shown in Figs. 1, 3, 5, and Fig. 7. The groundwater pollution measurement is simulated over a long time span, approximately 0-1.0 years, as Tables I–IV demonstrate. The surface plot of groundwater pollutants is shown in Figs. 2, 4, 6, and 8. The comparison of the analytical solution, the natural cubic spline, the Crank-Nicolson method, the upwind explicit method, and the Special A-D cubic spline method at 0.5 years is shown in Figs. 9.

VI. Conclusion

A one-dimensional groundwater pollutant concentration model is introduced. The techniques of the initial condition and boundary conditions of the groundwater pollutant concentration model are proposed. The numerical solutions of pollutants in groundwater are estimated using the natural cubic spline, the Crank-Nicolson method, the upwind explicit method, and the Special A-D cubic spline method approach. The upwind explicit method is closest to the analytical solution. The simulation that is being shown can be used to evaluate the quality of groundwater that has been contaminated for more than ten years. The proposed upwind explicit method analysis provides close to exact and properly accurate solutions. The suggested numerical model can simulate multiple scenarios in five to ten years.

References

[1] W. Timpitak and N. Pochai, "A Risk Assessment Model for Airborne Infection in a Ventilated Room using the Adaptive Runge-Kutta Method with Cubic Spline Interpolation", IAENG International Journal of Applied Mathematics, vol. 52, no. 4, pp791-798, 2022.

[2] P. Unyapoti and N. Pochai, "A Shoreline Evolution Model With the Wavelength Effect of Breaking Waves on Groin Structures", IAENG International Journal of Computer Science, vol. 49, no. 3, pp683-694, 2022.

[3] S. Manilam and N. Pochai "A Non-Dimensional Mathematical Model of Shoreline Evolution with a Groin Structure Using an Unconditionally Stable Explicit Finite Difference Technique", IAENG International Journal of Applied Mathematics, vol. 52, no. 2, pp473-481, 2022.

[4] W. Timpitak and N. Pochai, "A Mathematical Model of Risk Assessment on Airborne Infection in a Room with an Outlet Ventilation System", Engineering Letters, vol. 30, no. 2, pp898-903, 2022.

[5] W. Boonmeemapasuk and N. Pochai "A Risk Model of Airborne Transmission and Vaccine Efficacy in an Outpatient Room with a Ventilation System", Engineering Letters, vol. 30, no. 2, pp644-651, 2022.

[6] A. Vongkok and N. Pochai "Numerical simulations for reactive nitrogen compounds pollution measurements in a stream using Saulyev method", Italian journal of pure and applied mathematics, vol. 43, pp552-582, 2020.

[7] K. Subklay and N. Pochai "Numerical Simulation of Water-Quality Model on Flooding Using Revised Lax-Diffusive and Modified Siemieniuch-Gladwell Methods", Thai Journal of Mathematics, vol. 17, no. 3, pp685-702, 2019.

[8] K. Kulmart and N. Pochai "Simulation of Salinity Intrusion in Chao Phraya River by a Developed Mathematical Model and MacCormack Scheme with Cubic Spline Interpolation", Universal Journal of Mechanical Engineering 7(6): 386-397, 2019.

[9] N. Pongnoo and N. Pochai, "A Numerical Computation of Non dimensional Form of a Mathematical Model of Soil Salinity Profile in a Rice Field near Marine Shrimp Aquaculture Farm", Adv Studies Theor Phys, vol. 5, no. 4, pp185-191, 2011.

[10] N. Pongnoo and N. Pochai, "Numerical Treatment of a Couple Mathematical Models of Ground Water Flow in Rice Field near Marine Shrimp Aquaculture Farm", Appl Math Sci, vol. 6, no. 6, pp283-289, 2012.

[11] N. Pongnoo and N. Pochai, "A Numerical Treatment of a Mathematical Model of Groundwater Flow in Rice Field near Marine Shrimp Aquaculture Farm", Procedia Engineering 32, pp.1191-1197, 2012.

[12] N. Pongnu and N. Pochai, "Mathematical Simulation of Groundwater Management in a Drought Area Using an Implicit Finite Difference Method", Thai Journal of Mathematics., vol. 17, no. 3, pp663-672, 2019.

[13] J. Limthanakul and N. Pochai, "A Two-dimensional Mathematical Model for Long-term Contaminated Groundwater Pollution Measurement around a Land Fill", Mathematics and Statistics, vol. 8, no. 1, pp61-74, 2020.

[14] S. Yena and N. Pochai "Groundwater-quality Assessment Models with Total Nitrogen Transformation Effects", Mathematics and Statistics, vol. 8, no. 2, pp210-219, 2020.

[15] S. Yena and N. Pochai "Numerical Simulation of a Two-Dimensional Vertically Averaged Groundwater Quality Assessment in Homogeneous Aquifer Using Explicit Finite Difference Techniques", Mathematics and Statistics, vol. 8, no. 2, pp152-165, 2020.

[16] W. Timpitak and N. Pochai, "Numerical Simulations to a One-dimensional Groundwater Pollution Measurement Model Through Heterogeneous Soil", IAENG International Journal of Applied Mathematics, vol. 50, no. 3, pp558-565, 2020.

[17] J. Limthanakul and N. Pochai, "A Mathematical Model of Horizontal Averaged Groundwater Pollution Measurement with Several Substances due to Chemical Reaction", Mathematics and Statistics, vol. 8, no. 6, pp645-655, 2020.

[18] S. Khatbanjong and N. Pochai, "Numerical Groundwater Quality Assessment Model Using Two-level Explicit Methods", Engineering Letters, vol. 29, no. 1, pp183-190, 2021.

[19] N. Pongnoo and N. Pochai, "Mathematical Modelling of Groundwater Quality Measurement around a Landfill for Residential water Usage", Engineering Letters, vol. 30, no. 3, pp1006-1016, 2022.

[20] M. Thongmoon, S. Tangmanee, R. Mckibbin, A Comparison Of Splines Interpolations With Standard Finite Difference Methods For One-Dimensional Advection-Diffusion Equation, International Journal of Modern Physics, vol. 19, no. 08, 2008.

[21] M. Thongmoon, Numerical Experiment of Air Pollutant Concentration in the Street Tunnel, International Mathematical Forum, 5, no. 10, 449 – 465, 2010.

[22] N. Srisarakham and M. Thongmoon, A Note on Three-Step Iterative Method with Seventh Order of Convergence for Solving Nonlinear Equations, Thai Journal of Mathematics, vol. 14, no. 3, pp565–573, 2016.

[23] A. Kumar, D.K, Jaiswal, N. Kumar, Analytical solutions to one-dimensional advection-diffusion equation whit variable coefficients in semi-infinite media, J. Hydrol. 380, pp330-337, 2010.

[24] Svetislav SavovĆ, Alexander Djordjevich, Finite difference solution of the one-dimensional advection-diffusion equation with variable coefficients in semi-infinite media, International Journal of Heat and Mass Transfer 55, pp91-94, 2012.

[25] M. Dehghan, "Weighted finite difference techniques for the one-dimensional advection-diffusion equation," Applied Mathematics and Computation, vol. 147, no. 2, pp. 307–319, 2004.

[26] E.V. Shikin and A.I. Plis. Handbook on splines for the user. CRC Press Inc. 1995.

[27] A. A. Opanuga, Member, IAENG, E.A. Owoloko, H. I. Okagbue,O.O. Agboola, "Finite Difference Method and Laplace Transform for Boundary Value Problems," Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering 2017, WCE 2017, 5-7 July, 2017, London, U.K., pp65- 69.

P. Klankaew is an assistant researcher of Centre of Excellence in Mathematics, MHESI, Bangkok 10400, Thailand.

N. Pochai is a researcher of Centre of Excellence in Mathematics, MHESI, Bangkok 10400, Thailand.