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High-order Rogue Waves of the Nonlinear Schrödinger Equation

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Abstract—**Rogue waves are characterized by their unexpectedly large amplitudes and highly localized displacements, originating from either an equilibrium state or a relatively calm background. Extreme wave events have attracted significant attention due to their potential hazards to oceanic and optical systems. In this study, we utilize the bilinear Kadomtsev-Petviashvili (KP) reduction method and the W-treatment, to systematically investigate rogue wave solutions of the nonlinear Schrödinger (NLS) equation. Furthermore, we explore the evolution of rogue waves as parameters change. This approach allows for a robust derivation and thorough analysis of accurate rogue wave solutions, providing greater understanding of their formation and evolutionary processes.**

Keywords: the NLS equation, rogue waves, bilinear KP method, W-treatment

1 Introduction

The investigation of integrable properties and the formulation of exact solutions for nonlinear evolution equations (NLEEs) are crucial to our understanding nonlinear phenomena [1–4] in physics and mathematics. Rogue waves, alternatively termed as freak, monster, killer, extreme, or abnormal waves, are emerging as a significant focus within the physics community. Derived from the discipline of oceanography, this term delineates the occurrence of extensive, unpredictable waves on the ocean surface that arise suddenly and pose substantial risks to sea-faring vessels, including large ships and ocean liners. These impressive waves are not only indicative of oceanic anomalies, but they also parallel extreme phenomena observed in optical fibers [5–7]. Recently, rogue waves have drawn considerable interest in physics and nonlinear wave research, leading to a wealth of research.

The pursuit of comprehending and forecasting these waves has resulted in the derivation of analytical expressions for rogue waves within a plethora of integrable nonlinear wave equations. Of particular significance is the NLS equation [8, 9], which has played a crucial role in rogue wave studies. Furthermore, the derivative NLS equation [10,11], the three-wave interaction equation [12], and the Davey-Stewartson equations [13, 14] have significantly contributed to our understanding of these waves. In addition to these, a myriad of other equations [15–17] have enriched the growing body of knowledge on rogue waves, each providing distinct insights and augmenting the collective comprehension of these enigmatic natural phenomena.

In this study, we examine rogue waves within the context of the focusing NLS equation (1.1).

$$iu_t - u_{xx} - 2|u|^2 u = 0. \tag{1.1}$$

Upon applying the variable transformation $u \to ue^{-2it}$, the NLS equation (1.1) morphs into

$$iu_t - u_{xx} - 2(|u|^2 - 1)u = 0. \tag{1.2}$$

where

$$u = u(x,t) \to 1, \qquad x, t \to \pm\infty. \tag{1.3}$$

To present the rational solutions in the NLS equation, we introduce Schur polynomial $S_n(\boldsymbol{t})$ through a generating function in the following manner.

$$\sum_{n=0}^{\infty} S_n(\boldsymbol{t})\xi^n = exp(\sum_{k=1}^{\infty} t_k \xi^k),$$

where $\boldsymbol{t} = (t_1, t_2, \ldots)$.

Utilizing the subsequent transformation,

$$u = \frac{g}{f}, \tag{1.4}$$

the NLS equation (1.2) is initially converted into a bilinear form,

$$\begin{cases} (D_x^2 + 2)f \cdot f - 2|g|^2 = 0, \\ (D_x^2 - iD_t)g \cdot f = 0, \end{cases} \tag{1.5}$$

where f is a real variable and g is a complex variable, D corresponds to the Hirota bilinear operator [18], defined as follows:

$$D_x^m D_y^n f \cdot g = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m (\frac{\partial}{\partial y} - \frac{\partial}{\partial y'})^n f(x,y) g(x',y')|_{x'=x, y'=y}. \tag{1.6}$$

Subsequently, we examine a $(2+1)$-dimensional generalization of Eq.(1.5),

$$\begin{cases} (D_x D_y + 2)f \cdot f - 2gh = 0, \\ (D_x^2 - iD_t)g \cdot f = 0, \end{cases} \tag{1.7}$$

where h represents an additional complex variable.

The rogue waves of Eq.(1.1) have been thoroughly examined using the KP reduction method [13]. In this

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study, we address the resolution of rogue waves within Eq.(1.1) by synergistically integrating the KP reduction method with the treatment in Ref. [19]. This integrated approach presents a solution that is both coherent and articulate.

This paper proceeds as follows: we get two conclusions regarding the higher-order rogue waves of Eq.(1.2) in section 2. The specific proofs are provided in section 3. In section 4, we examine the dynamics of two types of rogue waves as parameters change. In conclusion, section 5 encapsulates the principal discoveries of this study and deliberates on their prospective implications for future research endeavors.

2 High-order rogue wave solutions

The general rogue waves of the NLS equation (1.2) are delineated by Theorem 1 and Theorem 2.

Theorem 1 *The NLS equation, represented as equation (1.2) under the boundary conditions of equation (1.3), accepts $N-th$ order rational and nonsingular rogue wave solutions*

$$u = \frac{\tau_1}{\tau_0}. \tag{2.1}$$

where

$$\tau_n = \det_{1\le k,l\le N}(m^n_{2k-1,2l-1}). \tag{2.2}$$

the matrix elements are delineated by

$$m^n_{kl} = \sum_{i=0}^{k}\sum_{j=0}^{l}\frac{a_i}{(k-i)!}\frac{a_j^*}{(l-j)!}(f(\mu)\partial\mu)^{k-i}(f(\nu)\partial\nu)^{l-j} \times(\frac{1}{\mu+\nu}(-\frac{\mu}{\nu})^n e^{(\mu+\nu)x-(\mu^2-\nu^2)it})|_{\mu=\nu=1}, \tag{2.3}$$

with

$$f(\mu)=\mu, \qquad f(\nu)=\nu, \tag{2.4}$$

"$*$" denotes complex conjugation, a_k represents the complex constants.

In Theorem 1, m_{kl} is articulated utilizing derivatives in relation to the auxiliary parameters μ and ν. Alternatively, matrix elements can be depicted solely through algebraic means by leveraging elementary Schur polynomials. Specifically, the matrix element $m^n_{k,l}$ referred to in Eq.(2.3) can be articulated as shown below.

Theorem 2 *The matrix element m_{kl} in Eq.(2.3) can be articulated through the purely algebraic expression*

$$m^n_{kl} = \sum_{v=0}^{\min(k,l)}\Phi_{kv}\Psi_{lv}, \tag{2.5}$$

where

$$\Phi_{kv} = (\frac{1}{2})^v\sum_{i=0}^{k-v}a_i S_{k-v-i}(\boldsymbol{x}^+ + \boldsymbol{r} + v\boldsymbol{s}),$$
$$\Psi_{lv} = (\frac{1}{2})^v\sum_{j=0}^{l-v}a_j^* S_{l-v-j}(\boldsymbol{x}^- + \boldsymbol{r} + v\boldsymbol{s}), \tag{2.6}$$

and vectors $\boldsymbol{x}^\pm = (x_1^\pm, x_2^\pm, \ldots), \boldsymbol{r} = (r_1, r_2, \ldots), \boldsymbol{s} = (s_1, s_2, \ldots)$ are defined by

$$x_l^\pm = \frac{x \mp 2^l it}{l!},$$
$$\sum_{l=1}^{\infty} r_l\xi^l = -\ln(e^{\frac{\xi}{2}}\cos\frac{\xi}{2}),$$
$$\sum_{l=1}^{\infty} s_l\xi^l = \ln\frac{2}{\xi}\tanh\frac{\xi}{2}. \tag{2.7}$$

3 Derivation and proof of of rogue-wave solutions

In this section, the proof of Theorems 1 and 2 as described. Our approach is based on the Hirota's bilinear representation of integrable equations [18]. We have also taken into consideration the frequent occurrence of such bilinear equations within the KP hierarchy [20]. Therefore, when specific reduction constraints are applied to the solutions of the KP hierarchy, they furnish solutions for the pristine integrable system. The method yields the determinant-type solutions. It is remarkable that it helps in constructing solutions of higher-dimensional integrable equations much more easily than those of lower-dimensional ones.

The derivation will be presented as follows. In order to get solutions to the Eq.(1.5), our consideration extends to a (2 + 1)-dimensional generalization Eq.(1.7). We commence by formulating a broad category of algebraic solutions of Eq.(1.7) using Gram determinants. Following this, we narrow down these solutions to ensure they meet the dimension-reduction condition.

$$f_x - f_y = C_1 f, \tag{3.1}$$

and

$$h = g^*, \qquad f: real, \tag{3.2}$$

where C_1 is some constant. In this scenario, the higher-dimensional bilinear equation (1.7) is simplified to

$$\begin{cases}(D_x^2+2)f\cdot f - 2gh = 0,\\ (D_x^2 - iD_t)g\cdot f = 0.\end{cases} \tag{3.3}$$

Then, we apply the condition (3.2) to the algebraic solution. Upon further analysis, the bilinear system (3.3) is reducible to the bilinear NLS equation (1.5). Consequently, Eq. (1.7) provides the general high-order rogue wave solutions to the NLS equation (1.2).

Next, we shall adhere to the aforementioned framework to deduce general rogue wave solutions for the NLS equation (1.2). First, we deduce algebraic solutions for the higher-dimensional bilinear equation (1.7). According to Lemma 1 in Ref. [13], we understand that the function $m_{kl}^{(n)}, \delta_k^{(n)}, \phi_l^{(n)}$ is functions of x_1, x_2, x_{-1} fulfilling the differential and difference relations as follows

$$\begin{aligned} m_{kl,x_1}^{(n)} &= \delta_k^{(n)}\phi_l^{(n)}, \\ m_{kl,x_2}^{(n)} &= \delta_k^{(n+1)}\phi_l^{(n)} + \delta_k^{(n)}\phi_l^{(n-1)}, \\ m_{kl,x_{-1}}^{(n)} &= -\delta_k^{(n-1)}\phi_l^{(n+1)}, \\ m_{kl}^{(n+1)} &= m_{kl}^{(n)} + \delta_k^{(n)}\phi_l^{(n+1)}, \\ \delta_{k,x_i}^{(n)} &= \delta_k^{(n+i)}, \phi_{l,x_i}^{(n)} = -\phi_l^{(n-i)}, (i=1,2,-1). \end{aligned} \quad (3.4)$$

It can be inferred that the determinant

$$\tau_n = \det_{1\le k,l\le N}(m_{kl}^{(n)}) \quad (3.5)$$

adheres to the following bilinear equations

$$\begin{cases} (D_{x_1}D_{x_{-1}} - 2)\tau_n\cdot\tau_n + 2\tau_{n+1}\tau_{n-1} = 0, \\ (D_{x_1}^2 - D_{x_2})\tau_{n+1}\cdot\tau_n = 0. \end{cases} \quad (3.6)$$

Functions $m^{(n)}, \delta^{(n)}, \phi^{(n)}$ are introduced for the purpose of this study,

$$\begin{aligned} m^{(n)} &= \frac{1}{\mu+\nu}(-\frac{\mu}{\nu})^n e^{\zeta+\eta}, \\ \delta^{(n)} &= \mu^n e^{\zeta}, \\ \phi^{(n)} &= (-\nu)^n e^{\eta}, \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} \zeta &= \frac{1}{\mu}x_{-1} + \mu x_1 + \mu^2 x_2, \\ \eta &= \frac{1}{\nu}x_{-1} + \nu x_1 + \nu^2 x_2. \end{aligned} \quad (3.8)$$

These functions can be readily confirmed to comply with both differential and difference relations

$$\begin{aligned} m_{x_1}^{(n)} &= \delta^{(n)}\phi^{(n)}, \\ m_{x_2}^{(n)} &= \delta^{(n+1)}\phi^{(n)} + \delta^{(n)}\phi^{(n-1)}, \\ m_{x_{-1}}^{(n)} &= -\delta^{(n-1)}\phi^{(n+1)}, \\ m^{(n+1)} &= m^{(n)} + \delta^{(n)}\phi^{(n+1)}, \\ \delta_{x_i}^{(n)} &= \delta^{(n+i)}, \phi_{x_i}^{(n)} = -\phi^{(n-i)}, (i=1,2,-1). \end{aligned} \quad (3.9)$$

Therefore, be defining

$$\begin{aligned} m_{kl}^{(n)} &= A_k B_l m^{(n)}, \\ \delta_k^{(n)} &= A_k \delta^{(n)}, \\ \phi_l^{(n)} &= B_l \phi^{(n)}, \end{aligned} \quad (3.10)$$

where A_k, B_l are differential operators with respect to μ and ν respectively as

$$\begin{aligned} A_k &= \sum_{i=0}^{k} \frac{a_i}{(k-i)!}[f_1(\mu)\partial_\mu]^{k-i}, \\ B_l &= \sum_{j=0}^{l} \frac{b_j}{(l-j)!}[f_2(\nu)\partial_\nu]^{l-j}, \end{aligned}$$

$f_1(\mu)$ and $f_2(\nu)$ are the arbitrary functions of μ and ν, a_i, b_j are the complex constants, thus these $m_{kl}^{(n)}, \delta_k^{(n)}, \phi_l^{(n)}$ satisfy the differential and difference relations since A_k, B_l commute with differentials ∂_{x_k}. Lemma 1 in Ref. [13] shows that for an arbitrary indices sequence $(k_1, k_2, \ldots, k_N; l_1, l_2, \ldots, l_N)$, the determinant

$$\tau_n = \det_{1\le u,v\le N}(m_{k_v,l_u}^{(n)}). \quad (3.11)$$

conforms to to the higher-dimensional bilinear equation.

The aforementioned results represent a comprehensive and highly adaptable category of algebraic solutions to the bilinear equation, characterized by a significant degree of freedom. Nevertheless, only a select few of these solutions can simultaneously meet both the dimension reduction and reality conditions. In the next step, we add constraints to the solutions to force them to satisfy the reality condition and the dimension reduction condition.

We commence with a general dimension reduction condition

$$\tau_{n,x_1} + \beta\tau_{n,x_{-1}} = C_1\tau_n, \quad (3.12)$$

where β and C_1 are undetermined constants. To compute the left-hand side of Eq.(3.12), we observe from the defining relations in Eq.(3.7) and (3.10) of $m^{(n)}$ and $m_{kl}^{(n)}$ that

$$m_{kl,x_1}^{(n)} + \beta m_{kl,x_{-1}}^{(n)} = A_k B_l (Q_1(\mu) + Q_2(\nu)) m^{(n)}, \quad (3.13)$$

where

$$Q_1(\mu) = \mu + \frac{\beta}{\mu} \qquad Q_2(\nu) = \nu + \frac{\beta}{\nu}. \quad (3.14)$$

Moving forward, we propose the adoption of novel variables W_1 and W_2 through

$$\begin{aligned} Q_1(\mu) &= W_1(\mu) + \frac{1}{W_1(\mu)}, \\ Q_2(\nu) &= W_2(\nu) + \frac{1}{W_2(\nu)}. \end{aligned} \quad (3.15)$$

With respect to these novel variables, we redefine functions $f_1(\mu), f_2(\nu)$ in differential operators A_k, B_l are

$$f_1(\mu) = \frac{W_1(\mu)}{W_1'(\mu)}, \quad f_2(\nu) = \frac{W_2(\nu)}{W_2'(\nu)}, \quad (3.16)$$

under the definition

$$f_1(\mu)\partial_\mu = \frac{W_1(\mu)}{W_1'(\mu)}\partial_\mu = W_1(\mu)\partial_{W_1(\mu)}, \quad (3.17)$$

thus

$$\begin{aligned} & A_k Q_1(\mu) m^{(n)} \\ &= (\sum_{i=0}^{k} \frac{a_i}{(k-i)!}(W_1\partial_{W_1})^{k-i})(W_1 + \frac{1}{W_1})m^{(n)} \\ &= \sum_{i=0}^{k} \frac{1}{i!}(W_1(\mu) + (-1)^i \frac{1}{W_1(\mu)})A_{k-i}m^{(n)}, \end{aligned} \quad (3.18)$$

$$B_l Q_2(\nu) m^{(n)} = \sum_{j=0}^{l} \frac{1}{j!}(W_2(\nu) + (-1)^j \frac{1}{W_2(\nu)}) B_{l-j} m^{(n)}, \quad (3.19)$$

$$m_{kl,x_1}^{(n)} + \beta m_{kl,x_{-1}}^{(n)} = \sum_{i=0}^{k} \frac{1}{i!}(W_1(\mu) + (-1)^i \frac{1}{W_1(\mu)}) m_{(k-i)l}^{n} + \sum_{j=0}^{l} \frac{1}{j!}(W_2(\nu) + (-1)^j \frac{1}{W_2(\nu)}) m_{k(l-j)}^{n}. \quad (3.20)$$

The selected values for μ and ν are represented as μ_0 and ν_0. Given that the treatment for both μ_0 and ν_0 is identical, we will focus solely on the consideration of μ_0 in subsequent discussions. Let $W_1(\mu_0) = 1$, ensuring that the odd-i terms in the aforementioned summation are effectively eliminated. Then $Q_1(\mu_0) = 2$. Differentiating $Q_1(\mu) = W_1(\mu) + \frac{1}{W_1(\mu)}$ with respect of μ, we get

$$W_1'(\mu) = \frac{Q_1'(\mu)}{1 - W_1^{-2}(\mu)}. \quad (3.21)$$

From the constraints $Q_1(\mu_0) = 2$, $Q_1'(\mu_0) = 0$, we can derive $\mu_0 = 1, \beta = 1$, An analogous treatment of ν_0 lead to $\nu_0 = 1$.

Given the aforementioned selections for μ and ν, Eq.(3.20) can be simplified to

$$m_{kl,x_1}^{(n)} + m_{kl,x_{-1}}^{(n)}|_{\mu=1,\nu=1} = 2 \sum_{i=0,i:even}^{k} \frac{1}{i!} m_{(k-i)l}^{(n)}|_{\mu=1,\nu=1} + 2 \sum_{j=0,j:even}^{l} \frac{1}{j!} m_{k(l-j)}^{(n)}|_{\mu=1,\nu=1}, \quad (3.22)$$

This key identity demonstrates that at the selected (μ, ν) values, $m_{kl,x_1}^{(n)} + m_{kl,x_{-1}}^{(n)}$ can be expressed as linear combinations of $m_{kl}^{(n)}$ and other $m_{\hat{k}\hat{l}}^{(n)}$ terms of lower row/column indices with differences of 2. Under the structural relation, the determinant (3.11) with index selections $(k_1, k_2, \ldots, k_N; l_1, l_2, \ldots, l_N)$ satisfies

$$\tau_n = \det_{1 \le k,l \le N} (m_{(2k-1)(2l-1)}^{(n)}), \quad (3.23)$$

following the computational methodology presented in Ref. [13], it would demonstrate that the function τ_n adheres to the dimension reduction property

$$\tau_{n,x_1} + \tau_{n,x_{-1}} = 4N\tau_n. \quad (3.24)$$

Upon substituting the dimension reduction condition into the higher-dimensional bilinear equation (3.6), we obtain

$$\begin{cases} (D_{x_1}^2 + 2)\tau_n \cdot \tau_n + 2\tau_{n+1}\tau_{n-1} = 0, \\ (D_{x_1}^2 - D_{x_2})\tau_{n+1} \cdot \tau_n = 0. \end{cases} \quad (3.25)$$

Now, we aim to get more explicit representations of $f_1(\mu)$ and $f_2(\nu)$ as shown in Eq.(3.16). Form Eq.(3.15), we can infer that

$$(W_1(\mu) - \frac{1}{W_1(\mu)})^2 = Q_1^2(\mu) - 4. \quad (3.26)$$

Differentiating the first equation in (3.15) with respect to μ, we derive

$$\frac{W_1'(\mu)}{W_1(\mu)}(W_1(\mu) - \frac{1}{W_1(\mu)}) = Q_1'(\mu). \quad (3.27)$$

By applying these two equations, as well as the definition of $f_1(\mu)$, we obtain

$$f_1(\mu) = \mu.$$

In a similar computational approach, the function $f_2(\nu)$ can also be get as follows.

$$f_2(\nu) = \nu.$$

Then, we define

$$x_1 = x, \quad x_2 = -it, \quad (3.28)$$

and let $x_{-1} = 0$. In addition, the complexity of conjugacy and regularity of solutions are attributable to Ref. [13]. The NLS Equation Eq.(1.2) with boundary conditions (1.3), accepts rational and nonsingular rogue wave solutions of the $N-$th order

$$q = \frac{\tau_1}{\tau_0}, \quad (3.29)$$

where

$$\tau_n = \det_{1 \le k,l \le N} (m_{(2k-1)(2l-1)}^{n}), \quad (3.30)$$

the elements of the matrix are determined by

$$m_{kl}^{n} = \sum_{i=0}^{k} \sum_{j=0}^{l} \frac{a_i}{(k-i)!} \frac{a_j^*}{(l-j)!} (\mu\partial\mu)^{k-i} (\nu\partial\nu)^{l-j} \times (\frac{1}{\mu+\nu}(-\frac{\mu}{\nu})^n e^{(\mu+\nu)x - (\mu^2-\nu^2)it})|_{\mu=\nu=1}. \quad (3.31)$$

In light of the preceding results, Theorem 1 has been established.

Subsequently, we develop a rigorous demonstration of Theorem 2 through careful simplification of the rogue wave solutions. We develop the generator G for the differential operators $(\mu\partial_\mu)^i(\nu\partial_\nu)^j$ as follows.

$$G = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\varepsilon^i}{i!} \frac{\xi^j}{j!} (\mu\partial_\mu)^i (\nu\partial_\nu)^j, \quad (3.32)$$

and utilizing Eq.(3.17), we get

$$G = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\varepsilon^i}{i!} \frac{\xi^j}{j!} (\partial_{\ln W_1})^i (\partial_{\ln W_2})^j = exp(\varepsilon\partial_{\ln W_1} + \xi\partial_{\ln W_2}). \quad (3.33)$$

Therefore, for any function $F(W_1, W_2)$, the following holds true[27]

$$GF(W_1, W_2) = F(e^{\varepsilon}W_1, e^{\xi}W_2). \tag{3.34}$$

Then, we implement the generator on $m^{(0)}$. Following the dimensional reduction ($x_{-1} = 0$) and the establishment of variable relations(3.28), the term $m^{(0)}$ is simplified from its initial form in (3.7) to

$$m^{(0)} = \frac{1}{\mu+\nu} e^{(\mu+\nu)x-(\mu^2-\nu^2)it}. \tag{3.35}$$

For effective application of Eq.(3.34), we reformulate the parameters μ and ν in $m^{(0)}$ as functional combinations of W_1 and W_2 as follows.

$$\begin{aligned} \mu + \frac{1}{\mu} &= W_1 + \frac{1}{W_1}, \\ \nu + \frac{1}{\nu} &= W_2 + \frac{1}{W_2}, \end{aligned} \tag{3.36}$$

then, we derive

$$\begin{aligned} \mu(W_1) &= W_1, \\ \nu(W_2) &= W_2. \end{aligned} \tag{3.37}$$

Now, implementing Eq.(3.34) on the function $m^{(0)}$ yields

$$\begin{aligned} Gm^{(0)} = &\frac{1}{\mu(e^{\varepsilon}W_1) + \nu(e^{\xi}W_2)} \exp((\mu(e^{\varepsilon}W_1) + \nu(e^{\xi}W_2))x \\ &- (\mu^2(e^{\varepsilon}W_1) - \nu^2(e^{\xi}W_2))it). \end{aligned} \tag{3.38}$$

Due to our earlier results $\mu = \nu = 1$, we have $W_1 = W_2 = 1$, thus

$$\begin{aligned} \frac{1}{m^{(0)}} Gm^{(0)} = &\frac{2}{\mu(e^{\varepsilon}) + \nu(e^{\xi})} \exp((\mu(e^{\varepsilon}) + \nu(e^{\xi}) - 2)x \\ &- (\mu^2(e^{\varepsilon}) - \nu^2(e^{\xi}))it). \end{aligned} \tag{3.39}$$

Furthermore, we develop the right-hand side as a bivariate Taylor series in the parameters ε and ξ. To expand the fraction in front of the exponential term, we observe $f(\varepsilon)$ and $g(\xi)$,

$$\begin{aligned} \frac{2}{f(\varepsilon)+g(\xi)} = &\exp(-\ln\frac{(f(\varepsilon)+g(0))(g(\xi)+f(0))}{2(f(0)+g(0))}) \\ &\times \sum_{v=0}^{\infty} (\frac{f(\varepsilon)-f(0)}{f(\varepsilon)+g(0)} \frac{g(\xi)-g(0)}{g(\xi)+f(0)})^v. \end{aligned} \tag{3.40}$$

Therefore, when we substitute

$$f(\varepsilon) = \mu(e^{\varepsilon}) = e^{\varepsilon}, g(\xi) = \nu(e^{\xi}) = e^{\xi}$$

into the equation above, we obtain the following result,

$$\begin{aligned} &\exp(-\ln\frac{(f(\varepsilon)+g(0))(g(\xi)+f(0))}{2(f(0)+g(0))}) \\ &= \exp(-\ln\frac{e^{\varepsilon}+1}{2} - \ln\frac{e^{\xi}+1}{2}) \\ &= \exp(-\ln(e^{\frac{\varepsilon}{2}}\cosh\frac{\varepsilon}{2}) - \ln(e^{\frac{\xi}{2}}\cosh\frac{\xi}{2})) \\ &= \exp(\sum_{k=0}^{\infty} \gamma_k(\varepsilon^k + \xi^k)), \end{aligned} \tag{3.41}$$

where $-\ln(e^{\frac{\xi}{2}}\cosh\frac{\xi}{2}) = \sum_{k=1}^{\infty} \gamma_k \xi^k$.

$$\begin{aligned} &\sum_{v=0}^{\infty} (\frac{f(\varepsilon)-f(0)}{f(\varepsilon)+g(0)} \frac{g(\xi)-g(0)}{g(\xi)+f(0)})^v \\ &= \sum_{v=0}^{\infty} (\frac{e^{\varepsilon}-1}{e^{\varepsilon}+1} \frac{e^{\xi}-1}{e^{\xi}+1})^v \\ &= \sum_{v=0}^{\infty} (\frac{\varepsilon\xi}{4})^v \exp(v\ln\frac{2\tanh\frac{\varepsilon}{2}}{\varepsilon} + v\ln\frac{2\tanh\frac{\xi}{2}}{\xi}) \\ &= \sum_{v=0}^{\infty} (\frac{\varepsilon\xi}{4})^v \exp(v\sum_{k=0}^{\infty} S_k(\varepsilon^k + \xi^k)), \end{aligned} \tag{3.42}$$

where

$$\ln(\frac{2}{\xi}\tanh\frac{\xi}{2}) = \sum_{k=1}^{\infty} S_k \xi^k.$$

In relation to the exponential term in Eq.(3.39), we find that

$$\begin{aligned} &\exp((\mu(e^{\varepsilon}) + \nu(e^{\xi}) - 2)x - (\mu^2(e^{\varepsilon}) - \nu^2(e^{\xi}))it) \\ &= \exp((e^{\varepsilon} + e^{\xi} - 2)x - (e^{2\varepsilon} - e^{2\xi})it) \\ &= \exp(\sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!}(x - 2^k it) + \sum_{k=1}^{\infty} \frac{\xi^k}{k!}(x + 2^k it)). \end{aligned} \tag{3.43}$$

Combining all these results, Eq.(3.39) is simplified to

$$\begin{aligned} \frac{1}{m^{(0)}} Gm^{(0)} = &\sum_{v=0}^{\infty} (\frac{\varepsilon\xi}{4})^v \exp(\sum_{k=1}^{\infty} (r_k + x_k^+ + vS_k)\varepsilon^k \\ &+ \sum_{k=1}^{\infty} (r_k + x_k^- + vS_k)\xi^k), \end{aligned} \tag{3.44}$$

where $x_k^{\pm} = \frac{x \mp 2^k it}{k!}$. Subsequently, by equating the coefficients of $\varepsilon^k \xi^l$ on both sides, we arrive at the following derivation.

$$\begin{aligned} &\frac{1}{m^{(0)}} \frac{1}{k!l!} (f_1(\mu)\partial_\mu)^k (f_2(\nu)\partial_\nu)^l m^{(0)}|_{\mu=\nu=1} \\ &= \sum_{v=0}^{\min(k,l)} (\frac{1}{4})^v S_{k-v}(x^+ + r_k + vs_k) S_{l-v}(x^- + r_k + vs_k). \end{aligned} \tag{3.45}$$

Thus, we get

$$\begin{aligned} &\frac{1}{m^{(0)}} A_k B_l m^{(0)}|_{\mu=\nu=1} \\ &= \sum_{i=0}^{k} \sum_{j=0}^{l} a_i a_j^* \sum_{v=0}^{\min(k-i,l-j)} [\frac{1}{4^v} S_{k-i-v}(x_i^+ + r_i + vs_i) \cdot \\ &\quad S_{l-j-v}(x_i^- + r_i + vs_i)] \\ &= \sum_{v=0}^{\min(k,l)} \frac{1}{4^v} \sum_{i=0}^{k-v} \sum_{j=0}^{l-v} [a_i a_j^* S_{k-i-v}(x_i^+ + r_i + vs_i) \cdot \\ &\quad S_{l-j-v}(x_i^- + r_i + vs_i)] \\ &= \sum_{v=0}^{\min(k,l)} \Phi_{kv} \Psi_{lv}, \end{aligned} \tag{3.46}$$

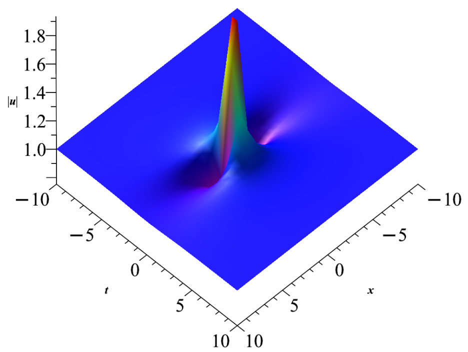

Figure 1: The three-dimensional structure when $a_1 = 0$.

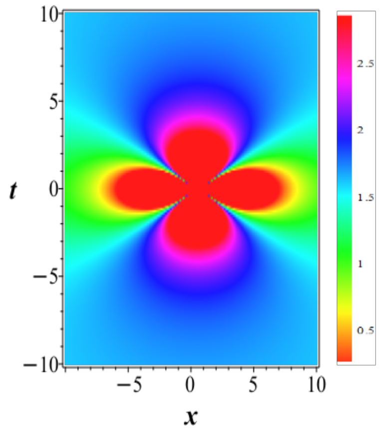

Figure 2: The density profiles when $a_1 = 0$.

where

$$\Phi_{kv} = (\frac{1}{2})^v \sum_{i=0}^{k-v} a_i S_{k-v-i}((\boldsymbol{x})^+ + \boldsymbol{r} + v\boldsymbol{s}),$$

$$\Psi_{l\nu} = (\frac{1}{2})^v \sum_{j=0}^{l-v} a_j^* S_{l-v-j}((\boldsymbol{x})^- + \boldsymbol{r} + \nu\boldsymbol{s}). \quad (3.47)$$

Theorem 2 is proved.

4 Analysis of rogue waves

In the section, we explore the dynamical properties of the obtained rogue wave solutions. The first-order rogue wave emerges when we set $N = 1$ and $a_1 = 0$ in Theorem 2, yielding

$$u(x,t) = \frac{m_{11}^{(1)}}{m_{11}^{(0)}} = \frac{(x - 2it + \frac{1}{2})(x + 2it - \frac{3}{2}) + \frac{1}{4}}{(x - 2it - \frac{1}{2})(x + 2it - \frac{1}{2}) + \frac{1}{4}}. \quad (4.1)$$

Fig.1 illustrates the three-dimensional structure and Fig.2 shows the density profile of the rogue wave when $a_1 = 0$. The corresponding density profiles are depicted in Fig. 3-4 for values of $a_1 = -5$ and $a_1 = 5$, respectively. It is readily apparent that the symmetry center of the rogue wave shifts as the parameter a_1 varies.

To get the second-order rogue wave, we set $N = 2, a_1 = a_2 = 0$ in Theorem 2, when $a_3 = 1$,

$$u(x,t) = 1 + \frac{\Lambda}{\Delta}, \quad (4.2)$$

where

$$\Lambda = -216x - 144x^2 + 96x^3 - 48x^4 - 1152t^2 + 1152xt^2 - 1152x^2t^2 - 3840t^4 + 24it(128t^4 + 64x^2t^2 + 8x^4 + 12x - 64xt^2 - 16x^3 + 32t^2 - 12) + 144 + 1152ixt - 576it, \quad (4.3)$$

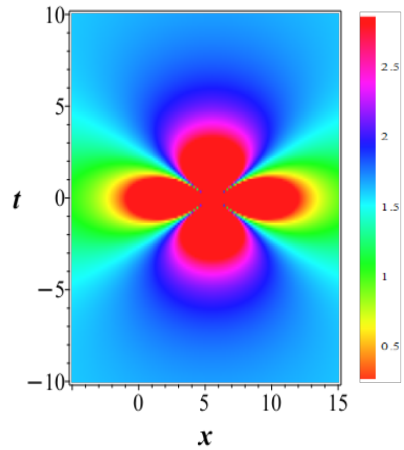

Figure 3: The density profiles when $a_1 = -5$.

$$\Delta = 153 - 36x + 216x^2 - 168x^3 + 72x^4 - 48x^5 + 16x^6 + 96t^2(2x^4 - 4x^3 + 3x + 3) + 384t^4(2x^2 - 2x + 5) + 1024t^6 + 1152xt^2 - 576t^2. \quad (4.4)$$

The second-order rogue wave solution can be similarly captured by varying the parameter a_3. Figures 5-7 illustrate the three-dimensional wave structures corresponding to a_3 values of 1, 2i, and -2i, respectively. In all cases, the solution manifests as three distinct intensity peaks that emerge at different spatial and/or temporal locations. Each peak essentially represents a first-order Peregrine rogue wave. The results presented above can be extrapolated to encompass higher order rogue waves.

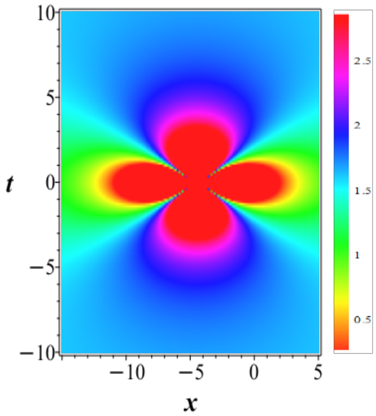

Figure 4: The density profiles when $a_1 = 5$.

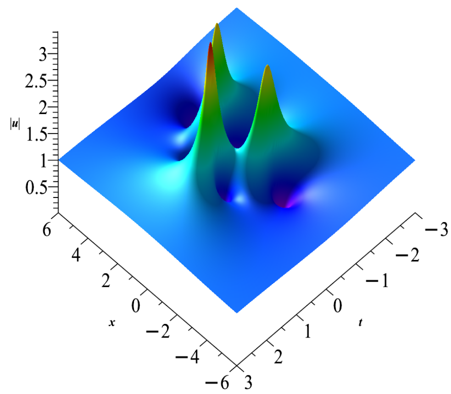

Figure 6: The three-dimensional structure when $a_3 = 2i$.

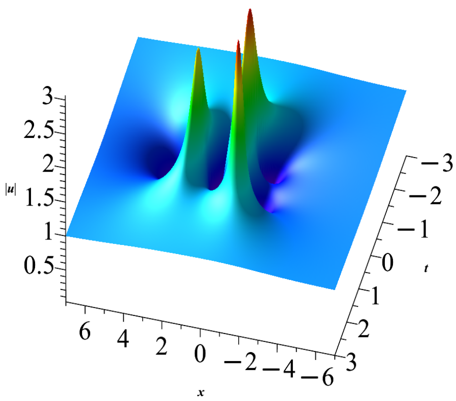

Figure 5: The three-dimensional structure when $a_3 = 1$.

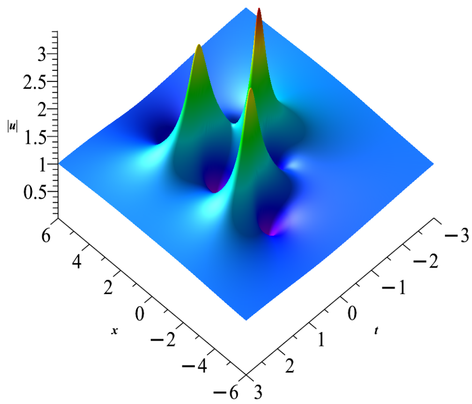

Figure 7: The three-dimensional structure when $a_3 = -2i$.

Through special selection of the free parameters, a_k, it is possible to generate more complex spatio-temporal patterns. However, due to spatial limitations, we have not included this in our current discussion.

5 Conclusions

In this research, we explore the intriguing phenomenon of rogue waves, utilizing the advanced mathematical methodologies of the bilinear KP reduction approach and the W-technique. In the course of our research, we have broadened Yang et al.'s framework [13] by generalizing the matrix elements m_{ij}^n. The original parameters μ and ν were substituted with universal functions $f(\mu)$ and $f(\nu)$ of μ and ν, effectively widening the scope of application. Our study provides not only explicit analytical expressions for the rogue waves but also insight into their inherent structural characteristics. These findings elucidate the rogue wave dynamics in nonlinear dispersive systems, potentially enhancing wave prediction and mitigation techniques. Using this method, solutions for other types of nonlinear integrable systems remain to be explored. The solutions for other types of nonlinear integrable systems using the method presented in this work remain to be explored.

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