Discrete Fractional Alpha Mixed Difference and Its Sums

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***Abstract*—This paper investigates higher-order mixed alpha operators within the framework of finite difference equations. We developed theorems and corollaries for integer order (m^{th} order) anti-difference principle using the $(q,h)_\alpha$ difference operator. Furthermore, we have obtained theorems, supported by appropriate examples. We lastly examine a GDP model that is controlled by a fractional differential equation that incorporates the mixed alpha difference operator.**

***Index Terms*—Closed form, Summation form, Discrete integration, Discrete Delta operator, Mixed alpha difference operator, Fractional calculus, Fractional sum.**

I. Introduction

DIFFERENCE equations are meant for discrete process where as the differential equations deals with continuous system. The discrete case methodology to the continuous situation in order to identify closed-form of fractional order continuous and discrete integration is discussed in [1]. In [3], the author's goal is to determine the relationship between the two fractional operators as well as q-analogue of the fractional calculus, which are divisible as q-integrals and q-derivatives. [4] The author of this paper examines the application of a forward hybrid delta operator with shift value to obtain a generalized infinite series of the fractional hybrid summation formula and the numerical closed form solution of the fractional order hybrid difference equation. The authors in [5] define q-difference operator Δ_q and give some important findings on the inverse of the $n^{th}-$order q-difference operator using the second-kind Stirling number and the extended polynomial factorial. Authors in [6] presents an equation for a three-dimensional q-difference operator. Additionally, they used the three-dimensional q-difference operator to deduce other theorems. The authors introduce the operator and its inverse in [7], from which the sequence and series of x-Fibonacci, together along with a number of findings and theorems are obtained.

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Since the authors in [8] were aware of the fundamental general characteristics of linear difference equation solutions and because each of these normal forms makes the successive approximation approach easily applicable, they have chosen to use the transforming substitution. The q-symmetric variational calculus was introduced in [9] which offers a fresh perspective on the study of quantum calculus. For symmetric quantum variational problems, the researchers established a sufficient optimality condition as well as a required Euler-Lagrange type optimality condition. In order to facilitate the explicit solution of discrete equations using fractional difference operators on the left and right, the authors in [10] proposed the contemporary theory of fractional h-difference equations, the discrete case methodology to the continuous situation in order to identify closed-form of fractional order continuous and discrete integration which is enhanced with practical tools. Also authors in [2], [12] developed ℓ-nabla integration of f and discrete fractional integration for factorials and geometric functions.

[11]Through the lens of discrete fractional calculus, the author explains monotonicity and convexity. Additionally, the nonlocal nature of the fractional difference produces unexpected results regarding monotonicity and convexity in an introductory calculus course. Moreover, the fractional difference's nonlocal character opens up intriguing possibilities for biological modeling.

In [13], the researcher yields an extremely intriguing q-constant. The fractional difference is defined in a new way in [14]. Based on this concept, other qualities were defined, such as the significant Leibniz rule and a broad exponential law. Equations involving second-order linear differences are then solved using those findings. [15] This work presents a class of difference equations that can be solved in closed form. The Asymptotic actions of their answers in a specific example is studied using the obtained formulae for the solutions.

[16] The author distinguishes between two fundamentally distinct scenarios based on whether or not all of the characteristic equation's roots are finite and distinct from zero. The authors in [17] studied the derivation of elementary difference operators (factorial polynomials) and difference equations utilizing the operators Δ and E. In [18], the researcher used the Generalized infinite series of fractional Fibonacci summation formula and the fractional order Fibonacci difference equation which can be solved numerically and in closed form using the forward Fibonacci delta operator with many parameters and its inverse on real valued functions. The goal of article [19] is to examine whether complete function solutions with entire function

coefficients exist for linear n-th order homogeneous and non-homogeneous q-difference equations. If such solutions do exist, then the characteristics of those solutions will be examined in relation to the relationship between the order apparent of the solutions and the coefficients. The author in [20] is responsible for developing the analytic theory of a q-difference system as well as the analytic theory in the case where there are no restrictions on the characteristic equation's roots. The golden section and the connections between the Fibonacci and Lucas numbers are introduced by the author [21]. In [22], the author uses difference operators to generate functions and approximate summations with appropriate examples. Authors in [23] discussed about a fundamental study of non-state-dependent differential-difference equations for simulating the dynamics of economic growth, including GDP evolution, was presented by Benhabib and Rustichini (1991). In order to better understand long-term GDP growth patterns, numerical methods have been developed in response to subsequent studies that have highlighted the complexity of solving state-dependent models analytically, particularly in vintage capital frameworks. [24] Fractional differential equations with generalized derivatives have been studied recently for modeling complicated dynamical systems, such as GDP and population increase. Existence, uniqueness, and numerical solutions have been successfully established using strategies like fixed point theorems and iterative approaches like Picard iteration.

II. Preliminaries

The literature review study of this research is mostly focused on [3]. The basic concepts of factorial polynomials, the fractional sum of order $\nu > 0$, difference and anti-difference delta operators, and First-order principle of anti-difference, each of which will be used in the subsequent chapters are provided in this section.
The authors in [3] used the following definitions: For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, The polynomial function of Factorial $t^{(n)}$ is defined by

$$t^{(n)} = \prod_{r=0}^{n-1} (t-r). \tag{1}$$

Also, for $\nu \in (-\infty, \infty)$, the ν^{th} factorial polynomial is defined by $t^{(\nu)} = \frac{\Gamma(1+t)}{\Gamma(1+t-\nu)}$, $1+t-\nu$ and

$$1 + t \notin -\mathbb{N}_0 = \{0, -1, -2, \cdots\}. \tag{2}$$

Defined at s, the ν^{th} fractional Taylor monomial is

$$h_\nu(s,t) = \frac{(t-s)^{(\nu)}}{\Gamma(1+\nu)}, \tag{3}$$

where $(t-s)^{(\nu)}$ is obtained by changing t into $t-s$ in (2).

Definition II.1. [3] Let $q, h, n \in \mathbb{N}$ and $t \in \mathbb{R}$. The (q,h) polynomial function of factorial $k_{q,h}^{(n)}$ is described by

$$t_{q,h}^{(n)} = t\prod_{r=1}^{n-1}(t - (q^r + rh)). \tag{4}$$

Theorem II.2. *[3] Let $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $m, n \in \mathbb{N}$, $t \in \mathbb{R}$ and then the higher order anti-difference principle of (q,h) difference operator is given by*

$$\underset{(q,h)}{\overset{-m}{\Delta}} x(t) - \sum_{d=0}^{m-1} \frac{n^{(d)}}{d!} \underset{(q,h)}{\overset{-(m-d)}{\Delta}} x\Big(\big(t - h\sum_{j=0}^{n-1} q^j\big)\Big/q^n\Big)$$
$$= \sum_{r=m-1}^{n-1} \frac{r^{(m-1)}}{(m-1)!} x\Big(\big(t - h\sum_{s=0}^{r} q^s\big)\Big/q^{r+1}\Big). \tag{5}$$

Theorem II.3. *[3] Let $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $h \in \mathbb{R} - \{0\}$, $q \in \mathbb{R} - \{0, 1\}$, $t \in \mathbb{N}$ and $s, \nu \in \mathbb{R}$. Then the ν^{th} order anti-difference principle of (q,h) difference equation in terms of $x_t(\nu, s, q/h)$ is obtained by*

$$\underset{(q,h)}{\overset{-\nu}{\Delta}} x(s) - \frac{[x_{t+1}(\nu, s, q/h)]^2}{x_{t+1}(\nu, s, q/h) - x_{t+2}(\nu, s, q/h)} = x_t(\nu, s, q/h). \tag{6}$$

Theorem II.4. *[3] Let $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $s, \alpha \in \mathbb{R}$, $m, t \in \mathbb{N}$ Then the higher order of $(q,h)_\alpha$ difference operator is given by*

$$\underset{(q,h)_\alpha}{\overset{-m}{\Delta}} x(s) - \frac{[x_{t+1}(m, s, (q/h)_\alpha)]^2}{x_{t+1}(m, s, (q/h)_\alpha) - x_{t+2}(m, s, (q/h)_\alpha)}$$
$$= \sum_{r=0}^{t} x_r(m, s, (q/h)_\alpha). \tag{7}$$

Definition II.5. [3] Let $x, y : \mathscr{M}_h^q \to \mathbb{R}$ be a function and $\alpha \in \mathbb{R}$. Then the difference operator $(q,h)_\alpha$ is defined as

$$\underset{(q,h)_\alpha}{\Delta} x(t) = (tq+h)x - \alpha x(t), \ t \in \mathbb{R}. \tag{8}$$

III. Anti-Difference Principle of Alpha Mixed Operator

In this, we develop some theorems and corallaries for integer order summation formula using the $(q,h)_\alpha$ mixed alpha difference operator.

Theorem III.1. *Let $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $k \in \mathbb{R}$, $n \in \mathbb{N}$, and $q > 1$, $h > 0$ then the first order anti-difference principle of $(q,h)_\alpha$ mixed alpha difference operator is given by* $\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} x(t) - \underset{(q,h)_\alpha}{\overset{-1}{\Delta}} \alpha^n x\Big(\big(t - h\sum_{j=0}^{n-1} q^j\big)\Big/q^n\Big)$

$$= \sum_{r=0}^{n-1} \alpha^r x\Big(\big(t - h\sum_{s=0}^{r} q^s\big)\Big/q^{r+1}\Big). \tag{9}$$

Proof: *The proof follows from the definition*

$$x(t) = y(tq+h) - y(t) \tag{10}$$

and

$$y(t) = x((t-h)/q) + x\big((t - h\sum_{r=0}^{1} q^r)/q^2\big)$$
$$+ x\big((t - h\sum_{r=0}^{2} q^r)/q^3\big) + x\big((t - h\sum_{r=0}^{3} q^r)/q^4\big)$$
$$+ x\big((t - h\sum_{r=0}^{4} q^r)/q^5\big) + \ldots + x\big((t - h\sum_{r=0}^{n-1} q^r)/q^n\big) \tag{11}$$

From the relations (10) *and* (11)*, we get*

$$y(t) = x((t-h)/q) + \alpha x\big((t - h\sum_{r=0}^{1} q^r)/q^2\big)$$

$$+\alpha^2 x\big((t - h\sum_{r=0}^{2} q^r)/q^3\big) + \alpha^3 x\big((t - h\sum_{r=0}^{3} q^r)/q^4\big)$$

$$+\alpha^4 x\big((t - h\sum_{r=0}^{4} q^r)/q^5\big) + \ldots + \alpha^{n-1} x\big((t - h\sum_{r=0}^{n-1} q^r)/q^n\big)$$

$$+ \alpha^n g\big((t - h\sum_{r=0}^{n-1} q^r)/q^n\big). \quad (12)$$

Since $\overset{-1}{\underset{(q,h)_\alpha}{\Delta}} x(t) = y(t)$*, then* (12) *becomes*
$\overset{-1}{\underset{(q,h)_\alpha}{\Delta}} x(t) - \alpha^n \overset{-1}{\underset{(q,h)_\alpha}{\Delta}} x\big((t - h\sum_{r=0}^{n-1} q^r)/q^n\big)$

$$= x((t-h)/q) + \alpha x\big((t - h\sum_{r=0}^{1} q^r)/q^2\big)$$

$$+\alpha^2 x\big((t - h\sum_{r=0}^{2} q^r)/q^3\big) + \ldots$$

$$+\alpha^{n-1} x\big((t - h\sum_{r=0}^{n-1} q^r)/q^n\big)$$

which completes the proof.

Corollary III.2. *Let* $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $q \in \mathbb{R} - \{0, 1\}$, $n \in \mathbb{N}$, $t, \alpha \in \mathbb{R}$,*and* $q > 1 > 0$ *then the first order anti-difference principle of* $(q, h)_\alpha$ *alpha q-difference operator is given by*

$$\overset{-1}{\underset{(q,0)_\alpha}{\Delta}} x(t) - \overset{-1}{\underset{(q,0)_\alpha}{\Delta}} \alpha^n x(t/q^n) = \sum_{r=0}^{n-1} \alpha^r x(t/q^{r+1}). \quad (13)$$

Proof: *The Proof follows by Substituting* $h = 0$ *in the equation* (9) *and we get equation* (13)*.*

Corollary III.3. *Let* $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $h \in \mathbb{R} - \{0\}$, $n \in \mathbb{N}$, $t, \alpha \in \mathbb{R}$, *and* $q > 1 > 0$ *then the first order anti-difference principle of* $(q, h)_\alpha$ *alpha h-difference operator is given by*

$$\overset{-1}{\underset{(1,h)_\alpha}{\Delta}} x(t) - \overset{-1}{\underset{(1,h)_\alpha}{\Delta}} \alpha^n x(t - nh) = \sum_{r=0}^{n-1} \alpha^r x(t - (r+1)h). \quad (14)$$

Proof: *The Proof follows by Substituting* $q = 1$ *in the equation* (9) *and we get equation* (14)*.*

Remark III.4. The operators $\overset{-1}{\underset{(q,0)_\alpha}{\Delta}}$ and $\overset{-1}{\underset{(1,h)_\alpha}{\Delta}}$ are denoted as $I_{q(\alpha)}$ and $\Delta^{-1}_{h(\alpha)}$ operators.

Remark III.5. The operators $\overset{-1}{\underset{(q,0)_\alpha}{\Delta}}$ and $\overset{-1}{\underset{(1,h)_\alpha}{\Delta}}$ are the first order $q_{(\alpha)}$ and $h(\alpha)$ difference operators respectively. That is, $\overset{-1}{\underset{(q,0)_\alpha}{\Delta}} = \overset{-1}{\underset{q_{(\alpha)}}{\Delta}}$ and $\overset{-1}{\underset{(1,h)_\alpha}{\Delta}} = \overset{-1}{\underset{h(\alpha)}{\Delta}}$.

IV. Higher Order Alpha Mixed Finite Difference Equations

In this section, we developed theorems and corollaries for integer order (m^{th} order) anti-difference principle using the $(q, h)_\alpha$ difference operator.

Theorem IV.1. *Let* $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $t \in \mathbb{R}$, $n, m \in \mathbb{N}$ *Then the higher order anti-difference principle of* $(q, h)_\alpha$ *mixed alpha difference operator is given by*

$$\overset{-m}{\underset{(q,h)_\alpha}{\Delta}} x(t) - \sum_{d=0}^{m-1} \frac{n^{(d)}}{d!} \alpha^{n-d} \overset{-(m-d)}{\underset{(q,h)_\alpha}{\Delta}} x\Big((t - h\sum_{j=0}^{n-1} q^j)/q^n\Big)$$

$$= \sum_{r=m-1}^{n-1} \frac{r^{(m-1)}}{(m-1)!} \alpha^{r-(m-1)} x\Big((t - h\sum_{s=0}^{r} q^s)/q^{r+m}\Big). \quad (15)$$

Proof: *The similar proof in Theorem II.2 is applying in the* $\overset{-1}{\underset{(q,h)_\alpha}{\Delta}}$ *operator repeatedly on both sides of equation* (9)*.*

Corollary IV.2. *Let* $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $t, \alpha \in \mathbb{R}$, $q \in \mathbb{R} - \{0\}$, $m, n \in \mathbb{N}$ *Then the higher order anti-difference principle of* $(q, h)_\alpha$ *mixed alpha difference operator is given by*
$\overset{-m}{\underset{(q,0)_\alpha}{\Delta}} x(t) - \sum_{r=0}^{x-1} \frac{n^{(d)}}{d!} \alpha^{n-d} \overset{-(m-d)}{\underset{(q,0)_\alpha}{\Delta}} x(t/q^n)$

$$= \sum_{r=m-1}^{n-1} \frac{r^{(m-1)}}{(m-1)!} \alpha^{r-(m-1)} x(t/q^{r+1}). \quad (16)$$

Proof: *The Proof follows by substituting* $h = 0$ *in the equation* (15) *then we have* (16)*.*

Corollary IV.3. *Let* $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $t, \alpha \in \mathbb{R}$, $h \in \mathbb{R} - \{0\}$, $m, n \in \mathbb{N}$ *Then the higher order anti-difference principle of* $(q, h)_\alpha$ *mixed alpha difference operator is given by*
$\overset{-x}{\underset{(1,h)_\alpha}{\Delta}} x(t) - \sum_{d=0}^{m-1} \frac{n^{(d)}}{d!} \alpha^{n-d} \overset{-(m-d)}{\underset{(1,h)_\alpha}{\Delta}} x(t - nh)$

$$= \sum_{r=m-1}^{n-1} \frac{r^{(m-1)}}{(m-1)!} \alpha^{r-(m-1)} x(t - (r+1)h). \quad (17)$$

Proof *The Proof follows by substituting* $q = 1$ *in the equation* (15) *then we get* (17)*.*

Corollary IV.4. *Consider the conditions given in Theorem IV.1. Then the m-th order of* $(q, h)_\alpha$ *mixed alpha difference equation is given by*

$$\overset{-m}{\underset{(q,h)_\alpha}{\Delta}} x(t) - \sum_{r=n-m}^{n-1} \frac{n^{(r-n+m)} \alpha^{n-r}}{(r-n+m)!} \overset{-(n-r)}{\underset{(q,h)_\alpha}{\Delta}}$$

$$x\Big(\big(t - h\sum_{j=0}^{n-1} q^j\big)\Big/q^n\Big)$$

$$= \sum_{r=0}^{n-m} \frac{(m+r-1)^{(m-1)}}{(m-1)!} \alpha^{r-(m-1)}$$

$$x\Big(\big(t - h\sum_{s=0}^{m+r-1} q^s\big)\Big/q^{m+r}\Big). \quad (18)$$

Proof: *The proof is finished by substitution*
$\sum_{d=0}^{m-1} (n^{(d)}/d!) \alpha^{n-d} \overset{-(m-d)}{\underset{(q,h)_\alpha}{\Delta}} x\Big(\big(t - h\sum_{j=0}^{n-1} q^j\big)\Big/q^n\Big)$ *by*

$\sum_{r=n-m}^{n-1}(n^{(r-n+m)}/(r-n+m)!)\alpha^{n-r}$

$\underset{(q,h)_\alpha}{\overset{-(n-r)}{\Delta}} x\Big(\big(t-h\sum_{j=0}^{n-1}q^j\big)\Big/q^n\Big)$ *and*

$\sum_{r=m-1}^{n-1}(r^{(m-1)}/(m-1)!)\alpha^{r-(m-1)}$

$x\Big(\big(t-h\sum_{s=0}^{r}q^s\big)\Big/q^{r+1}\Big)$ *by*

$\sum_{r=0}^{n-m}((m+r-1)^{(m-1)}/(m-1)!)\alpha^{r-(m-1)}$

$x\Big(\big(t-h\sum_{s=0}^{m+r-1}q^s\big)\Big/q^{m+r}\Big)$ *in equation* (15).

Theorem IV.5. *Let $x,y:\mathscr{M}_h^q\to\mathbb{R}$, $q\in\mathbb{R}-\{0,1\}$, $h\in\mathbb{R}-\{0\}$, $m\in\mathbb{N}$ and $t,\alpha\in\mathbb{R}$. Then for the infinite series of mth order mixed alpha difference operator principle is,*

$$\underset{(q,h)_\alpha}{\overset{-m}{\Delta}} x(t)=\sum_{r=m-1}^{\infty}\frac{r^{(m-1)}}{(m-1)!}\alpha^{r-(m-1)}$$
$$x\Big(\big(t-h\sum_{j=0}^{r}q^j\big)\Big/q^{r+m}\Big). \qquad (19)$$

Proof: *Taking $\lim_{n\to\infty}$ in* (15) *and consider $\underset{(q,h)_\alpha}{\overset{-m}{\Delta}} x(0)=0$, we arrive* (19).

V. Mixed Alpha Geometric Factorials and its Difference Equations

In this section, we develop the m^{th} order difference equation for $(q,h)_\alpha$ mixed alpha difference operator using the factorial coefficient functions.

Definition V.1. Let $s,q,h,t,\alpha\in\mathbb{R}$, $m\in\mathbb{N}$ such that $s-h\sum_{j=0}^{t}q^j/q^{t+m}\in\mathscr{M}_h^q$ and $x:T_q\to\mathbb{R}$ be a function. Then the factorial-coefficient of x at t on $(m,s,(q/h)_\alpha)$ is defined as

$$x_t(m,s,(q/h)_\alpha)=\alpha^t\frac{(t+m-1)^{(m-1)}}{(m-1)!}$$
$$x\Big(\big(s-h\sum_{j=0}^{t}q^j\big)\Big/q^{t+m}\Big). \qquad (20)$$

Corollary V.2. *Let $x,y:\mathscr{M}_h^q\to\mathbb{R}$, $s,\alpha\in\mathbb{R}$, $t\in\mathbb{N}$, where $h>0$, $q>1$ and $\sum_{r=t+1}^{\infty}\alpha^r x\Big(\big(s-h\sum_{j=0}^{r}q^j\big)\Big/q^{r+1}\Big)$ is convergent. Then*

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} x(s)-\sum_{r=t+1}^{\infty}x_r(1,s,(q/h)_\alpha)=\sum_{r=0}^{t}x_r(1,s,(q/h)_\alpha). \qquad (21)$$

Proof: *Assuming $\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} x(0)=0$ and Taking $\lim_{n\to\infty}$ in equation* (12) *, then*

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} x(t)=x((t-h)/q)+\alpha x\big((t-h\sum_{r=0}^{1}q^r)/q^2\big)$$
$$+\alpha^2x\big((t-h\sum_{r=0}^{2}q^r)/q^3\big)$$
$$+\ldots+\alpha^r x\big((t-h\sum_{p=0}^{r}q^p)/q^{r+1}\big)$$
$$+\alpha^{r+1}x\big((t-h\sum_{p=0}^{r+1}q^p)/q^{r+2}\big)+\cdots \qquad (22)$$

Replacing $'t'$ by $'s'$ and $'r'$ by $'t'$ in (22)*, we obtain*

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} x(s)=x\big((s-h)/q\big)+\alpha x\big((s-h\sum_{r=0}^{1}q^r)/q^2\big)$$
$$+\alpha^2x\big((s-h\sum_{r=0}^{2}q^r)/q^3\big)$$
$$+\ldots+\alpha^t x\big((s-h\sum_{r=0}^{t}q^r)/q^{t+1}\big)$$
$$+\alpha^{t+1}x\big((s-h\sum_{r=0}^{t+1}q^r)/q^{t+2}\big)+\cdots,$$

Hence the proof completes.

Lemma V.3. *Let $s\in\mathbb{R}$, $t\in\mathbb{N}$, $\alpha,h\in\mathbb{R}>0$, $q\in\mathbb{R}-\{0,1\}$ and the infinite series $\sum_{r=t+1}^{\infty}\alpha^r u\Big(\big(s-h\sum_{j=0}^{r}q^j\big)\Big/q^{r+1}\Big)$ is convergent. Then the Alpha (q,h) geometric function is defined as $\sum_{r=t+1}^{\infty}x_r(m,s,(q/h)_\alpha)=$*

$$\frac{[x_{t+1}(m,s,(q/h)_\alpha)]^2}{x_{t+1}(m,s,(q/h)_\alpha)-x_{t+2}(m,s,(q/h)_\alpha)}. \qquad (23)$$

The following Theorem V.4 is the higher order finite series formula for the $(q,h)_\alpha$ difference operator.

Theorem V.4. *Consider the conditions given in Corollary V.2. Then, the first order anti-difference principle of $(q,h)_\alpha$ difference operator is given by*

$$\underset{(q,h)_\alpha}{\overset{-1}{\Delta}} x(s)-\frac{[x_{t+1}(1,s,(q/h)_\alpha)]^2}{x_{t+1}(1,s,(q/h)_\alpha)-x_{t+2}(1,s,(q/h)_\alpha)}$$
$$=\sum_{r=0}^{t}x_r(1,s,(q/h)_\alpha) \qquad (24)$$

Proof: *The proof completes by substituting the equation* (23) *in* (21).

VI. Alpha Mixed Gamma Geometric Factorials in Fractional Order Difference

In this part, we elaborate on the fractional order anti-difference principle, building upon the foundation laid out in the Lemma V.3 for integer order. using this, we are able to establish the fundamental theorems for alpha mixed fractional difference equations.

Definition VI.1. Let $s,q,t,\nu,\alpha\in\mathbb{R}$ such that $\big(s-h\sum_{j=0}^{t}q^j\big)\Big/q^{t+\nu}\in\mathscr{M}_h^q$ and $x:\mathscr{M}_h^q\to\mathbb{R}$ be a function. Then the Gamma factorial-coefficient of x at t on $(\nu,s,(q/h)_\alpha)$ is defined as

$$x_t(\nu,s,(q/h)_\alpha)=\alpha^t\frac{\Gamma(t+\nu)}{\Gamma(t+1)\Gamma(\nu)}x\Big(\big(s-h\sum_{j=0}^{t}q^j\big)\Big/q^{t+\nu}\Big). \qquad (25)$$

Definition VI.2. Let $s, q, t, \nu, \alpha \in \mathbb{R}$ such that $s - h\sum_{j=0}^{t} q^j/q^{t+\nu} \in \mathscr{M}_h^q$ and $x_t(\nu, s, (q/h)_\alpha)$ given in (25). Then the $(q/h)_\alpha$-Geometric factorial function is defined as $\sum_{r=t+1}^{\infty} x_r(\nu, s, (q/h)_\alpha)$

$$= \frac{[x_{t+1}(\nu, s, (q/h)_\alpha)]^2}{x_{t+1}(\nu, s, (q/h)_\alpha) - x_{t+2}(\nu, s, (q/h)_\alpha)}. \tag{26}$$

Theorem VI.3. *Let $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $h \neq 0 \in \mathbb{R}$, $q \in \mathbb{R} - \{0, 1\}$, $t, \nu, \alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. Then the ν^{th} order of $(q, h)_\alpha$ difference equation is given by*

$$\underset{(q,h)_\alpha}{\overset{-\nu}{\Delta}} x(t) - \sum_{d=n-\nu}^{n-1} \frac{\Gamma(n+1)\alpha^{n-d}}{\Gamma(2n-d-\nu+1)\Gamma(d-n+\nu-1)}$$

$$\underset{(q,h)_\alpha}{\overset{-(n-d)}{\Delta}} x\Big(\Big(t - h\sum_{j=0}^{n-1} q^j\Big)\Big/q^n\Big)$$

$$= \sum_{r=0}^{n-\nu} \frac{\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(r+1)}\alpha^{r-(\nu-1)} x\Big(\Big(t - h\sum_{s=0}^{\nu+r-1} q^s\Big)\Big/q^{\nu+r}\Big). \tag{27}$$

Proof: *The proof follows by applying the Corollary IV.4 and by equation (2) using the $(q, h)_\alpha$ difference operator.*

Theorem VI.4. *Let $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $q \in \mathbb{R} - \{0, 1\}$, $h \in \mathbb{R} - \{0\}$, $\alpha, s \in \mathbb{R}$, $t \in \mathbb{N}$ and $\nu \in \mathbb{R}$. Then the ν^{th} order of $(q, h)_\alpha$ difference operator is given by*

$$\underset{(q,h)_\alpha}{\overset{-\nu}{\Delta}} x(s) - \frac{[x_{t+1}(\nu, s, (q/h)_\alpha)]^2}{x_{t+1}(\nu, s, (q/h)_\alpha) - x_{t+2}(\nu, s, (q/h))_\alpha}$$

$$= x_t(\nu, s, (q/h)_\alpha). \tag{28}$$

where $\mathscr{A} = \Gamma(t+\nu+1)/\Gamma(\nu)\Gamma(t+2)$ and $\mathscr{B} = \Gamma(t+\nu+2)/\Gamma(\nu)\Gamma(t+3)$.

Proof: *The proof follows from Theorem II.4, Theorem II.3 and by (2).*

Theorem VI.5. *Let $x, y : \mathscr{M}_h^q \to \mathbb{R}$, $\alpha, \nu, t \in \mathbb{R}$, $q > 0$, $h > 0$, $(\nu + r - 1)/2 \in \mathbb{N}$ and $(\nu + r - 3)/2 \in \mathbb{N}$. Then the ν^{th} of $(q, h)_\alpha$ difference operator for infinite series is given by*

$$\underset{(q,h)_\alpha}{\overset{-\nu}{\Delta}} x(t) = \sum_{r=0}^{\infty} \frac{\Gamma(r+\nu)}{\Gamma(\nu)\Gamma(r+1)}\alpha^{r-\nu+1}$$

$$x\Big(\Big(t - h\sum_{s=0}^{(\nu+r-1)/2} q^{2r}(1+q)\Big)\Big/q^{\nu+r}\Big), \tag{29}$$

and

$$\underset{(q,h)_\alpha}{\overset{-\nu}{\Delta}} x(t) = \sum_{r=0}^{\infty} \frac{\Gamma(r+\nu)}{\Gamma(\nu)\Gamma(r+1)}\alpha^{r-\nu+1}$$

$$x\Big(\Big(t - h\sum_{s=0}^{(\nu+r-3)/2} q^{2r}(1+q) + q^{\nu+r-1}\Big)\Big/q^{\nu+r}\Big). \tag{30}$$

Proof: *The proof completes by generalizing the Theorem IV.5 to any real order ($\nu \in \mathbb{R}$) using (2).*

VII. Results and Discussion

This section covers the value analysis of the difference operators $(q, h)_\alpha$. For instance, by taking the values $s = 8.3$ and $t = 50$ in Theorem VI.4, then Figure: 1, 2 shows that the values of the $(q, h)_\alpha$ difference operator gradually climb and then eventually decline, indicating that it will converge, if the ν and α values increase.

As shown in figures: 1, 2, the values of the (q, h) difference operator are decreasing with time for every $\nu > 0 \in \mathbb{R}$, suggesting that it will converge.

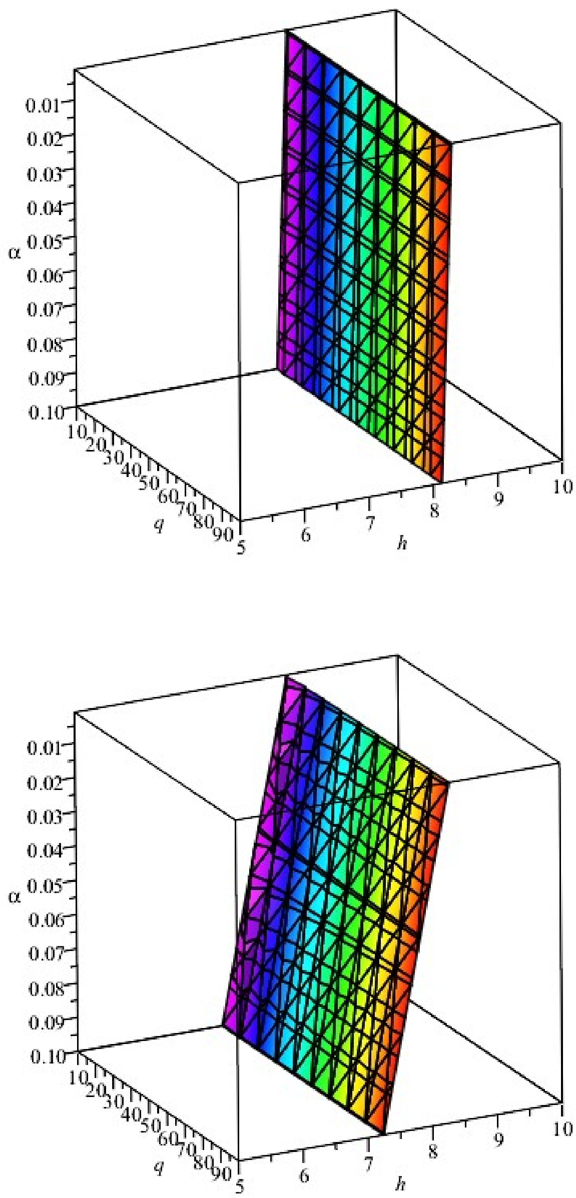

Fig. 1: Solution for Theorem VI.4 with ν values 0.2 and 1.3, where q varies from 10 to 100, h varies from 5 to 10 and α varies from 1×10^{-2} to 0.1.

The general solution from the Theorem VI.4 and Figures: 1, 2 provides the values for q and h for any genuine q. Consequently, the value stability for $(q, h)_\alpha$ operators may be predicted with ease.

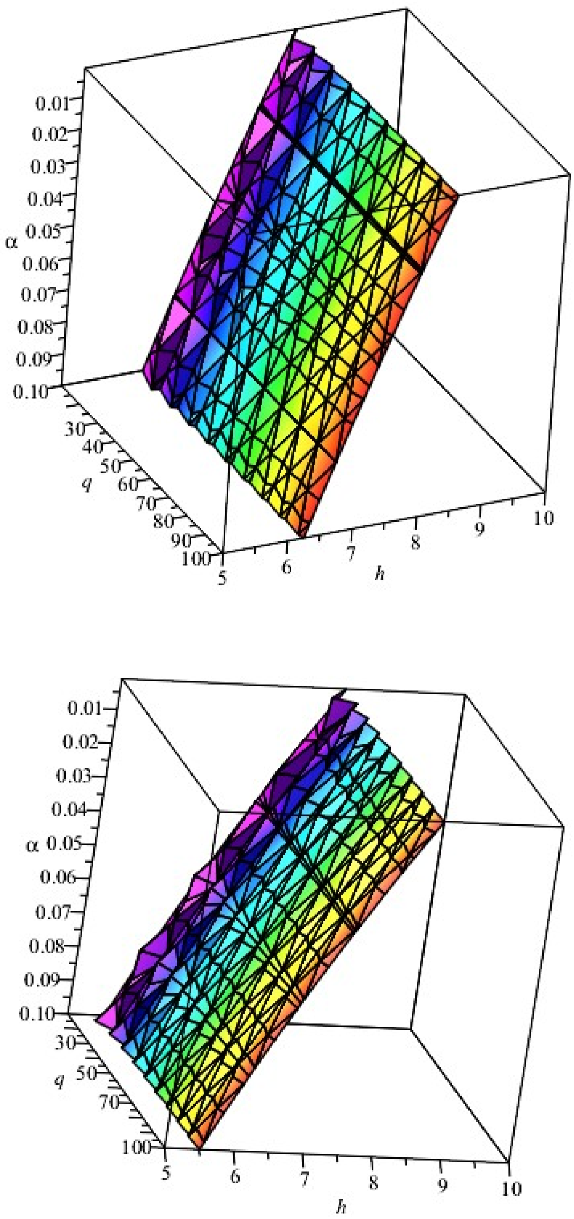

Fig. 2: Solution for Theorem VI.4 with ν values 2.7 and 3.9, where q varies from 10 to 100, h varies from 5 to 10 and α varies from 1×10^{-2} to 0.1.

VIII. GDP Growth Using Mixed Alpha Difference Operator

In this section we discuss about the GDP growth application that uses discrete time increments and a difference equation model to show how a nation's GDP changes over time. This kind of model is often used in economics to simulate and forecast GDP dynamics.

A mixed alpha difference equation-based GDP growth application is a more advanced theoretical framework that is commonly utilized in fractional or memory-dependent dynamic systems. A fractional difference equation, which is a generalization of normal difference equations, is commonly referred to as a "mixed alpha difference equation" because the order of the difference is a fractional value $\alpha \in (0,1)$ rather than a whole integer.

This method adds memory effects to GDP modeling, which makes current GDP more realistic for economic processes by relying on a weighted sum of historical GDP values rather than just the previous value.

In contemporary economic theory, modeling GDP growth with mixed difference operators especially those incorporating fractional (non-integer) order differences has become a potent tool. Mixed or fractional difference equations enable the inclusion of historical memory and long-range dependence in the dynamics of economic growth, in contrast to classic difference equations that assume GDP increases in discrete, memoryless stages.

The late 20th and early 21st centuries saw a rise in the use of fractional calculus in economic modeling. Among the first to explicitly introduce long-memory processes in economic time series analysis were Granger and Joyeux (1980) and Hosking (1981). They showed that persistent temporal dependencies in data, like GDP growth and inflation, may be modeled using fractional differencing.

In order to improve on conventional growth models, the idea of mixed alpha difference equations where the order of differencing falls between 0 and 1 has gained popularity. These models are particularly helpful in examining GDP evolution in uncertain, slowly adapting environments because they use fractional memory kernels, which reflect both short-term dynamics and long-term dependencies.

The following Theorem follows from the Theorem III.1

Theorem VIII.1. *Let $GDP(n) = GDP_0(1+r)^n$, GDP of a Country at time n, where GDP_0 is the initial GDP and r is a constant growth rate.*

$$\underset{(q,h)\alpha}{\Delta} GDP(n) = \sum_{k=0}^{\infty} \alpha^k \left[GDP\left(\frac{n - h\sum_{j=0}^{k} q^j}{q^{k+1}} \right) - GDP\left(\frac{n - h\sum_{j=0}^{k-1} q^j}{q^k} \right) \right] \quad (31)$$

Proof: Let $GDP_0 = 1000$ (initial GDP), $r = 0.05 (5\%\ growth\ rate)$, $h = 1, q = 1.1$, $n = 5, \alpha = 1$.

Step:1 Calculate GDP at different Time Periods,

$$GDP(0) = 1000$$
$$GDP(1) = 1000(1+0.05)^1 = 1050$$
$$GDP(2) = 1000(1+0.05)^2 = 1102.5$$
$$GDP(3) = 1000(1+0.05)^3 = 1157.625$$
$$GDP(4) = 1000(1+0.05)^4 = 1215.50625$$
$$GDP(5) = 1000(1+0.05)^5 = 1276.28256$$

Step:2 Applying the Mixed Alpha Difference Operator

$$\underset{(1,1)}{\Delta} GDP(5) = \sum_{k=0}^{\infty} \alpha^k \left[GDP\left(\frac{5 - 1\sum_{j=0}^{k} 1.1^j}{1.1^{k+1}} \right) - GDP\left(\frac{5 - 1\sum_{j=0}^{k-1} 1.1^j}{1.1^k} \right) \right]$$

Taking $k = 0$

$$GDP\left(\frac{5-1.1}{1.1^1}\right) - GDP(4) = GDP(5) - GDP(4) = 1276.2815625 - 1215.50625 = 60.7753125$$

Taking $k = 1$

$$GDP\left(\frac{5-1.1}{1.1^2}\right) - GDP(4) = GDP(5) - GDP(4) = 1276.2815625 - 1215.50625 = 60.7753125$$

Similarly, find the other values of k.

After calculating a few terms, we can derive the overall change in GDP over time based on the discrete fractional approach. Each term in the sum represents a form of accleration or deceleration of GDP growth relative to prior values influenced by the parameters h and q. ■

The theorem illustrates how the GDP growth dynamics could be analyzed using mixed alpha difference operators, allowing for the representation of varying growth rates over time. By expanding on this theorem, more complex economic models could utilize fractional differences to capture changes in economic indicators like GDP, leading to forecasting and policy-making strategies.

The following Theorem follows from the Theorem IV.1.

Theorem VIII.2. *Let $GDP(n) = GDP_0(1+r)^n$, represents the GDP at time n, where GDP_0 is the initial GDP and r is a constant growth rate.*

$$\underset{(q,h)\alpha}{\Delta} GDP(n) = \sum_{d=0}^{m-1} \frac{n(n-1)\cdots(n-d+1)}{d!} \times \underset{(q,h)\alpha}{\Delta} GDP\left(\frac{n - h\sum_{j=0}^{d-1} q^j}{q^d}\right) \quad (32)$$

Proof: Let $GDP_0 = 1000$ (initial GDP), $r = 0.05$ (5% growth rate), $h = 1$, $q = 1.1$, $n = 5$, $m = 3$, $\alpha = 1$

Step:1 Calculate GDP at different Time Periods

$$GDP(0) = 1000$$
$$GDP(1) = 1000(1+0.05)^1 = 1050$$
$$GDP(2) = 1000(1+0.05)^2 = 1102.5$$
$$GDP(3) = 1000(1+0.05)^3 = 1157.625$$
$$GDP(4) = 1000(1+0.05)^4 = 1215.50625$$
$$GDP(5) = 1000(1+0.05)^5 = 1276.28256$$

Step:2 Applying $m = 3$ in equation (32) we get

$$\underset{(1,1)}{\Delta} GDP(5) = \sum_{d=0}^{2} \frac{5(4)(3)(2)}{d!} \times \underset{(1,1)}{\Delta} GDP\left(\frac{5 - 1\sum_{j=0}^{d-1} 1^j}{1.1^d}\right)$$

Calculating $d = 0, 1, 2, \cdots$

For $d = 0 \Rightarrow$

$$\underset{(1,1)}{\Delta} GDP\left(\frac{5-1(0)}{1.1^0}\right) = \underset{(1,1)}{\Delta} GDP(5) = GDP(5) - GDP(4) = 1276.2815625 - 1215.50625 = 60.7753125$$

For $d = 1 \Rightarrow$

$$\underset{(1,1)}{\Delta} GDP\left(\frac{5-1(1.1^0)}{1.1^1}\right) = \underset{(1,1)}{\Delta} GDP(4.5454) = 1000(1+0.05)^{4.5454} = 1215.50625$$

Next calculate the difference $\underset{(1,1)}{\Delta} GDP(4.5454)$ requires steps similar to above.

Similarly, Compute $\underset{(1,1)}{\Delta} GDP\left(\frac{5-1(1.1^0+1.1^1)}{1.1^2}\right)$ calculate as above for GDP.

Each term from $d = 0, 1, 2$ contributes to the comprehensive growth profile that takes into account previous growth trajectories and shifts. ■

This theorem demonstrates how the higher order anti-difference can be utilized to derive insights into GDP growth by incorporating past values and assesing their influence on current performance.

IX. Conclusion

The factorial-coefficient and gamma geometric factorial methods were used to construct the mixed alpha's difference operator with integer and fractional order theorems. We examined the stability for alpha mixed difference operator. Lastly, real-world systems like population growth and gross domestic product models can be effectively analyzed by using the mixed alpha difference operator, which improves fractional differential equation modeling capabilities and more precisely captures complicated dynamics.

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