Improved Young and Cauchy-Schwarz Inequalities

Xuesha Wu

Abstract—**This paper primarily investigates several inequalities involving Young and Cauchy-Schwarz. Initially, we derive two Young-type scalar inequalities by employing the definition of the hyperbolic cosine function and its Taylor series expansion. Building on these results, we further establish Young-type inequalities for matrices and the Hilbert-Schmidt norm. Additionally, by leveraging the convexity of a specific function, we derive matrix Cauchy-Schwarz inequalities for unitarily invariant norms, which improve the existing results.**

Index Terms—**Young-type inequality, Hilbert-Schmidt norm, Cauchy-Schwarz inequality, unitarily invariant norm**

I. INTRODUCTION

YOUNG-TYPE inequalities have wide applications in engineering, especially in fields such as signal processing, control theory, image processing, optimization problems and system analysis. For example, in image denoising, Young-type inequalities help derive the bounds for noise and signals, which in turn facilitates the design of optimal filters. Likewise, Cauchy-Schwarz inequalities are widely applied in areas such as machine learning, data processing, quantum mechanics, optimization theory, network communication and information theory. For instance, in information theory, Cauchy-Schwarz inequalities are used to analyze metrics such as signal-to-noise ratio and bit error rate in the information transmission process, aiding in the design of efficient encoding and decoding strategies. Therefore, the investigation of Young-type inequalities and Cauchy-Schwarz inequalities holds significant practical and theoretical importance.

Throughout this paper, let $M_{m,n}$ represent the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. I stands for the proper dimension identity matrix. Denote by $\|\cdot\|$ any unitarily invariant norm on M_n, where $||UAV|| = ||A||$ holds for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt (or Frobenius) norm is expressed as

$$\|A\|_2 = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} s_j^2(A))^{\frac{1}{2}}$$

where $s_j(A)$ for $j = 1, 2, \cdots, n$ represents j-th largest singular value of A with $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. These singular values are equivalent to the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, which are arranged in decreasing order and counted with their respective multiplicities. A^* represents the conjugate transpose of the matrix A.

It is clear that the Hilbert-Schmidt norm is unitarily invariant.

Manuscript received March 16, 2025; revised June 1, 2025.

This work was supported by the China Higher Education Association (Grant No.23JS0401).

Xuesha Wu is a professor at the School of General and International Education, Chongqing Polytechnic University of Electronic Technology, Chongqing, 401331, China (e-mail: xuesha_wu@163.com).

For positive semidefinite matrices $A, B \in M_n$ and $0 \leq v \leq 1$, the $v-$weighted geometric mean of matrices A and B is given by

$$A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}},$$

when $v = \frac{1}{2}$, the geometric mean is denoted by $A\sharp B$.

If $A, B \in M_n$ are positive definite, Kittaneh and Manasrah [1] obtained

$$\begin{aligned} &2g_0(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B \\ &\leq A + B \\ &\leq 2h_0(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B, \end{aligned} \tag{1}$$

where $0 \leq v \leq 1$, $g_0 = \min\{v, 1-v\}$ and $h_0 = \max\{v, 1-v\}$.

Zou [2] established enhanced versions of inequalities (1) as follows

$$\begin{aligned} &2g_0(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B \\ &\leq A + B \\ &\leq \alpha(v)(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B \\ &\leq 2h_0(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B, \end{aligned} \tag{2}$$

where $\alpha(v) = 2(1 - 2(v - v^2))$.

Subsequently, Liu and Yang [3] demonstrated stronger versions of inequalities (2) as follows

$$\begin{aligned} &2g_0(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B \\ &\leq A + B \\ &\leq \beta(v)(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B \\ &\leq \alpha(v)(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B, \end{aligned} \tag{3}$$

where $\beta(v) = \frac{3}{2} - 2(v - v^2)$, $\alpha(v) = 2(1 - 2(v - v^2))$.

Recently, Hu and Liu [4] presented refined versions of inequalities (3), which can be expressed as

$$\begin{aligned} &2g_0(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B \\ &\leq A + B \\ &\leq \gamma(v)(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B \\ &\leq \beta(v)(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B, \end{aligned} \tag{4}$$

where $\gamma(v) = \frac{5}{4} - (v - v^2)$, $\beta(v) = \frac{3}{2} - 2(v - v^2)$.

Bhatia and Davis [5] obtained that if $A, B, X \in M_n$ such that A and B are positive semidefinite, then

$$\begin{aligned} & 2||A^{\frac{1}{2}}XB^{\frac{1}{2}}|| \\ & \leq ||A^vXB^{1-v} + A^{1-v}XB^v|| \\ & \leq ||AX + XB||, \end{aligned} \tag{5}$$

where $0 \leq v \leq 1$.

The second inequality of (5) is commonly referred to as Heinz inequality.

Kittaneh and Manasrah [1], He and Zou [6] respectively obtained that if $A, B, X \in M_n$ such that A and B are positive semidefinite, then

$$\begin{aligned} & ||AX + XB||_2^2 \\ & \leq ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 \\ & \quad +2h_0||AX - XB||_2^2, \end{aligned} \tag{6}$$

where $0 \leq v \leq 1$, $h_0 = \max\{v, 1-v\}$.

Zou [2] demonstrated an improvement version of inequality (6) as follows

$$\begin{aligned} & ||AX + XB||_2^2 \\ & \leq ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 \\ & \quad +\alpha(v)||AX - XB||_2^2, \end{aligned} \tag{7}$$

where $\alpha(v) = 2(1 - 2(v - v^2))$.

Liu and Yang [3] established a stronger version of inequality (7) as follows

$$\begin{aligned} & ||AX + XB||_2^2 \\ & \leq ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 \\ & \quad +\beta(v)||AX - XB||_2^2, \end{aligned} \tag{8}$$

where $\beta(v) = \frac{3}{2} - 2(v - v^2)$.

Hu and Liu [4] provided a refinement version of inequality (8) as follows

$$\begin{aligned} & ||AX + XB||_2^2 \\ & \leq ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 \\ & \quad +\gamma(v)||AX - XB||_2^2, \end{aligned} \tag{9}$$

where $\gamma(v) = \frac{5}{4} - (v - v^2)$.

Kittaneh and Manasrah [7] showed that if $A, B, X \in M_n$ such that A and B are positive semidefinite, then

$$\begin{aligned} & ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 \\ & +2g_0||AX - XB||_2^2 \\ & \quad \leq ||AX + XB||_2^2, \end{aligned} \tag{10}$$

where $0 \leq v \leq 1$ and $g_0 = \min\{v, 1-v\}$, inequality (10) is the inverse of inequality (6).

Horn and Mathias [8, 9] derived that if $A, B \in M_n$ and any real number $r > 0$, then

$$|||A^*B|^r||^2 \leq ||(AA^*)^r|| \cdot ||(BB^*)^r||, \tag{11}$$

which is a matrix Cauchy-Schwarz inequality for unitarily invariant norms.

For $A, B, X \in M_n$ and $r > 0$, Bhatia and Davis [10] achieved an enhanced version of inequality (11) in the following form

$$|||A^*XB|^r||^2 \leq |||AA^*X|^r|| \cdot |||XBB^*|^r||. \tag{12}$$

If $A, B, X \in M_n$ such that A and B are positive semidefinite, (12) is equivalent to

$$|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2 \leq |||AX|^r|| \cdot |||XB|^r||. \tag{13}$$

Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then, for any real number $r > 0$, the function

$$\psi(\nu) = |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r||$$

is convex on $[0, 1]$ and achieves its minimum at $\nu = \frac{1}{2}$. As a result, $\psi(\nu)$ is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$, moreover, $\psi(\nu) = \psi(1-\nu)$ for $\nu \in [0, 1]$ (See [11]). Leveraging the convexity of the function $\psi(\nu)$, Hiai and Zhan [11] provided an enhanced version of inequality (13) in the following form

$$\begin{aligned} & |||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2 \\ & \leq |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \leq |||AX|^r|| \cdot |||XB|^r||. \end{aligned} \tag{14}$$

Hu [12] employed the convexity of $\psi(\nu)$ to derive an enhancement of the second inequality presented in (14)

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \leq 2\nu_0|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2 \\ & \quad +(1-2\nu_0)|||AX|^r|| \cdot |||XB|^r||, \end{aligned} \tag{15}$$

where $0 \leq \nu \leq 1$, $\nu_0 = \min\{\nu, 1-\nu\}$.

Recently, using the convexity of $\psi(\nu)$, He et al. [13] gave refinements of inequality (15), which can be expressed as (I) if $\nu \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, then

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \leq 4\nu_0|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r|| \\ & \quad +(1-4\nu_0)|||AX|^r|| \cdot |||XB|^r||, \end{aligned} \tag{16}$$

(II) if $\nu \in [\frac{1}{4}, \frac{3}{4}]$, then

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \leq 2(1-2\nu_0)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r|| \\ & \quad +(4\nu_0-1)||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2, \end{aligned} \tag{17}$$

where $\nu_0 = \min\{\nu, 1-\nu\}$.

For more papers on Young and Cauchy-Schwarz inequalities, please refer to references [14-17] and their corresponding bibliographies. For article on improving inequalities of matrix unitarily invariant norms by utilizing the convex functions, see reference [18].

This note, building upon the previous discussions, aims to enhance inequalities involving both Young and Cauchy-Schwarz. The structure of the note is organized as follows: Section 2 introduces four Young-type scalar inequalities. In Section 3, using Young-type scalar inequalities obtained in the second section, we derive the matrix and Hilbert-Schmidt norm forms of Young-type inequalities, which lead to improvements of inequalities (4) and (9). Section 4 focuses on refining inequalities (16) and (17) by means of convexity properties. Finally, the conclusion of this paper is given in Section 5.

II. Young-type Scalar Inequalities

Now we present the first theorem of this note.

Theorem 1. Let $a, b > 0$, $0 \le v \le 1$. Then

$$a + b \le a^v b^{1-v} + a^{1-v} b^v + \eta(v)(\sqrt{a} - \sqrt{b})^2, \tag{18}$$

where $\eta(v) = \frac{13}{12} - \frac{1}{3}(v - v^2)$.

Proof. To establish inequality (18), it is sufficient to demonstrate that the following inequality holds

$$(1 - \eta(v))(a + b) + 2\eta(v)\sqrt{ab} \le a^v b^{1-v} + a^{1-v} b^v.$$

Taking $a = e^x, b = e^y$, by the definition of the hyperbolic cosine function, it follows that

$$\begin{aligned}&(\frac{1}{3}(v - v^2) - \frac{1}{12})cosh(\frac{x-y}{2})\\ &+(\frac{13}{12} - \frac{1}{3}(v - v^2))\\ &\le cosh((1-2v)(\frac{x-y}{2})).\end{aligned} \tag{19}$$

Taking $w = \frac{x-y}{2}$, by the Taylor series expansion of $coshw$, it follows that inequality (19) is equivalent to

$$\begin{aligned}&(\frac{1}{3}(v - v^2) - \frac{1}{12})(1 + \frac{w^2}{2!} + \frac{w^4}{4!} + \cdots)\\ &+(\frac{13}{12} - \frac{1}{3}(v - v^2))\\ &\le 1 + \frac{(1-2v)^2 w^2}{2!} + \frac{(1-2v)^4 w^4}{4!} + \cdots.\end{aligned} \tag{20}$$

Since $0 \le v \le 1$, we have

$$\frac{1}{3}(v - v^2) - \frac{1}{12} \in [-\frac{1}{12}, 0].$$

Therefore, inequality (20) clearly holds.
This completes the proof.

Corollary 1. Let $a, b > 0$, $0 \le v \le 1$. Then

$$(a + b)^2 \le (a^v b^{1-v} + a^{1-v} b^v)^2 + \eta(v)(a - b)^2, \tag{21}$$

where $\eta(v) = \frac{13}{12} - \frac{1}{3}(v - v^2)$.

Proof. Inequality (18) leads to the conclusion that

$$\begin{aligned}&(\sqrt{a} + \sqrt{b})^2 - (a^{\frac{v}{2}} b^{\frac{1-v}{2}} + a^{\frac{1-v}{2}} b^{\frac{v}{2}})^2\\ &= a + b - (a^v b^{1-v} + a^{1-v} b^v)\\ &\le (\frac{13}{12} - \frac{1}{3}(v - v^2))(\sqrt{a} - \sqrt{b})^2.\end{aligned}$$

Thus

$$(a + b)^2 \le (a^v b^{1-v} + a^{1-v} b^v)^2 + (\frac{13}{12} - \frac{1}{3}(v - v^2))(a - b)^2.$$

This completes the proof.

Below, we will present improvements of Theorem 1 and Corollary 1.

Theorem 2. Let $a, b > 0$, $0 \le v \le 1$. Then

$$a + b \le a^v b^{1-v} + a^{1-v} b^v + \zeta(v)(\sqrt{a} - \sqrt{b})^2, \tag{22}$$

where $\zeta(v) = \frac{21}{20} - \frac{1}{5}(v - v^2)$.

Proof. The proof is similar to Theorem 1. We leave it to the readers.

Remark 1. Theorem 2 is more precise than Theorem 1.

It follows that

$$\begin{aligned}&\eta(v) - \zeta(v)\\ &= \frac{13}{12} - \frac{1}{3}(v - v^2) - \frac{21}{20} + \frac{1}{5}(v - v^2)\\ &= \frac{1}{30}(2v - 1)^2\\ &\ge 0.\end{aligned} \tag{23}$$

Corollary 2. Let $a, b > 0$, $0 \le v \le 1$. Then

$$(a + b)^2 \le (a^v b^{1-v} + a^{1-v} b^v)^2 + \zeta(v)(a - b)^2, \tag{24}$$

where $\zeta(v) = \frac{21}{20} - \frac{1}{5}(v - v^2)$.

Proof. The proof is similar to Corollary 1. We leave it to the readers.

Remark 2. Corollary 2 is clearly more precise than Corollary 1, by (23).

III. Young-type Inequalities for Matrices

In the following, we will apply Theorem 1 to derive Young-type inequalities for matrices, which provide improved versions of inequalities (4).

Theorem 3. Let $A, B \in M_n$ be positive definite. Then

$$\begin{aligned}&2g_0(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B\\ &\le A + B\\ &\le \eta(v)(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B\\ &\le \gamma(v)(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B,\end{aligned} \tag{25}$$

where $v \in [0, 1]$, $g_0 = \min\{v, 1 - v\}, \eta(v) = \frac{13}{12} - \frac{1}{3}(v - v^2)$ and $\gamma(v) = \frac{5}{4} - (v - v^2)$.

Proof. From inequalities (1), it can be concluded that the first inequality in (25) is satisfied. For the second inequality in (25), given that $P \in M_n$ is positive definite, by the spectral decomposition theorem, there exists a unitary matrix $U \in M_n$ such that

$$P = UGU^*,$$

where $G = diag(\lambda_1, \lambda_2, \cdots, \lambda_n), \lambda_i > 0, 1 \le i \le n$. For $a > 0$ and $b = 1$, inequality (18) implies that we obtain

$$a + 1 \le a^v + a^{1-v} + \eta(v)(\sqrt{a} - 1)^2,$$

and so

$$G + I \le G^v + G^{1-v} + \eta(v)(G^{\frac{1}{2}} - I)^2. \tag{26}$$

Multiplying both sides of inequality (26) by U and U^*, we obtain

$$P+I \le P^v + P^{1-v} + \eta(v)(P^{\frac{1}{2}} - I)^2.$$

Let $P = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, and thus the second inequality in (25) is satisfied.

Next, we prove that the third inequality in (25) holds.

It is straightforward to observe that for $0 \le v \le 1$, the following holds

$$\begin{aligned} &\gamma(v) - \eta(v) \\ &= \frac{5}{4} - (v - v^2) - \frac{13}{12} + \frac{1}{3}(v - v^2) \\ &= \frac{2}{3}(v - \frac{1}{2})^2 \\ &\ge 0. \end{aligned} \tag{27}$$

Therefore, the third inequality in (25) holds.
This completes the proof.

Remark 3. Obviously, Theorem 3 is a refinement of the inequalities (4).

Next, we will use Corollary 1 to obtain a matrix Young-type inequality for the Hilbert-Schmidt norm.

Theorem 4. Let $A, B, X \in M_n$ such that A, B are positive semidefinite. Then

$$\begin{aligned} &||AX + XB||_2^2 \\ &\le ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 \\ &\quad +\eta(v)||AX - XB||_2^2, \end{aligned} \tag{28}$$

where $v \in [0,1]$ and $\eta(v) = \frac{13}{12} - \frac{1}{3}(v - v^2)$.

Proof. We first prove that when $A, B \in M_n$ are positive definite, by the spectral decomposition theorem, there exist unitary matrices $U, Q \in M_n$ such that

$$A = UG_1U^*, B = QG_2Q^*,$$

where

$$G_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_n), G_2 = diag(\mu_1, \mu_2, \cdots, \mu_n),$$

$\lambda_i, \mu_i > 0, 1 \le i \le n.$
Let $D = U^*XQ = (d_{ij})$, then

$$\begin{aligned} &A^vXB^{1-v} + A^{1-v}XB^v \\ &= (UG_1U^*)^vX(QG_2Q^*)^{1-v} \\ &\quad +(UG_1U^*)^{1-v}X(QG_2Q^*)^v \\ &= UG_1^v(U^*XQ)G_2^{1-v}Q^* + UG_1^{1-v}(U^*XQ)G_2^vQ^* \\ &= U(G_1^vDG_2^{1-v} + G_1^{1-v}DG_2^v)Q^*, \end{aligned}$$

hence

$$\begin{aligned} &||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 \\ &= ||G_1^vDG_2^{1-v} + G_1^{1-v}DG_2^v||_2^2 \\ &= \sum_{i,j=1}^n (\lambda_i^v\mu_j^{1-v} + \lambda_i^{1-v}\mu_j^v)^2|d_{ij}|^2. \end{aligned}$$

Similarly, we have

$$||AX + XB||_2^2 = \sum_{i,j=1}^n (\lambda_i + \mu_j)^2|d_{ij}|^2$$

and

$$||AX - XB||_2^2 = \sum_{i,j=1}^n (\lambda_i - \mu_j)^2|d_{ij}|^2.$$

From inequality (21), we derive

$$\begin{aligned} &\sum_{i,j=1}^n (\lambda_i^v\mu_j^{1-v} + \lambda_i^{1-v}\mu_j^v)^2|d_{ij}|^2 \\ &+\eta(v)\sum_{i,j=1}^n (\lambda_i - \mu_j)^2|d_{ij}|^2 \\ &\ge \sum_{i,j=1}^n (\lambda_i + \mu_j)^2|d_{ij}|^2. \end{aligned}$$

If $A, B \in M_n$ are positive semidefinite. Define $A_\varepsilon = A + \varepsilon I$, $B_\varepsilon = B + \varepsilon I$, where ε is an arbitrary positive real number. Consequently, A_ε and B_ε are positive definite matrices,

$$\begin{aligned} &||A_\varepsilon X + XB_\varepsilon||_2^2 \\ &\le ||A_\varepsilon^vXB_\varepsilon^{1-v} + A_\varepsilon^{1-v}XB_\varepsilon^v||_2^2 \\ &\quad +\eta(v)||A_\varepsilon X - XB_\varepsilon||_2^2. \end{aligned}$$

Let $\varepsilon \to 0$. Therefore, inequality (28) holds.
This completes the proof.

Remark 4. Theorem 4 is sharper than inequality (9), by (27).

Next, we will use Theorem 2 and Corollary 2 to present improvements of Theorem 3 and Theorem 4.

Theorem 5. Let $A, B \in M_n$ be positive definite. Then

$$\begin{aligned} &2g_0(A + B - 2A\sharp B) + A\sharp_vB + A\sharp_{1-v}B \\ &\le A + B \\ &\le \zeta(v)(A + B - 2A\sharp B) + A\sharp_vB + A\sharp_{1-v}B \\ &\le \eta(v)(A + B - 2A\sharp B) + A\sharp_vB + A\sharp_{1-v}B, \end{aligned}$$

where $v \in [0,1]$, $g_0 = \min\{v, 1-v\}, \zeta(v) = \frac{21}{20} - \frac{1}{5}(v - v^2)$ and $\eta(v) = \frac{13}{12} - \frac{1}{3}(v - v^2)$.

Proof. The proof of Theorem 5 is similar to Theorem 3. We leave it to the readers.

Theorem 6. Let $A, B, X \in M_n$ such that A, B are positive semidefinite. Then

$$\begin{aligned} &||AX + XB||_2^2 \\ &\le ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 \\ &\quad +\zeta(v)||AX - XB||_2^2, \end{aligned}$$

where $v \in [0,1]$ and $\zeta(v) = \frac{21}{20} - \frac{1}{5}(v - v^2)$.

Proof. The proof of Theorem 6 is similar to Theorem 4. We leave it to the readers.

Remark 5. Theorem 6 is clearly more precise than Theorem 4, by (23).

IV. Cauchy-Schwarz Inequalities for Matrices

In this section, we utilize the convexity of $\psi(\nu)$ to establish matrix Cauchy-Schwarz inequalities for unitarily invariant norms, which lead to improved forms of inequalities (16) and (17). To initiate our discussion, we first introduce the following lemma.

Lemma 1 ([12]). Let g be a real valued convex function on the interval $[a,b]$ which contains (x_1,x_2). Then for $x_1 \le x \le x_2$, we have

$$g(x) \le \frac{g(x_2)-g(x_1)}{x_2-x_1}x - \frac{x_1 g(x_2) - x_2 g(x_1)}{x_2-x_1}.$$

Theorem 7. Let $A, B, X \in M_n$ such that A and B are positive semidefinite. We have

(I) if $\nu \in [0,\frac{1}{8}] \cup [\frac{7}{8},1]$, then

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (1-8\nu_0)|||AX|^r|| \cdot |||XB|^r|| \\ & \quad +8\nu_0|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \cdot |||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r||, \end{aligned} \tag{29}$$

(II) if $\nu \in [\frac{1}{8},\frac{1}{4}] \cup [\frac{3}{4},\frac{7}{8}]$, then

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (2-8\nu_0)|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \cdot |||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r|| \\ & \quad +(8\nu_0-1)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r||, \end{aligned} \tag{30}$$

(III) if $\nu \in [\frac{1}{4},\frac{3}{8}] \cup [\frac{5}{8},\frac{3}{4}]$, then

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (3-8\nu_0)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r|| \\ & \quad +(8\nu_0-2)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r|| \cdot |||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r||, \end{aligned} \tag{31}$$

(IV) if $\nu \in [\frac{3}{8},\frac{5}{8}]$, then

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (4-8\nu_0)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r|| \cdot |||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r|| \\ & \quad +(8\nu_0-3)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2, \end{aligned} \tag{32}$$

where $\psi(\nu) = |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r||$, $r>0$ and $\nu_0 = \min\{\nu, 1-\nu\}$.

Proof. If $0 \le \nu \le \frac{1}{8}$, by Lemma 1 and the convexity of $\psi(\nu)$, we obtain

$$\psi(\nu) \le \frac{\psi(\frac{1}{8})-\psi(0)}{\frac{1}{8}-0}\nu - \frac{0\psi(\frac{1}{8})-\frac{1}{8}\psi(0)}{\frac{1}{8}-0},$$

which is equal to

$$\psi(\nu) \le (1-8\nu)\psi(0) + 8\nu\psi(\frac{1}{8}).$$

Thus

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (1-8\nu)|||AX|^r|| \cdot |||XB|^r|| \\ & \quad +8\nu|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \cdot |||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r||, \end{aligned}$$

that is

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (1-8\nu_0)|||AX|^r|| \cdot |||XB|^r|| \\ & \quad +8\nu_0|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \cdot |||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r||. \end{aligned} \tag{33}$$

If $\frac{7}{8} \le \nu \le 1$, by Lemma 1 and the convexity of $\psi(\nu)$, we obtain

$$\psi(\nu) \le \frac{\psi(1)-\psi(\frac{7}{8})}{1-\frac{7}{8}}\nu - \frac{\frac{7}{8}\psi(1)-\psi(\frac{7}{8})}{1-\frac{7}{8}},$$

which is equal to

$$\psi(\nu) \le (8-8\nu)\psi(\frac{7}{8}) + (8\nu-7)\psi(1).$$

Thus

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (8-8\nu)|||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r|| \cdot |||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \\ & \quad +(8\nu-7)|||AX|^r|| \cdot |||XB|^r||, \end{aligned}$$

that is

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (1-8\nu_0)|||AX|^r|| \cdot |||XB|^r|| \\ & \quad +8\nu_0|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \cdot |||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r||. \end{aligned} \tag{34}$$

If $\frac{1}{8} \le \nu \le \frac{1}{4}$, by Lemma 1 and the convexity of $\psi(\nu)$, we obtain

$$\psi(\nu) \le \frac{\psi(\frac{1}{4})-\psi(\frac{1}{8})}{\frac{1}{4}-\frac{1}{8}}\nu - \frac{\frac{1}{8}\psi(\frac{1}{4})-\frac{1}{4}\psi(\frac{1}{8})}{\frac{1}{4}-\frac{1}{8}},$$

which is equal to

$$\psi(\nu) \le (2-8\nu)\psi(\frac{1}{8}) + (8\nu-1)\psi(\frac{1}{4}).$$

Thus

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (2-8\nu)|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \cdot |||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r|| \\ & \quad +(8\nu-1)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r||, \end{aligned}$$

that is

$$\begin{aligned} & |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ & \le (2-8\nu_0)|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \cdot |||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r|| \\ & \quad +(8\nu_0-1)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r||. \end{aligned} \tag{35}$$

If $\frac{3}{4} \le \nu \le \frac{7}{8}$, by Lemma 1 and the convexity of $\psi(\nu)$, we obtain

$$\psi(\nu) \le \frac{\psi(\frac{7}{8})-\psi(\frac{3}{4})}{\frac{7}{8}-\frac{3}{4}}\nu - \frac{\frac{3}{4}\psi(\frac{7}{8})-\frac{7}{8}\psi(\frac{3}{4})}{\frac{7}{8}-\frac{3}{4}},$$

which is equal to

$$\psi(\nu) \le (7-8\nu)\psi(\frac{3}{4}) + (8\nu-6)\psi(\frac{7}{8}).$$

Thus

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(7-8\nu)|||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r||\cdot|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r||$$
$$+(8\nu-6)|||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r||\cdot|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r||,$$

that is

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(2-8\nu_0)|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r||\cdot|||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r|| \quad (36)$$
$$+(8\nu_0-1)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r||\cdot|||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r||.$$

If $\frac{1}{4}\leq\nu\leq\frac{3}{8}$, similarly, we obtain

$$\psi(\nu)\leq\frac{\psi(\frac{3}{8})-\psi(\frac{1}{4})}{\frac{3}{8}-\frac{1}{4}}\nu-\frac{\frac{1}{4}\psi(\frac{3}{8})-\frac{3}{8}\psi(\frac{1}{4})}{\frac{3}{8}-\frac{1}{4}},$$

which is equal to

$$\psi(\nu)\leq(3-8\nu)\psi(\frac{1}{4})+(8\nu-2)\psi(\frac{3}{8}).$$

Thus

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(3-8\nu)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r||\cdot|||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r||$$
$$+(8\nu-2)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r||\cdot|||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r||,$$

that is

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(3-8\nu_0)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r||\cdot|||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r|| \quad (37)$$
$$+(8\nu_0-2)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r||\cdot|||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r||.$$

If $\frac{5}{8}\leq\nu\leq\frac{3}{4}$, similarly, we obtain

$$\psi(\nu)\leq\frac{\psi(\frac{3}{4})-\psi(\frac{5}{8})}{\frac{3}{4}-\frac{5}{8}}\nu-\frac{\frac{5}{8}\psi(\frac{3}{4})-\frac{3}{4}\psi(\frac{5}{8})}{\frac{3}{4}-\frac{5}{8}},$$

which is equal to

$$\psi(\nu)\leq(6-8\nu)\psi(\frac{5}{8})+(8\nu-5)\psi(\frac{3}{4}).$$

Thus

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(6-8\nu)|||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r||\cdot|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r||$$
$$+(8\nu-5)|||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r||\cdot|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r||,$$

that is

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(3-8\nu_0)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r||\cdot|||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r|| \quad (38)$$
$$+(8\nu_0-2)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r||\cdot|||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r||.$$

If $\frac{3}{8}\leq\nu\leq\frac{1}{2}$, similarly, we obtain

$$\psi(\nu)\leq\frac{\psi(\frac{1}{2})-\psi(\frac{3}{8})}{\frac{1}{2}-\frac{3}{8}}\nu-\frac{\frac{3}{8}\psi(\frac{1}{2})-\frac{1}{2}\psi(\frac{3}{8})}{\frac{1}{2}-\frac{3}{8}},$$

which is equivalent to

$$\psi(\nu)\leq(4-8\nu)\psi(\frac{3}{8})+(8\nu-3)\psi(\frac{1}{2}).$$

Thus

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(4-8\nu)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r||\cdot|||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r||$$
$$+(8\nu-3)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2,$$

that is

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(4-8\nu_0)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r||\cdot|||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r|| \quad (39)$$
$$+(8\nu_0-3)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2.$$

If $\frac{1}{2}\leq\nu\leq\frac{5}{8}$, similarly, we obtain

$$\psi(\nu)\leq\frac{\psi(\frac{5}{8})-\psi(\frac{1}{2})}{\frac{5}{8}-\frac{1}{2}}\nu-\frac{\frac{1}{2}\psi(\frac{5}{8})-\frac{5}{8}\psi(\frac{1}{2})}{\frac{5}{8}-\frac{1}{2}},$$

which is equal to

$$\psi(\nu)\leq(5-8\nu)\psi(\frac{1}{2})+(8\nu-4)\psi(\frac{5}{8}).$$

Thus

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(5-8\nu)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2$$
$$+(8\nu-4)|||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r||\cdot|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r||,$$

that is

$$|||A^{\nu}XB^{1-\nu}|^r||\cdot|||A^{1-\nu}XB^{\nu}|^r||$$
$$\leq(4-8\nu_0)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r||\cdot|||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r|| \quad (40)$$
$$+(8\nu_0-3)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2.$$

It follows from (33)-(40) and $r>0$, $\nu_0=\min\{\nu,1-\nu\}$ that Theorem 7 holds.
This completes the proof.

Corollary 3. Theorem 7 is sharper than inequalities (16) and (17).

Proof. By the convexity of $\psi(\nu)$ and (16), (17), if $\nu\in[0,\frac{1}{8}]\cup[\frac{7}{8},1]$, then

$$\begin{aligned}\psi(\nu) &\leq (1-8\nu_0)\psi(0)+8\nu_0\psi(\frac{1}{8})\\ &\leq (1-8\nu_0)\psi(0)+8\nu_0(\frac{1}{2}\psi(\frac{1}{4})+\frac{1}{2}\psi(0))\\ &= (1-4\nu_0)\psi(0)+4\nu_0\psi(\frac{1}{4}).\end{aligned}$$

If $\nu \in [\frac{1}{8}, \frac{1}{4}] \cup [\frac{3}{4}, \frac{7}{8}]$, then

$$\begin{aligned} \psi(\nu) &\leq (2-8\nu_0)\psi(\frac{1}{8})+(8\nu_0-1)\psi(\frac{1}{4}) \\ &\leq (2-8\nu_0)(\frac{1}{2}\psi(\frac{1}{4})+\frac{1}{2}\psi(0)) \\ &\quad +(8\nu_0-1)\psi(\frac{1}{4}) \\ &= (1-4\nu_0)\psi(0)+4\nu_0\psi(\frac{1}{4}). \end{aligned}$$

If $\nu \in [\frac{1}{4}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{3}{4}]$, then

$$\begin{aligned} \psi(\nu) &\leq (3-8\nu_0)\psi(\frac{1}{4})+(8\nu_0-2)\psi(\frac{3}{8}) \\ &\leq (3-8\nu_0)\psi(\frac{1}{4}) \\ &\quad +(8\nu_0-2)(\frac{1}{2}\psi(\frac{1}{2})+\frac{1}{2}\psi(\frac{1}{4})) \\ &= 2(1-2\nu_0)\psi(\frac{1}{4})+(4\nu_0-1)\psi(\frac{1}{2}). \end{aligned}$$

If $\nu \in [\frac{3}{8}, \frac{5}{8}]$, then

$$\begin{aligned} \psi(\nu) &\leq (4-8\nu_0)\psi(\frac{3}{8})+(8\nu_0-3)\psi(\frac{1}{2})) \\ &\leq (4-8\nu_0)(\frac{1}{2}\psi(\frac{1}{2})+\frac{1}{2}\psi(\frac{1}{4})) \\ &\quad +(8\nu_0-3)\psi(\frac{1}{2}) \\ &= 2(1-2\nu_0)\psi(\frac{1}{4})+(4\nu_0-1)\psi(\frac{1}{2}). \end{aligned}$$

Consequently, Theorem 7 is a refinement of inequalities (16) and (17).
This completes the proof.

Based on inequalities (14) and (29)-(32), we obtain the following refinements of inequality (14).

Corollary 4. Let $A, B, X \in M_n$ such that A and B are positive semidefinite. We have
(I) if $\nu \in [0, \frac{1}{8}] \cup [\frac{7}{8}, 1]$, then

$$\begin{aligned} &|||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ &\leq (1-8\nu_0)|||AX|^r|| \cdot |||XB|^r|| \\ &\quad +8\nu_0|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \cdot |||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r|| \\ &\leq |||AX|^r|| \cdot |||XB|^r||, \end{aligned}$$

(II) if $\nu \in [\frac{1}{8}, \frac{1}{4}] \cup [\frac{3}{4}, \frac{7}{8}]$, then

$$\begin{aligned} &|||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ &\leq (2-8\nu_0)|||A^{\frac{1}{8}}XB^{\frac{7}{8}}|^r|| \cdot |||A^{\frac{7}{8}}XB^{\frac{1}{8}}|^r|| \\ &\quad +(8\nu_0-1)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r|| \\ &\leq |||AX|^r|| \cdot |||XB|^r||, \end{aligned}$$

(III) if $\nu \in [\frac{1}{4}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{3}{4}]$, then

$$\begin{aligned} &|||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ &\leq (3-8\nu_0)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r|| \\ &\quad +(8\nu_0-2)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r|| \cdot |||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r|| \\ &\quad +(8\nu_0-1)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r|| \\ &\leq |||AX|^r|| \cdot |||XB|^r||, \end{aligned}$$

(IV) if $\nu \in [\frac{3}{8}, \frac{5}{8}]$, then

$$\begin{aligned} &|||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r|| \\ &\leq (4-8\nu_0)|||A^{\frac{3}{8}}XB^{\frac{5}{8}}|^r|| \cdot |||A^{\frac{5}{8}}XB^{\frac{3}{8}}|^r|| \\ &\quad +(8\nu_0-3)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r||^2 \\ &\quad +(8\nu_0-1)|||A^{\frac{1}{4}}XB^{\frac{3}{4}}|^r|| \cdot |||A^{\frac{3}{4}}XB^{\frac{1}{4}}|^r|| \\ &\leq |||AX|^r|| \cdot |||XB|^r||, \end{aligned}$$

where $\psi(\nu) = |||A^\nu XB^{1-\nu}|^r|| \cdot |||A^{1-\nu}XB^\nu|^r||$, $r > 0$ and $\nu_0 = \min\{\nu, 1-\nu\}$.

V. Conclusion

This paper primarily explores some inequalities involving Young and Cauchy-Schwarz. We begin by deriving two Young-type scalar inequalities, employing $coshw$ and its Taylor series expansion. Based on the obtained inequalities (18), (21), (22) and (23), we then present Young-type inequalities for matrices and Hilbert-Schmidt norm. Furthermore, by leveraging the convexity of $\psi(\nu)$, we establish Cauchy-Schwarz inequalities for unitarily invariant norms of matrices, which enhance inequalities (16) and (17). At the same time, we present a corollary of Theorem 7. These topics will be further investigated in future studies.

References

[1] F. Kittaneh and Y. Manasrah, "Reverse Young and Heinz inequalities for matrices", *Linear and Multilinear Algebra*, vol. 59, no. 9, pp. 1031-1037, 2011.

[2] L. Zou, "On some matrix inequalities", *Acta Mathematica Sinica*, vol. 55, no. 4, pp. 715-720, 2012.

[3] X. Liu and X. Yang, "Some inequalities for weighted geometric mean and norm of matrices", *Applied Mathematics A Journal of Chinese Universities*, vol. 33, no. 3, pp. 373-378, 2018.

[4] X. Hu and W. Liu, "Some improved inequalities for matrices", *Journal of Mathematics*, vol. 43, no. 1, pp. 38-42, 2023.

[5] R. Bhatia and C. Davis, "More matrix forms of the arithmetic–geometric mean inequality", *SIAM Journal on Matrix Analysis and Applications*, vol. 14, no. 1, pp. 132-136, 1993.

[6] C. He and L. Zou, "Some inequalities involving unitarily invariant norms", *Mathematical Inequalities & Applications*, vol. 15, no. 4, pp. 767-776, 2012.

[7] F. Kittaneh and Y. Manasrah, "Improved Young and Heinz inequalities for matrices", *Journal of Mathematical Analysis and Applications*, vol. 361, no. 1, pp. 262-269, 2010.

[8] R. A. Horn and R. Mathias, "Cauchy–Schwarz inequalities associated with positive semidefinite matrices", *Linear Algebra and Its Applications*, vol. 142, pp. 63-82, 1990.

[9] R. A. Horn and R. Mathias, "An analog of the Cauchy–Schwarz inequality for Hadamard products and unitarily invariant norms", *SIAM Journal on Matrix Analysis and Applications*, vol. 11, no. 4, pp. 481-498, 1990.

[10] R. Bhatia and C. Davis, "A Cauchy–Schwarz inequality for operators with applications", *Linear Algebra and Its Applications*, vol. 223/224, pp. 119-129, 1995.

[11] F. Hiai and X. Zhan, "Inequalities involving unitarily invariant norms and operator monotone functions", *Linear Algebra and Its Applications*, vol. 341, no. 1-3, pp. 151-169, 2002.

[12] X. Hu, "Some inequalities for unitarily invariant norms", *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 615-623, 2012.

[13] Z. He, J. Liu, and Q. Wang, "Refinements of Cauchy–Schwarz norm inequality for operators", *Communication on Applied Mathematics and Computation*, vol. 32, no. 3, pp. 644-650, 2018.

[14] D. Van and D. Huy, "Further new refinements and reverses of real power form for Young-type inequalities via famous constants and applications", *Operators and Matrices*, vol. 17, no. 2, pp. 485-515, 2023.

[15] C. Yang and G. Zhang, "Some refinements of Young type inequalities", *Journal of Mathematical Inequalities*, vol. 18, no. 2, pp. 519-531, 2024.

[16] M. Sababheh, C. Conde, and H. Moradi, "On the matrix Cauchy-Schwarz inequality", *Operators and Matrices*, vol. 17, no. 2, pp. 525-538, 2023.

[17] M. Jena and N. Das, "Generalized Cauchy-Schwarz type inequalities and their applications", *Filomat*, vol. 38, no. 21, pp. 7647-7655, 2024.

[18] X. Hu, Y. Wang, Y. Yi, and J. Xue, "Refinements of singular value and unitarily invariant norm inequalities", *IAENG International Journal of Applied Mathematics*, vol. 55, no. 4, pp. 720-726, 2025.