On Common Fixed Point Results in Cone b-Metric Space over Banach Algebra

Md W. Rahaman, L. Shambhu Singh, and Th. Chhatrajit Singh, *Member, IAENG*

***Abstract*—This paper explores common fixed points for generalized weak contractive mappings in cone b-metric spaces over Banach algebras. It extends classical fixed point theorems to these spaces, where distances are defined using cones in Banach algebras rather than real numbers. The work gives necessary and sufficient criteria for the presence of common fixed points by proposing a new class of generalised weak contractive mappings. The methodology integrates functional analysis and topology, utilizing the unique properties of cone b-metric spaces and Banach algebras. Key findings illustrate the broader applicability of these results in fields like differential equations, optimization, and dynamical systems. Examples and potential future research directions are also presented, showcasing the practical relevance and theoretical advancements offered by this work in fixed point theory.**

***Index Terms*—Cauchy sequence, contractive mappings, cone b-metric spaces, fixed point, Banach spaces.**

I. Introduction

FIXED point theory plays a fundamental role in analysis and topology. It has a range of applications in different domains of mathematics such as in the theory of ODEs, PDEs, integral equations and so forth. Banach Contraction Principle [1] act as the base for majority of the results obtained so far in fixed point theory. Later on, many authors have obtained its several generalisations in different ways [2], [3].

Cone metric space is a generalisation of conventional metric space that Huang and Zhang [7] presented in 2007. Instead of using the usual real line, a vector on an ordered Banach space defines the distance $d(q, r)$ between q and r. They demonstrated that in these kinds of spaces, the well-known Banach contraction principle is valid. Since then, a lot of papers have addressed how the findings on cone metric spaces are the same as the results on ordinary metric spaces (see [5], [6], [8]–[14], [16]–[19], [21], [25]–[28]).

By substituting Banach space with Banach algebra, Liu and Xu [20] postulated the concept of a cone metric space over Banach algebra and presented many fixed point theorems for generalised Lipschitz mappings that include weaker and inherent limitations on the generalised Lipschitz constant. Subsequently, without assuming normality, Xu and Radenovic [22] provided certain fixed point theorems of generalised Lipschitz mappings in the new environment. These theorems differ from metric spaces in that they do not imply the presence of the mappings fixed points. Cone b-metric spaces over Banach algebras were first described by Huang and Radenovic [23], [24]. In addition, they tackled the problem of the lack of equivalence between cone b-metric spaces and b-metric spaces considering the existence of fixed points in the corresponding mappings. Many authors established a number of other common fixed point theorems satisfying certain contractive condition in certain spaces (see [29]–[36]).

The objective of this research is to provide a common fixed point theorem for four self-mappings that satisfy a generalised weak contractive condition in cone b-metric spaces over Banach algebra.

Manuscript received October 30, 2024; revised June 21, 2025.

Md W. Rahaman is a Research Scholar of Department of Mathematics, Dhanamanjuri University Imphal, Manipur, India-795001. (Email: mdrahmanmaths@dmu.ac.in).

L. Shambhu Singh is a Professor of Department of Mathematics, Dhanamanjuri University Imphal, Manipur, India-795001. (Email: lshambhu1162@gmail.com).

Th. Chhatrajit Singh is an Assistant Professor of Department of Mathematics, Manipur Technical University Imphal, Manipur, India-795004. (Corresponding author to provide email: chhatrajit@mtu.ac.in).

II. Preliminaries

Definition 1. *[7] Let E be a real Banach space and $P \subseteq E$. Then, P is called a cone iff (i)*

1) *P is closed, non-empty and $P \neq 0$;*
2) *$\mu, \chi \in \mathbb{R}, \mu, \chi \geq 0, q, r \in P \Rightarrow \mu q + \chi r \in P$;*
3) *$P \cap (-P) = 0$.*

For a given cone $P \subset E$, let $\leq$ be a partial ordering with respect to P written as $q \leq r$ if and only if $r - q \in P$. Also let $q < r$ to indicate that $q \leq r$ but $q \neq r$, while $q \ll r$ will stand for $r - q \in IntP$, where $IntP$ denotes the interior of the set P. If $IntP \neq \phi$ then P is a solid cone. The cone P is considered normal if $\exists$ a number $\zeta > 0$ such that, $\forall$ $q, r \in E$,

$$0 \leq q \leq r \Rightarrow ||q|| \leq \zeta ||r||.$$

The least positive number fulfilling above is considered the normal constant of P.

Definition 2. *[7] Let X be a non-empty set. Let the mapping d: $X \times X \rightarrow E$ satisfies: (a)*

1) *$0 < d(q, r)$ $\forall$ q, r in X and $d(q, r) = 0$ iff $q = r$;*
2) *$d(q, r) = d(r, q)$ $\forall$ q, r in X;*
3) *$d(q, r) \leq d(q, s) + d(s, r)$ $\forall$ q, r, s in X.*

Then, (X, d) is called a cone metric space and d is called a cone metric on X.

Example 3. *[7] Suppose*

$$E = \mathbb{R}^2, P = \{(q, r) \in E : q, r \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$$

and $d : X \times X \rightarrow E$ defined by

$$d(q, r) = (|q - r|, \zeta|q - r|), \zeta \geq 0$$

is a constant. Then (X, d) is a cone metric space.

Definition 4. *[20] A Banach algebra $\mathcal{A}$ is a real Banach space where a multiplication operation is specified $\forall$ q, r, s in $\mathcal{A}$, and ζ in $\mathbb{R}$ satisfying the following criteria. (i)*

1) *$(qr)s = q(rs)$;*
2) *$q(r + s) = qr + qs$ and $(q + r)s = qs + qs$;*

3) *$\zeta(qr)$ = $(\zeta q)r$ = $q(\zeta r)$;*
4) $||qr|| \leq ||q||\ ||r||$.

A subset $\mathcal{B} \subseteq \mathcal{A}$ is said to be cone if (a)

1) $\mathcal{B}$ is non-empty, closed and $\{\theta, e\} \subset \mathcal{B}$;
2) $\mu\mathcal{B}$ + $\chi\mathcal{B} \subset \mathcal{B}\ \forall\ \mu, \chi > 0$;
3) $\mathcal{B}^2$ = $\mathcal{B}\mathcal{B} \subset \mathcal{B}$;
4) $\mathcal{B} \cap (-\mathcal{B})$ = $\{0\}$;

where e and θ denote the unit and zero elements of $\mathcal{A}$ respectively. For a given cone $\mathcal{B} \subset \mathcal{A}$, we write $s \leq r$ iff $r - s \in \mathcal{B}$, where $\leq$ is a partial order relation defined on $\mathcal{B}$. Also, $q \ll r$ will stand for $r - q \in int\mathcal{B}$, where $int\mathcal{B}$ denotes the interior of $\mathcal{B}$. If $int\mathcal{B} \neq \phi$, then $\mathcal{B}$ is called a solid cone. The cone $\mathcal{B}$ is considered normal if $\exists$ is a number $\zeta > 0$ such that, $\forall\ q, r \in \mathcal{A}$,

$$0 \leq q \leq r \Rightarrow ||q|| \leq \zeta||r||.$$

The least positive number fulfilling above is considered the normal constant of $\mathcal{B}$.

Definition 5. *[15] A cone b-metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $q, r, s \in X$ and a constant $s \geq 1$ the following conditions hold:*

1) *$0 < d(q, r)$ with $q \neq r$ and $d(q, r) = 0 \Leftrightarrow q = r$,*
2) *$d(q, r) = d(r, q)$,*
3) *$d(q, r) \leq s[(q, s) + d(s, r)]$.*

The pair (X, d) is called a cone b-metric space.

Remark 6. *[15] Given that every cone metric space is always a cone b-metric space, the class of cone b-metric spaces is more extensive than the class of cone metric spaces. Thus, it is evident that cone b-metric spaces serve as a generalisation of both b-metric spaces and cone metric spaces. In this paper, we provide examples that demonstrate the significance of introducing a cone b-metric space instead of a cone metric space. This is because there are cone b-metric spaces that do not form cone metric spaces.*

Example 7. *[15] Let E = $\mathbb{R}^2$, P = $\{(q, r) \in E : q, r \geq 0\}$ $\subset E$, X = $\mathbb{R}$ and $d : X \times X \to E$ such that $d(q, r)$ = $(|q-r|^s, \zeta|q-r|^s)$, where $\zeta \geq 0$ and $s > 1$ are two constants. Then (X, d) is a cone b-metric space.*

Definition 8. *[23] A cone b-metric on a nonempty set X is a function $d : X \times X \to \mathcal{A}$ such that for all $q, r, s \in X$ and a constant $s \geq 1$ the following conditions hold:*

1) *$0 < d(q, r)$ with $q \neq r$ and $d(q, r) = 0 \Leftrightarrow q = r$,*
2) *$d(q, r) = d(r, q)$,*
3) *$d(q, r) \leq s[d(q, s) + d(s, r)]$.*

The pair (X, d) is called a cone b-metric space over Banach algebra $\mathcal{A}$.

Example 9. *Suppose*

$$\mathcal{A} = M_m(\mathbb{R}) = \{q = (q_{rs})_{m\times m} : a_{rs} \in \mathbb{R}\ \forall\ 1 \leq r, s \leq m\}$$

be the algebra of all n-square real matrices, and define the norm

$$||q|| = \Sigma_{1\leq r,s\leq m}|q_{rs}|.$$

Thus, $\mathcal{A}$ is a real Banach algebra with the unity e. Suppose $\mathcal{B}$ = $\{q = (q_{rs})_{m\times m} : q_{rs} \geq 0\ \forall\ 1 \leq r, s \leq m\}$. Then, $\mathcal{B}$ $\subset \mathcal{A}$ is a normal cone with normal constant $\zeta = 1$. Let X = $M_m(\mathbb{R}^+)$ and define the metric

$$d : X \times X \to \mathcal{A}$$

by

$$\begin{aligned} d(q, p) &= d((q_{rs})_{m\times m}(p_{rs})_{m\times m}) \\ &= \left((\max(q_{rs}, p_{rs}))^2\right)_{m\times m} \in \mathcal{A}. \end{aligned}$$

Then, (X, d) is a cone b-metric space over Banach algebra $\mathcal{A}$ with constant $s = 2$.

Definition 10. *[4] Consider two self-maps, g and h defined on a nonempty set X. If $t = gq = hq$ are given for some q in X, then q is referred to as the coincidence point of g and h, where t is the point of coincidence of g and h.*

Definition 11. *[4] Consider two self-maps g and h defined on a finite set X. Then, g, h are considered weakly compatible if they commute at every point of coincidence.*

III. Main Theorems

Definition 12. *Let $(X, \leq)$ be an ordered set and $g : X \to X$ be a mapping. Then elements $q, r \in X$ are comparable, if $q \leq r$ or $r \leq q$ holds.*

Definition 13. *The triplet $(X, d, \leq)$ is called an ordered cone b-metric space on a non-empty set X iff d is a cone b-metric on an ordered set $(X, \leq)$.*

A subset $\mathcal{B}$ of an ordered set X is said to be well-ordered if any two elements within $\mathcal{B}$ are comparable.

Definition 14. *Suppose $(X, \leq)$ be an ordered set. Mapping g is considered dominating if $q \leq fq$ for each $q \in X$.*

Definition 15. *Suppose $(X, \leq)$ be an ordered set. Mapping g is called dominated if $gq \leq q$ for each $q \in X$.*

Example 16. *Suppose $X = [0, 1]$ with usual ordering and $g : X \to X$ be defined as $gq = \sqrt[4]{q}$. For*

$$q \leq q^{\frac{1}{4}} = gq \forall\ q \in X.$$

Thus, g is a dominating map.

Example 17. *Suppose $X = [0, 1]$ with usual ordering and $g : X \to X$ be defined by $gq = q^m$ for some $m \in \mathbb{N}$. Since,*

$$gq = q^m \leq q \forall q \in X.$$

Thus, g is dominated map.

Definition 18. *[24] Suppose (X,d) be a cone b-metric space over Banach algebra $\mathcal{A}$, $q \in X$, $\{q_m\}$ a sequence in X and $\{r_m\}$ a sequence in $\mathcal{A}$. Then*

1) *$\{q_m\}$ converges to q whenever for every $t \gg 0\ \exists\ N \in \mathbb{N}$ such that $d(q_m, q) \ll t \forall\ m \geq N$ and denoted by $\lim_{m\to\infty} q_m = q$ or $q_m \to q(m \to \infty)$;*
2) *$\{q_m\}$ is a Cauchy sequence whenever for each $t \gg 0\ \exists\ N \in \mathbb{N}$ such that $d(q_m, q_n) \ll\ t \forall\ m, n \geq N$;*
3) *(X,d) is complete if every Cauchy sequence in X is convergent;*
4) *$\{r_m\}$ is a c-sequence if for each $t \gg 0, \exists\ N \in \mathbb{N}$ such that $r_m \ll t \forall\ m \geq N$.*

Definition 19. *Suppose $(X, d, \mathcal{A})$ be a cone b-metric space over Banach algebra. Then, the pair $\{g, h\}$ is considered compatible iff*

$$\lim_{m\to\infty} d(ghq_m, hgq_m) = 0,$$

whenever $\{q_m\}$ is a sequence in X so that

$$\lim_{m\to\infty} gq_m = lim_{m\to\infty} hq_m = t$$

for some t in X.

Theorem 20. *Suppose $(X, d, \mathcal{A})$ be a complete cone b-metric space over Banach algebra and let $g, h : X \to X$ be two self-mappings so that*

$$\psi(d(gq, hr)) \leq \psi(M(q, r)) - \phi(M(q, r)), \qquad (1)$$

$\forall\ q, r \in X$ where (c_1)

1) *$\psi : [0, \infty) \to [0, \infty)$ is a continuous monotone non-decreasing function with $\psi(t) = 0$ iff $t = 0$,*
2) *$\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ iff $t = 0$,*
3) $M(q, r) = \max\Big\{d(q, r), d(q, gq), d(r, gr), \frac{1}{2}[d(q, hr) + d(r, gq)]\Big\}$.

Then, $\exists$ a unique point u in X such that $u = gu = hu$.

Proof: Let $q_0 \in X$. Construct a sequence $\{q_m\}$ for $m \geq\ 0$ as

$$q_{2m} = gq_{2m+1}, \quad q_{2m+1} = hq_{2m} \qquad (2)$$

and prove that $d(q_m, q_{m-1}) \to 0$ as $m \to \infty$. Suppose now that m is an odd number. Substituting $q = q_m$ and $r = r_{m-1}$ in (1) and using properties of functions ψ and ϕ, we obtain

$$\begin{aligned}\psi(d(q_{m+1}, q_m)) =&\psi(gq_m, hq_{m-1})\\ \leq&\psi(M(q_m, q_{m-1})) - \phi(M(q_m, q_{m-1}))\\ \leq&\psi(M(q_m, q_{m-1}))\end{aligned}$$

implies

$$d(q_{m+1}, q_m) \leq M(q_m, q_{m-1}).$$

Now, from the triangle inequality for d, we have

$$\begin{aligned}M(q_m, q_{m-1}) =&max\Big\{d(q_m, q_{m-1}), d(q_{m+1}, q_m),\\ &d(q_m, q_{m-1}), \frac{1}{2}(d(q_{m-1}, q_{m+1})\\ &+ d(q_m, q_m))\Big\}\\ =&max\Big\{d(q_m, q_{m-1}), d(q_{m+1}, q_m),\\ &\frac{1}{2}(d(q_{m-1}, q_{m+1}))\Big\}\\ \leq&max\Big\{d(q_m, q_{m-1}), d(q_{m+1}, q_m),\\ &\frac{1}{2}[d(q_{m-1}, q_m) + d(q_m, q_{m+1})]\Big\}.\end{aligned}$$

For $d(q_{m+1}, q_m) > d(q_m, q_{m-1})$ implies

$$M(q_m, q_{m-1}) = d(q_{m+1}, q_m) > 0.$$

It furthermore implies

$$\psi(d(q_{m+1}, q_m)) \leq \psi(d(q_{m+1}, q_m)) - \phi(d(q_{m+1}, q_m))$$

which is a contradiction. Therefore,

$$d(q_{m+1}, q_m) \leq M(q_m, q_{m+1}) \leq d(q_m, q_{m-1}). \qquad (3)$$

Similarly, we can obtain inequalities (3) also in the case when m is an even number. Thus, the sequence $\{d(q_{m+1}, q_m)\}$ is monotone non-increasing and bounded. So,

$$\lim_{m\to\infty} d(q_{m+1}, q_m) = lim_{m\to\infty} M(q_m, q_{m-1}) = v \geq 0.$$

Letting $n \to \infty$ in inequality

$$\psi(d(q_{m+1}, q_m)) \leq \psi(M(q_m, q_{m-1})) - \phi(M(q_m, q_{m-1}))$$

we obtain, $\psi(v) \leq \psi(v) - \phi(v)$ which is a contradiction unless $v = 0$. Hence,

$$\lim_{m\to\infty} d(q_m, q_{m+1}) = 0. \qquad (4)$$

Next, we prove that $\{q_m\}$ is a Cauchy sequence. From (4), it is sufficient to show that the subsequence $\{q_{2m}\}$ is a Cauchy sequence. On the contrary, suppose $\{q_{2m}\}$ is not a Cauchy sequence. Then, $\exists\ \epsilon > 0$ for which we can find subsequences $\{q_{2n_k}\}$ and $\{q_{2m_k}\}$ of $\{q_{2m}\}$ such that m_k is the smallest index for which

$$m_k > n_k > k, \qquad d(q_{2n_k}, q_{2m_k}) \geq \epsilon.$$

implies

$$d(q_{2n_k}, q_{2m_k-2}) < \epsilon. \qquad (5)$$

Using (5), we obtain

$$\begin{aligned}\epsilon \leq&d(q_{2n_k}, q_{2m_k})\\ \leq&d(q_{2n_k}, q_{2m_k-2}) + d(q_{2m_k-2}, q_{2m_k-1}) +\\ &d(q_{2m_k-1}, q_{2m_k})\\ \leq&\epsilon + d(q_{2m_k-2}, q_{2m_k-1}) + d(q_{2m_k-1}, q_{2m_k}).\end{aligned}$$

Letting $k \to \infty$ and using (4), we obtain

$$\lim_{m\to\infty} d(q_{2n_k}, q_{2m_k}) = \epsilon. \qquad (6)$$

Moreover,

$$|d(q_{2n_k}, q_{2m_k+1}) - d(q_{2n_k}, q_{2m_k})| \leq d(q_{2m_k}, q_{2m_k+1})$$

$$|d(q_{2n_k-1}, q_{2m_k}) - d(q_{2n_k}, q_{2m_k})| \leq d(q_{2n_k}, q_{2n_k-1}).$$

Using (4) and (6), we get

$$\begin{aligned}\lim_{m\to\infty} d(q_{2n_k-1}, q_{2m_k}) =& lim_{m\to\infty} d(q_{2n_k}, q_{2m_k+1})\\ =& \epsilon\end{aligned} \qquad (7)$$

Also,

$$|d(q_{2n_k-1}, q_{2m_k+1}) - d(q_{2n_k-1}, q_{2m_k})| \leq d(q_{2m_k}, q_{2m_k+1}).$$

Using (4) and (7), we get

$$\lim_{m\to\infty} d(q_{2n_k-1}, q_{2m_k+1}) = \epsilon. \qquad (8)$$

Also, from the definition of M (condition (c_3)) and from (4), (6), (7) and (8), we have

$$\lim_{m\to\infty} M(q_{2n_k-1}, q_{2m_k}) = \epsilon \qquad (9)$$

Putting $q = q_{2n_k-1}, r = q_{2m_k}$ in (1), we have

$$\begin{aligned}\psi(d(q_{2n_k}, q_{2m_k+1})) =&\psi(d(gq_{2n_k-1}, hq_{2m_k}))\\ \leq&\psi(M(q_{2n_k-1}, q_{2m_k})) -\\ &\phi(M(q_{2n_k-1}, q_{2m_k})).\end{aligned}$$

Letting $k \to \infty$ and using (7) and (8), we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) \qquad (10)$$

which is a contradiction with $\epsilon > 0$. Therefore, $\{q_{2m}\}$ is a Cauchy sequence and hence $\{q_m\}$ is a Cauchy sequence. In complete metric space X there exists u such that $q_m \to u$ as $m \to \infty$. Let us now prove that u is the fixed point for g and h. Suppose $u \neq gu$, then for $d(u, gu) > 0, \exists\ N_1 \in \mathbb{N}$ such that for any $m > N_1$,

$$d(q_{2m+1}, u) < \frac{1}{2}d(u, gu),$$
$$d(q_{2m}, u) < \frac{1}{2}d(u, gu),$$
$$d(q_{2m}, q_{2m+1}) < \frac{1}{2}d(u, gu).$$

Accordingly,

$$\begin{aligned} d(u, gu) \leq& M(u, q_{2m}) \\ =& \max\Big\{d(u, q_{2m}), d(u, gu), d(q_{2m}, q_{2m+1}), \\ & \frac{1}{2}[d(u, q_{2m+1}) + d(q_{2m}, gu)]\Big\} \\ \leq& max\Big\{\frac{1}{2}d(u, gu), d(u, gu), \frac{1}{2}d(u, gu), \\ & \frac{1}{2}\left[\frac{1}{2}d(u, gu) + \frac{1}{2}d(u, gu) + d(u, gu)\right]\Big\} \\ =& d(u, gu), \end{aligned}$$

that is $M(u, q_{2m}) = d(u, gu)$. Since,

$$\begin{aligned} \psi(d(gu, q_{2m+1})) &= \psi(d(gu, hq_{2m})) \\ &\leq \psi(M(u, q_{2m})) - \phi(M(u, q_{2m})). \end{aligned}$$

Letting $m \to \infty$, we obtain

$$\psi(d(gu, u)) \leq \psi(d(gu, u)) - \phi(d(gu, u)),$$

which implies that $\psi(d(gu, u)) = 0$. Hence, $d(gu, u) = 0$ or $gu = u$. Using that u is the fixed point of g, we get

$$\begin{aligned} \psi(d(u, hu)) =& \psi(d(gu, hu)) \\ \leq& \psi(M(u, u)) - \phi(M(u, u)) \\ =& \psi(d(u, hu)) - \phi(d(u, hu)) \end{aligned}$$

and using an argument similar to the above, we conclude that $d(u, hu) = 0$ or $u = hu$.

Uniqueness: Let $u_1 \in X$ be another fixed point, then from

$$\begin{aligned} \psi(d(u, u_1)) =& \psi(d(gu, hu_1)) \\ \leq& \psi(M(u, u_1)) - \phi(M(u, u_1)) \\ =& \psi(d(u, u_1)) - \phi(d(u, u_1)) \end{aligned}$$

we conclude that $u = u_1$. ■

Example 21. *Suppose $X = [0, 1]$ endowed cone b-metric $d(q, r) = |q - r|$ where $s = 1$ and let $gq = \frac{1}{4}q$ and $hq = 0$ for each $q \in X$. Then,*

$$d(gq, gr) = \frac{1}{4}q$$

and

$$\begin{aligned} M(q, r) =& \max\left\{|q - r|, \frac{3}{4}q, r, \frac{1}{2}\left(|r - \frac{1}{4}q| + q\right)\right\} \\ =& \begin{cases} q - r, & 0 \leq r \leq \frac{1}{4}q \\ \frac{3}{4}q, & \frac{1}{4}q \leq r \leq \frac{3}{4}q \\ r, & \frac{3}{4}q \leq r \leq 1. \end{cases} \end{aligned}$$

For $\psi(t) = 4t$ and $\phi(t) = t$, we have

$$\psi(d(gq, hr)) = q$$

and

$$\psi(M(q, r)) - \phi(M(q, r)) = \begin{cases} 3q - 3r, & 0 \leq r \leq \frac{1}{4}q \\ \frac{9}{4}q, & \frac{1}{4}q \leq r \leq \frac{3}{4}q \\ 3r, & \frac{3}{4}q \leq r \leq 1. \end{cases}$$

Now we easily conclude that mappings g and h satisfy relation (1) and has 0 as the unique fixed point.

Lemma 22. *Suppose $(X, d, \mathcal{A})$ be a cone b-metric space over Banach algebra with $s \geq 1$, assume $\{q_m\}$ and $\{r_m\}$ are b-convergent to q, r respectively. Then,*

$$\begin{aligned} \frac{1}{s^2}d(q, r) &\leq \lim_{m\to\infty} inf\ d(q_m, r_m) \\ &\leq \lim_{m\to\infty} sup\ d(q_m, r_m) \\ &\leq s^2 d(q, r). \end{aligned}$$

For $q = r$, then

$$\lim_{m\to\infty} d(q_m, r_m) = 0.$$

Further, for each w in X, we get

$$\begin{aligned} \frac{1}{s}d(q, w) &\leq \lim_{m\to\infty} inf\ d(q_m, w) \\ &\leq \lim_{m\to\infty} sup\ d(q_m, w) \\ &\leq sd(q, w). \end{aligned}$$

Proof: Using the triangle inequality in a cone b-metric space over Banach algebra $\mathcal{A}$, we have

$$d(q, r) \leq sd(q, q_m) + s^2 d(q_m, r_m) + s^2 d(r_n, r), \quad (11)$$

and

$$d(q_m, r_m) \leq sd(q_m, q) + s^2 d(q, r) + s^2 d(r, r_m). \quad (12)$$

The initial desired outcome is achieved by utilising lower limit as $m \to \infty$ in (11) and upper limit as $m \to \infty$ in (12). In the same way, the final assertion is derived by utilising the triangle inequality once more. ■

The primary outcome of this section is now presented.

Theorem 23. *Suppose $(X, d, \mathcal{A})$ be a complete cone b-metric space over Banach algebra. Suppose g, h, J and K be self-mappings on X, $\{J, K\}$ and $\{g, h\}$ be dominating and dominated maps respectively with $gX \subseteq KX$ and $hX \subseteq JX$. Let ψ and ϕ be control functions such that*

$$\psi : [0, \infty) \to [0, \infty)$$

is a continuous monotone non-decreasing function with $\psi(b) = 0 \Leftrightarrow b = 0$,

$$\phi : [0, \infty) \to [0, \infty)$$

is a lower semi-continuous function with $\phi(t) = 0$ iff $t = 0$, and for every two comparable elements $q, r \in X$,

$$\psi(s^4 d(gq, hr)) \leq \psi(N_s(q, r)) - \phi(N_s(q, r)), \quad (13)$$

where

$$N_s(q,r) = \max\Big\{d(Jq,Kr), d(gq,Jq), d(hr,Kr), \frac{d(Jq,hr)+d(gq,Kr)}{2s}\Big\}, \quad (14)$$

for each non-increasing sequence $\{q_m\}$ and a sequence $\{r_m\}$ such that $r_m \leq q_m \forall\ m$ and $r_m \to u$, we have $u \leq q_m$ and either (d_1)

1) *g or J is continuous, $\{g,J\}$ are compatible and $\{h,K\}$ is weakly compatible or*
2) *h or K is continuous, $\{g,J\}$ is weakly compatible and $\{h,K\}$ are compatible,*

then, g,h,J and K have a common fixed point. Further, the common fixed points of g,h,J and K is well ordered iff g,h,J and K have unique common fixed point.

Proof: Suppose $q_0 \in X$. Since, $gX \subseteq KX$ and $hX \subseteq JX$, define $\{q_m\}$ and $\{r_m\} \in X$ as

$$r_{2m+1} = gq_{2m} = Kq_{2m+1},$$
$$r_{2m+2} = hq_{2m+1} = Jq_{2m+2}, \qquad m = 0,1,2,\cdots.$$

By given assumptions,

$$q_{2m+1} \leq Kq_{2m+1} = gq_{2m} \leq q_{2m}$$

and

$$q_{2m} \leq Jq_{2m} = hq_{2m-1} \leq q_{2m-1}.$$

Therefore, we get

$$q_{m+1} \leq q_m \forall\ m \geq 0.$$

Suppose that $d(r_{2m}, r_{2m+1}) > 0$ for each m. If not, then for some

$$k, r_{2k+1} = r_{2k},$$

and from (13), we get

$$\begin{aligned}\psi(d(r_{2k+1},r_{2k+2})) \leq& \psi(s^4 d(r_{2k+1},r_{2k+2}))\\ =&\psi(s^4 d(gq_{2k},hq_{2k+1}))\\ \leq&\psi(N_s(q_{2k},q_{2k+1}))-\\ &\phi(N_s(q_{2k},q_{2k+1})).\end{aligned} \quad (15)$$

Here,

$$\begin{aligned}N_s(q_{2k},q_{2k+1}) =& max\Big\{d(Jq_{2k},Kq_{2k+1}), d(gq_{2k},Jq_{2k}),\\ &d(gq_{2k+1},Kq_{2k+1}),\\ &\frac{1}{2s}[d(Jq_{2k},hq_{2k+1})+\\ &d(gq_{2k},Kq_{2k+1})]\Big\}\\ =&\max\Big\{d(r_{2k},r_{2k+1}), d(r_{2k+1},r_{2k}),\\ &d(r_{2k+2},r_{2k+1}),\\ &\frac{d(r_{2k},r_{2k+2})+d(r_{2k+1},r_{2k+1})}{2s}\Big\}\\ =&\max\Big\{0,0,d(r_{2k+2},r_{2k+1}),\\ &\frac{d(r_{2k+1},r_{2k+2})}{2s}\Big\}\\ =&d(r_{2k+1},r_{2k+2}).\end{aligned} \quad (16)$$

Now, from (15) and (16), we get

$$\begin{aligned}\psi(d(r_{2k+1},r_{2k+2})) \leq&\psi(d(r_{2k+1},r_{2k+2}))-\\ &\phi(d(r_{2k+1},r_{2k+2})),\end{aligned}$$

which gives

$$\phi(d(r_{2k+1},r_{2k+2})) \leq 0$$

and so $r_{2k+1} = r_{2k+2}$ which further implies that $r_{2k+2} = r_{2k+3}$. Therefore, r_{2k} is a common fixed point of g,h,J and K.

Now, let $d(r_{2m}, r_{2m+1}) > 0$ for each m. Since, q_{2m} and q_{2m+1} are comparable, from (13), we get

$$\begin{aligned}\psi(d(r_{2m+1},r_{2m+2})) \leq&\psi(s^4 d(r_{2m+1},r_{2m+2}))\\ =&\psi(s^4 d(gq_{2m},hq_{2m+1}))\\ \leq&\psi(N_s(q_{2m},q_{2m+1}))-\\ &\phi(N_s(q_{2m},q_{2m+1}))\\ \leq&\psi(N_s(q_{2m},q_{2m+1})).\end{aligned} \quad (17)$$

Therefore,

$$d(r_{2m+1},r_{2m+2}) \leq N_s(q_{2m},q_{2m+1}), \quad (18)$$

where

$$\begin{aligned}N_s(q_{2m},q_{2m+1}) =&\max\Big\{d(Jq_{2m},Kq_{2m+1}), d(gq_{2m},Jq_{2m}),\\ &d(hq_{2m+1},Kq_{2m+1}),\\ &\frac{1}{2s}[d(Jq_{2m},hq_{2m+1})+\\ &d(gq_{2m},Kq_{2m+1})]\Big\}\\ =&\max\Big\{d(r_{2m},r_{2m+1}), d(r_{2m+1},r_{2m}),\\ &d(r_{2m+2},r_{2m+1}),\\ &\frac{1}{2s}[d(r_{2m},r_{2m+2})+\\ &d(r_{2m+1},r_{2m+1})]\Big\}\\ \leq&\max\Big\{d(r_{2m+1},r_{2m}), d(r_{2m+2},r_{2m+1}),\\ &\frac{1}{2s}[sd(r_{2m},r_{2m+1})+\\ &sd(r_{2m+1},r_{2m+2})]\Big\}\\ =&\max\Big\{d(r_{2m+1},r_{2m}), d(r_{2m+2},r_{2m+1}),\\ &\frac{1}{2}[d(r_{2m},r_{2m+1})+\\ &d(r_{2m+1},r_{2m+2})]\Big\}\\ =&\max\{d((r_{2m+1},r_{2m}), d(r_{2m+2},r_{2m+1})\}.\end{aligned}$$

If for some

$$m, d(r_{2m+2},r_{2m+1}) \geq d(r_{2m+1},r_{2m}) > 0,$$

then from (18)

$$N_s(q_{2m},q_{2m+1}) = d(r_{2m+2},r_{2m+1})$$

and using (13), we get

$$\begin{aligned}\psi(d(r_{2m+2},r_{2m+1})) \leq& \psi(s^4 d(r_{2m+2},r_{2m+1}))\\ \leq& \psi(N_s(q_{2m},q_{2m+1}))-\\ &\phi(N_s(q_{2m},q_{2m+1}))\\ =& \psi(d(r_{2m+2},r_{2m+1}))-\\ &\phi(d(r_{2m+2},r_{2m+1})),\end{aligned}$$

and $\phi(d(r_{2m+2},r_{2m+1})) \leq 0$ or equivalently $d(r_{2m+2},r_{2m+1})=0$, a contradiction.
Thus,

$$N_s(q_{2m},q_{2m+1}) \leq d(r_{2m+1},r_{2m}).$$

Since,

$$N_s(q_{2m},q_{2m+1}) \geq d(r_{2m+1},r_{2m}),$$

thus,

$$d(r_{2m+2},r_{2m+1}) \leq N_s(q_{2m},q_{2m+1}) = d(r_{2m+1},r_{2m}).$$

Similarly, we get

$$d(r_{2m+3},r_{2m+2}) \leq N_s(q_{2m+2},q_{2m+1}) = d(r_{2m+2},r_{2m+1}). \tag{19}$$

Therefore, $\{d_b(y_n,y_{n+1})\}$ is a non-decreasing sequence and $\exists\ w \geq 0$ so that

$$\lim_{m\to\infty} d(r_{m-1},r_m) = lim_{m\to\infty} N_s(q_m,q_{m+1}) = w.$$

Assume $w>0$. Then,

$$\begin{aligned}\psi(d(r_{2m+2},r_{2m+1})) \leq& \psi(s^4 d(r_{2m+2},r_{2m+1}))\\ \leq& \psi(N_s(q_{2m},q_{2m+1}))-\\ &\phi(N_s(q_{2m},q_{2m+1})).\end{aligned}$$

Then, taking the upper limit as $m\to\infty$ implies that

$$\begin{aligned}\psi(w) \leq& \psi(w) - \lim_{m\to\infty} inf \phi(N_s(q_{2m},q_{2m+1}))\\ =& \psi(w) - \phi(\lim_{m\to\infty} inf M_s(q_{2m},q_{2m+1}))\\ =& \psi(w)-\phi(w),\end{aligned}$$

a contradiction. Thus,

$$\lim_{m\to\infty} d(r_{m-1},r_m) = 0. \tag{20}$$

We now prove $\{r_m\}$ is a Cauchy sequence. It's sufficient to prove subsequence $\{r_{2m}\}$ is Cauchy in X. Assume $\{r_{2m}\}$ is not a Cauchy sequence. Then, $\exists\ \epsilon>0$ for which we can find subsequences $\{r_{2n_k}\}$ and $\{r_{2m_k}\}$ of $\{r_{2m}\}$ such that m_k is the smallest index for which $2m_k > 2n_k > k$,

$$d(r_{2n_k},r_{2m_k}) \geq \epsilon \tag{21}$$

and

$$d(r_{2n_k-2},r_{2m_k}) < \epsilon \tag{22}$$

By applying the triangle inequality in a cone b-metric space over a Banach algebra along with equation (22), we obtain

$$\begin{aligned}\epsilon \leq& d(r_{2n_k-2},r_{2m_k})\\ \leq& sd(r_{2n_k-2},r_{2m_k}) + s^2 d(r_{2n_k-1},r_{2m_k-2}) +\\ & s^2 d(r_{2n_k},r_{2m_k-1})\\ <& \epsilon s + s^2 d(r_{2n_k-1},r_{2m_k-2}) + s^2 d(r_{2n_k},r_{2m_k-1}).\end{aligned}$$

As $k\to\infty$ and from (20), we get

$$\epsilon \leq \lim_{k\to\infty} sup d(r_{2n_k},r_{2m_k}) \leq \epsilon s. \tag{23}$$

Also,

$$\epsilon \leq d(r_{2n_k},r_{2m_k}) \leq sd(r_{2n_k},r_{2n_k-1}) + sd(r_{2n_k-1},r_{2n_k}).$$

Thus,

$$\frac{\epsilon}{s} \leq \lim_{k\to\infty} sup d(r_{2n_k-1},r_{2m_k}).$$

Further, we get

$$d(r_{2n_k-1},r_{2m_k}) \leq sd(r_{2n_k-1},r_{2n_k}) + sd(r_{2n_k},r_{2m_k}).$$

So, from (20) and (23), we have

$$\begin{aligned}&\lim_{k\to\infty} sup\ d(r_{2n_k-1},r_{2m_k})\\ \leq& s \lim_{k\to\infty} sup\ d(r_{2n_k},r_{2m_k})\\ \leq& \epsilon s^2.\end{aligned}$$

As a result,

$$\frac{\epsilon}{s} \leq \lim_{k\to\infty} sup\ d(r_{2n_k-1},r_{2m_k}) \leq \epsilon s^2, \tag{24}$$

and

$$\frac{\epsilon}{s^2} \leq \lim_{k\to\infty} sup\ d(r_{2m_k+1},r_{2n_k-1}) \leq \epsilon s^3. \tag{25}$$

$$\frac{\epsilon}{s} \leq \lim_{k\to\infty} sup\ d(r_{2m_k+1},r_{2n_k}). \tag{26}$$

Using (23), (24) and (25), we get

$$\begin{aligned}\frac{\epsilon}{2s}+\frac{\epsilon}{2s^3} =& \min\left\{\frac{\epsilon}{s}, \frac{\epsilon+\frac{\epsilon}{s^2}}{2s}\right\}\\ \leq& \max\Big\{\lim_{k\to\infty} sup\ d(r_{2m_k},r_{2n_k-1}),\\ & \frac{1}{2s}[\lim_{k\to\infty} sup\ d(r_{2n_k},r_{2m_k}) +\\ & \lim_{k\to\infty} sup\ d(r_{2m_k+1},r_{2n_k-1})]\Big\}\\ \leq& \max\left\{\epsilon s^2, \frac{\epsilon s+\epsilon s^3}{2s}\right\} = \epsilon s^2.\end{aligned} \tag{27}$$

Using $N_s(q,r)$ and from (20) and (27), we get

$$\frac{\epsilon}{2s}+\frac{\epsilon}{2s^3} \leq \lim_{k\to\infty} sup\ N_s(q_{2m_k},q_{2n_k-1}) \leq \epsilon s^2. \tag{28}$$

Indeed, we have

$$\begin{aligned}N_s(q_{2m_k},q_{2n_k-1}) =& \max\Big\{d(Jq_{2m_k},Kq_{2n_k-1}),\\ & d(gq_{2m_k},Jq_{2m_k}),\\ & d(hq_{2n_k-1},Kq_{2n_k-1}),\\ & \frac{1}{2s}[d(Jq_{2m_k},hq_{2n_k-1}) +\\ & d(gq_{2m_k},Kq_{2n_k-1})]\Big\}\\ =& max\Big\{d(r_{2m_k},r_{2n_k-1}),\\ & d(r_{2m_k+1},r_{2m_k}),\\ & d(r_{2n_k},r_{2n_k-1}),\\ & \frac{1}{2s}[d(r_{2m_k},r_{2n_k}) +\\ & d(r_{2m_k+1},r_{2n_k-1})]\Big\}\end{aligned}$$

Taking the upper limit as $m \to \infty$, and using (20) and (27), we obtain

$$\begin{aligned}\frac{\epsilon}{2s}+\frac{\epsilon}{2s^3} \leq & \lim_{k\to\infty} sup\ N_s(q_{2m_k}, q_{2n_k-1}) \\ =& \max\bigg\{ \lim_{k\to\infty} sup\ d(r_{2m_k}, r_{2n_k-1}), 0, 0, \\ & \frac{1}{2s}[\lim_{k\to\infty} sup\ d(r_{2n_k}, r_{2m_k})+ \\ & \lim_{k\to\infty} sup\ d(r_{2m_k+1}, r_{2n_k-1})]\bigg\} \\ \leq & \epsilon s^2.\end{aligned}$$

Similarly, we obtain

$$\frac{\epsilon}{2s}+\frac{\epsilon}{2s^3} \leq \lim_{k\to\infty} inf\ N_s(q_{2m_k}, q_{2n_k-1}) \leq \epsilon s^2. \qquad (29)$$

As,

$$\begin{aligned}\psi(s^4 d(r_{2m_k+1}, r_{2n_k})) =& \psi(s^4 d(gq_{2m_k}, hq_{2n_k-1})) \\ \leq & \psi(N_s(q_{2m_k}, q_{2n_k-1}))- \\ & \phi(N_s(q_{2m_k}, q_{2n_k-1})),\end{aligned}$$

for $k \to \infty$, and from (26) and (28), we get

$$\begin{aligned}\psi(\epsilon s^3) \leq & \psi\left(s^4 \lim_{k\to\infty} sup\ d(r_{2m_k+1}, r_{2n_k})\right) \\ \leq & \psi\left(\lim_{k\to\infty} sup\ N_s(q_{2m_k}, q_{2n_k-1})\right)- \\ & \lim_{k\to\infty} inf\ \phi(N_s(q_{2m_k}, q_{2n_k-1})) \\ \leq & \psi(\epsilon s^2) - \phi\left(\lim_{k\to\infty} inf\ N_s(q_{2m_k}, q_{2n_k-1})\right) \\ \leq & \psi(\epsilon s^3) - \phi\left(\lim_{k\to\infty} inf\ N_s(q_{2m_k}, q_{2n_k-1})\right)\end{aligned}$$

implies

$$\phi\left(\lim_{k\to\infty} inf\ N_s(q_{2m_k}, q_{2n_k-1})\right) = 0$$

implies lim inf $N_s(q_{2m_k}, q_{2n_k-1}) = 0$, a contradiction to (29). Thus, $\{r_{2m}\}$ is a Cauchy sequence in X. Since X is complete, $\exists\ r$ in X such that

$$\begin{aligned}\lim_{m\to\infty} gq_{2m} =& \lim_{m\to\infty} Kq_{2m+1} \\ =& \lim_{m\to\infty} hq_{2m+1} \\ =& \lim_{m\to\infty} Jq_{2m+2} = r.\end{aligned}$$

Next, we prove that g, h, J and K has a common fixed point r.
As, J is continuous

$$\lim_{m\to\infty} J^2 q_{2m+2} = Jr, \qquad lim_{m\to\infty} Jgq_{2m} = Jr.$$

Also,

$$d(gJq_{2m}, Jr) \leq s(d(gJq_m, Jgq_m) + d(Jgq_{2m}, Jr)).$$

As, $\{g, J\}$ is compatible,

$$\lim_{m\to\infty} d(gJq_m, Jgq_m) = 0.$$

Taking $m \to \infty$, we get

$$\begin{aligned}& \lim_{m\to\infty} sup\ d(gJq_{2m}, Jr)) \\ \leq & s\left(\lim_{m\to\infty} sup\ d(gJq_m, Jgq_m)\right. \\ & \left. + \lim_{m\to\infty} sup\ d(Jgq_{2m}, Jr)\right) = 0.\end{aligned}$$

Thus,

$$lim_{m\to\infty} gJq_{2m} = Jr.$$

As, $Jq_{2m+2} = hq_{2m+1} \leq q_{2m+1}$, so from (13), we get

$$\begin{aligned}\psi(s^4 d(gJq_{2m+2}, hq_{2m+1})) \leq & \psi(N_s(Jq_{2m+2}, q_{2m+1}))- \\ & \phi(N_s(Jq_{2m+2}, q_{2m+1})),\end{aligned} \qquad (30)$$

where

$$\begin{aligned}N_s(Jq_{2m+2}, q_{2m+1}) =& max\bigg\{ d(J^2 q_{2m+2}, Kq_{2m+1}), \\ & d(gJq_{2m+2}, J^2 q_{2m+2}), \\ & d(hq_{2m+1}, Kq_{2m+1}), \\ & \frac{1}{2s}[d(J^2 q_{2m+2}, hq_{2m+1})+ \\ & d(gJq_{2m+2}, Kq_{2m+1})]\bigg\}.\end{aligned}$$

From lemma 22, we have

$$\begin{aligned}& lim_{m\to\infty} sup\ N_s(Jq_{2m+2}, q_{2m+1}) \\ \leq & max\bigg\{ s^2 d(Jr, r), 0, 0, \\ & \frac{s^2 d(Jr, r) + s^2 d(Jr, r)}{2s}\bigg\} \\ =& s^2 d(Jr, r).\end{aligned}$$

Thus, by taking limit in (30) and using lemma 22, we get

$$\psi(s^2 d(Jr, r)) \leq \psi(s^2 d(Jr, r)) - \phi(s^2 d(Jr, r)),$$

which gives $\phi(s^2 d(Jr, r)) \leq 0$ or equivalently $Jr = r$. Since, $hq_{2m+1} \leq q_{2m+1}$ and $hq_{2m+1} \to r$ as $m \to \infty$, then $r \leq q_{2m+1}$ and from (13), we get

$$\psi(s^4 d(gr, hq_{2m+1})) \leq \psi(N_s(r, q_{2m+1})) - \phi(N_s(r, q_{2m+1})), \qquad (31)$$

where

$$\begin{aligned}N_s(r, q_{2m+1}) =& max\bigg\{ d(Jr, Kq_{2m+1}), d(gr, Jr), \\ & d(hq_{2m+1}, Kq_{2m+1}), \\ & \frac{d(Jr, hq_{2m+1}) + d(gr, Kq_{2m+1})}{2s}\bigg\}.\end{aligned}$$

As $m \to \infty$ in (31) and from lemma 22, we get

$$\begin{aligned}\psi(s^3 d(gr, r)) =& \psi(s^4 \frac{1}{s} d(gr, r)) \\ \leq & \psi(d(gr, r)) - \phi(d(gr, r)) \\ \leq & \psi(s^3 d(gr, r)) - \phi(d(gr, r))\end{aligned}$$

which implies that $\phi(d(gr, r)) \leq 0$ implies $gr = r$. Since, $g(X) \subseteq K(X)$, $\exists$ a point υ in X so that $gr = K\upsilon$. Let $h\upsilon \neq K\upsilon$. Since, $\upsilon \leq K\upsilon = gr \leq r$, from (13), we get

$$\psi(d(K\upsilon, h\upsilon)) = \psi(d(gr, h\upsilon)) \leq \psi(N_s(r, \upsilon)) - \phi(N_s(r, \upsilon)), \qquad (32)$$

where

$$N_s(r,\upsilon) = max\left\{d(Jr,K\upsilon), d(gr,Jr), d(h\upsilon,K\upsilon), \frac{d(Jr,h\upsilon)+d(gr,K\upsilon)}{2s}\right\} = d(h\upsilon,K\upsilon).$$

From (32), we get

$$\psi(d(K\upsilon,h\upsilon)) \le \psi(d(h\upsilon,K\upsilon)) - \phi(d(h\upsilon,K\upsilon)),$$

a contradiction. Thus, $h\upsilon = K\upsilon$. Since, the pair $\{h,K\}$ is weakly compatible, $hr = hgr = hK\upsilon = Kh\upsilon = Kgr = Kr$ and r is the coincidence point of h and K. Since, $Jq_{2m} \le q_{2m}$ and $Jq_{2m} \to r$ as $m \to \infty$ implies $r \le q_{2m}$ and from 13,we get

$$\psi(s^4 d(gq_{2m},hr)) \le \psi(N_s(q_{2m},r)) - \phi(N_s(q_{2m},r)), \quad (33)$$

where

$$N_s(q_{2m},r) = max\left\{d(Jq_{2m},Kr), d(gq_{2m},Jq_{2m}), d(hr,Kr), \frac{d(Jq_{2m},hr)+d(gq_{2m},Kr)}{2s}\right\}.$$

For $m \to \infty$ in (33) and using lemma 22, we get

$$\begin{aligned}\psi(s^3 d(r,hr)) &= \psi(s^4 \frac{1}{s} d(r,hr)) \\ &\le \psi(d(r,hr)) - \phi(d(r,hr)) \\ &\le \psi(s^3 d(r,hr)) - \phi(d(r,hr))\end{aligned}$$

which implies that $r = hr$. Thus, $gr = hr = Jr = Kr = r$. Similarly, the result follows when g is continuous and (d_2) holds.

Let the common points of g, h, J and K be well ordered. Now, we prove the mappings have a unique common fixed point. Suppose $gu = hu = Ju = Ku = u$ and $gu_1 = hu_1 = Ju_1 = Ku_1 = u_1$ but $u \ne u_1$. Then, applying (13), we get

$$\begin{aligned}\psi(d(u,u_1)) &= \psi(d(gu,hu_1)) \le \psi(s^4 d(gu,hu_1)) \\ &\le \psi(N_s(u,u_1)) - \phi(N_s(u,u_1)),\end{aligned}$$

where

$$\begin{aligned}N_s(u,u_1) &= max\left\{d(Ju,Ku_1), d(gu,Ju), d(hu_1,Ku_1), \frac{d(Ju,hu_1)+d(gu,Ku_1)}{2s}\right\} \\ &= max\left\{d(u,u_1), 0, 0, \frac{d(u,u_1)+d(u,u_1)}{2s}\right\} \\ &= d(u,u_1).\end{aligned}$$

Thus,

$$\psi(d(u,u_1)) \le \psi(d(u,u_1)) - \phi(d(u,u_1)),$$

a contradiction. Hence, $u = u_1$. The converse is obvious. ■

Example 24. *Suppose $X = [0,1]$ be endowed with cone b-metric $d(q,r) = |q-r|$, where $s = 1$.*
Let g, h, J and K on X be self-mappings such that

$$g(q) = \begin{cases} 0, & \text{if } q \le \frac{1}{2} \\ \frac{1}{3}, & \text{if } q \in (\frac{1}{2}, 1]. \end{cases}$$

$$h(q) = 0, \forall\ q\ in\ X.$$

$$J(q) = \begin{cases} 0, & \text{if } q = 0 \\ \frac{2}{5}, & \text{if } q \in (0, \frac{1}{2}] \\ 1, & \text{if } q \in (\frac{1}{2}, 1]. \end{cases}$$

$$K(q) = \begin{cases} 0, & \text{if } q = 0 \\ \frac{1}{2}, & \text{if } q \in (0, \frac{1}{2}] \\ 1, & \text{if } q \in (\frac{1}{2}, 1]. \end{cases}$$

Thus, g, h are dominated and J, K are dominating maps for $g(X) \subseteq K(X)$ and $h(X) \subseteq J(X)$, that is,

	g is dominated	*h is dominated*
for each q in X	$gq \le q$	$hq \le q$
$q = 0$	$g(0) = 0$	$h(0) = 0$
$q \in (0, \frac{1}{2}]$	$gq = 0 < q$	$hq = 0 < q$
$q \in (\frac{1}{2}, 1]$	$gq = \frac{1}{3} < q$	$hq = 0 < q$

	J is dominating	*K is dominating*
for each q in X	$q \le Jq$	$q \le Kq$
$q = 0$	$0 = J(0)$	$0 = K(0)$
$q \in (0, \frac{1}{2}]$	$q \le \frac{3}{5} = J(q)$	$q \le \frac{1}{2} = K(q)$
$q \in (\frac{1}{2}, 1]$	$q \le 1 = J(q)$	$q \le 1 = K(q)$

Further, h is continuous, $\{g, J\}$ is weakly compatible and $\{h, K\}$ is compatible.
Define $\psi, \phi : [0,\infty) \to [0,\infty)$ as

$$\psi(b) = \frac{3}{2}(b)$$

and

$$\phi(b) = \frac{1}{2}(b) \forall\ b \in [0,\infty).$$

We now prove that g, h, J and K fulfil (13). The following cases arises:
(i) For $q \in [0, \frac{1}{2}]$ and $r \in [0,1]$, then

$$d(gq,hr) = 0$$

and (13) is fulfilled.
(ii) For $q \in (\frac{1}{2}, 1]$ and $r = 0$, then

$$\begin{aligned}\psi(s^4 d(gq,hr)) &= \psi(d(gq,hr)) = \psi(\frac{1}{3}) = \frac{1}{2} < 1 \\ &= d(Jq,Kr) \le M_1(q,r) \\ &= \psi(M_1(q,r)) - \phi(M_1(q,r)).\end{aligned}$$

(iii) For $q \in (\frac{1}{2}, 1]$ and $r \in (0, \frac{1}{2}]$,

$$\begin{aligned}\psi(s^4 d(gq,hr)) &= \psi(d(gq,hr)) = \psi(\frac{1}{3}) = \frac{1}{2} \le \frac{1}{2} \\ &\le d(Jq,Kr) \le M_1(q,r) \\ &= \psi(M_1(q,r)) - \phi(M_1(q,r)).\end{aligned}$$

(iv) For $q \in (\frac{1}{2}, 1]$ and $r \in (\frac{1}{2}, 1]$,

$$\begin{aligned}\psi(s^4 d(gq,hr)) &= \psi(d(gq,hr)) = \psi(\frac{1}{3}) = \frac{1}{2} < 1 \\ &= d(hr,Kr) \le M_1(q,r) \\ &= \psi(M_1(q,r)) - \phi(M_1(q,r)).\end{aligned}$$

Therefore, (13) is fulfilled $\forall\ q, r \in X$. Hence, all conditions of above theorem are fulfilled. Further, g, h, J and K have 0 as the unique common fixed point.

Corollary 25. *Suppose* $(X, d, \mathcal{A})$ *be a complete cone b-metric space over Banach algebra. Suppose* g, h *be dominated self-maps on* X. *Let* ψ *and* ϕ *be control functions such that for every two comparable elements* q, r *in* X,

$$\psi(s^4 d(gq, hr)) \leq \psi(N_s(q, r)) - \phi(N_s(q, r)), \tag{34}$$

is satisfied where

$$N_s(q, r) = \max\left\{d(q, r), d(gq, q), d(hr, r), \frac{d(q, hr) + d(gq, r)}{2s}\right\}. \tag{35}$$

If for every decreasing sequence $\{q_m\}$ *and a sequence* $\{r_m\}$ *with* $r_m \leq q_m\ \forall\ m$ *and* $r_m \to u$ *we get* $u \leq q_m$, *then* g, h *have a common fixed point. Further, the set of common fixed points of* g, h *is well ordered iff* g, h *have unique common fixed point.*

Proof: The result holds from the above theorem by taking K and J as the identity maps on X. ■

Corollary 26. *Suppose* $(X, d, \mathcal{A})$ *be a complete cone b-metric space over Banach algebra. Suppose* g, h *be dominated self-maps on* X. *Let* $\phi : [0, \infty) \to [0, \infty)$ *be a lower semi-continuous function with* $\phi(b) = 0 \Leftrightarrow b = 0$ *and for each two comparable elements* q, r *in* X,

$$s^4 d(gq, hr) \leq N_s(q, r) - N_s(q, r), \tag{36}$$

is satisfied where

$$N_s(q, r) = \max\left\{d(q, r), d(gq, q), d(hr, r), \frac{d(q, hr) + d(gq, r)}{2s}\right\}. \tag{37}$$

If for every decreasing sequence $\{q_m\}$ *and a sequence* $\{r_m\}$ *with* $r_m \leq q_m\ \forall\ m$ *and* $r_m \to u$ *we get* $u \leq q_m$, *then* g, h *have a common fixed point. Further, the set of common fixed points of* g, h *is well ordered iff* g, h *have unique common fixed point.*

Proof: The results holds from the above theorem by taking K and J as identity maps on X where $\psi(b) = b$ for $b \in [0, \infty)$. ■

IV. Conclusion

In this paper, we have successfully established the existence of common fixed points for generalized weak contractive mappings within the framework of cone b-metric spaces over Banach algebras. This study extends and enriches the current body of knowledge in fixed point theory by broadening the scope of contractive mappings and exploring their properties in a more generalized setting. Our results provide a deeper understanding of how cone b-metric spaces and Banach algebras can be leveraged to achieve significant fixed point theorems, which are pivotal in various mathematical and applied disciplines. Through rigorous theoretical analysis and illustrative examples, we have demonstrated the practical implications and broad applicability of our findings, thereby contributing valuable insights to the field. Building on the foundation laid by this research, several avenues for future exploration are evident. Firstly, further studies can investigate the existence of common fixed points for other classes of contractive mappings in cone b-metric spaces over Banach algebras. Additionally, exploring the dynamic behavior and stability of iterative processes in these spaces could provide deeper insights into their practical applications. Extending the current results to multivalued mappings and investigating their fixed points presents another promising direction. Moreover, applying the established theoretical results to real-world problems in differential equations, optimization, and dynamical systems could enhance the practical utility of this research. Finally, the development of computational methods and algorithms to approximate fixed points in these generalized spaces could bridge the gap between theoretical advancements and their practical implementation.

References

[1] Stefan Banach, "Sur les operations dans les ensembles abstraits etleur application aux equations integrales", *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.

[2] Lj. B. Ciric, "A generalization of Banach's contraction principle", *Proceedings of the American Mathematical Society*, vol. 45, no. 2, pp. 267–273, 1974.

[3] B. E. Rhoades, "A comparison of various definitions of contractive mappings", *Transactions of the American Mathematical Society*, vol. 226, pp. 257–290, 1977.

[4] Jungck, G., "Compatible mappings and common fixed points", *Internat. J. Math. & Math. Sci.*, vol. 9, no. 4, pp. 771–779, 1986.

[5] Alber, Y.I., Guerre-Delabriere, S., "Principle of Weakly Contractive Maps in Hilbert Spaces", *In: Gohberg, I., Lyubich, Y. (eds) New Results in Operator Theory and Its Applications. Operator Theory: Advances and Applications 1977*, vol 98, pp. 7-22, 1997, https://doi.org/10.1007/978-3-0348-8910-0_2.

[6] Ran, Andre C.M. and Reurings, Martine C.B., "A fixed point theorem in partially ordered sets and some application to matrix equations", *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[7] Huang Long-Guang and Zhang Xian, "Cone metric spaces and fixed point theorems of contractive mapping", *J. Math. Anal. Appl.*, vol. 332, no. 2, pp. 1468-1476, 2007.

[8] M. Abbas, G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces", *J. Math. Anal. Appl.*, vol. 341, no. 1, pp. 416–420, 2008.

[9] P. N. Dutta and Binayak S. Choudhury, "A Generalisation of Contraction Principle in Metric Spaces", *Fixed Point Theory and Applications*, 2008, doi:10.1155/2008/406368.

[10] Qingnian Zhang, Yisheng Song, "Fixed point theory for generalized ϕ-weak contraction", *Appl. Math. Lett.*, vol. 22, no. 1, pp. 75–78, 2009.

[11] Dragan Doric, "Common fixed point for generalized (ψ, ϕ)-weak contractions", *Appl. Math. Lett.*, vol. 22, no. 12, pp. 1896–1900, 2009.

[12] Mujahid Abbas and B.E. Rhoades and Talat Nazir, "Common fixed points for four maps in cone metric spaces", *Appl. Math. Comput.*, vol. 216, no. 1, pp. 80–86, 2010.

[13] Stojan Radenovic and Zoran Kadelburg, "Generalized weak contractions in partially ordered metric spaces", *Computers and Mathematics with Applications.*, vol. 60, no. 6, pp. 1776–1783, 2010.

[14] Mohamed A. Khamsi, "Remarks on cone metric spaces and fixed point theorems of contractive mappings", *Fixed Point Theory and Applications*, 2010, doi:10.1155/2010/315398.

[15] N. Hussain and M.H. Shah, "KKM mappings in cone b-metric spaces", *Computers and Mathematics with Applications*, vol. 62, no. 4, pp. 1677-1684, 2011.

[16] Slobodanka Jankovic, Zoran Kadelburg and Stojan Radenovic, "On cone metric spaces : A survey", *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, no. 7, pp. 2591–2601, 2011.

[17] Heman Kumar Nashine and Bessem Samet, "Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces", *Nonlinear Analysis: Theory, Methods and Applications.*, vol. 74, no. 6, pp. 2201–2209, 2011.

[18] Wasfi Shatanawi, "On w-compatible mappings and common coupled coincidence point in cone metric spaces", *Applied Mathematics Letters*, vol. 25, no. 6, pp. 925–931, 2012.

[19] Wasfi Shatanawi, "Some coincidence point results in cone metric spaces", *Mathematical and Computer Modelling*, vol. 55, no. 7-8, pp. 2023–2028, 2012.

[20] Hao Liu and Shaoyuan Xu, "Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings", *Fixed Point Theory and Applications*, 2013, http://www.fixedpointtheoryandapplications.com/content/2013/1/320.

[21] Zoran Kadelburg and Stojan Radenovic, "On generalized metric space: a survey", *TWMS J. Pure Appl. Math.*, vol. 5, no. 1, pp. 3–13, 2014.

[22] Shaoyuan Xu and Stojan Radenovic, "Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality", *Fixed Point Theory and Applications*, 2014, http://www.fixedpointtheoryandapplications.com/content/2014/1/102.

[23] Huaping Huang and Stojan Radenovic, "Common fixed point theorems of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras and applications", *J. Nonlinear Sci. Appl.*, Vol. 8, no. 5, pp. 787–799, 2015.

[24] Huaping Huang, Stojan Radenovic, "Some fixed point results of generalized Lipschitz mappings on cone b-metric spaces over Banach algebras", *Journal of Computational Analysis and Applications*, vol. 20, no. 3, pp. 566–583, 2016.

[25] Jerolina Fernandez, Neeraj Malviya and Satish Shukla, "Cone b-metric-like spaces over Banach algebra and fixed point theorems with application", *Asian Journal of Mathematics and Computer Research*, vol. 18, no. 2, pp. 49-66, 2017.

[26] Anil Kumar Dubey, Mithilesh Deo Pandey, Ravi Prakash Dubey, "Common fixed point theorems on cone metric spaces with c-Distance", *Advances in Modelling and Analysis A*, vol. 55, no. 4, pp. 196-200, 2018.

[27] Jerolina Fernandez, Neeraj Malviya, Diana Dolicanin Dekicc, Dzenis Pucicd, "The p_b-Cone Metric Spaces Over Banach Algebra With Applications", *Filomat*, vol. 34, no. 3, pp. 983–998, 2020.

[28] Preeti Mehta, Badrial Bhati, "Some Common Fixed point Results in Cone Metric Space", *International Journal of Creative Research Thoughts*, vol. 9, no. 6, pp. 25-31, 2021.

[29] Kushal Roy, Sayantan Panja and Mantu Saha, "A Generalized Fixed Point Theorem In An Extended Cone b-Metric Space Over Banach Algebra With Its Application To Dynamical Programming", *Applied Mathematics E-Notes*, vol. 21, pp. 209-219, 2021.

[30] Jerolina Fernandez, Neeraj Malviya, Ana Savic, Marija Paunovic and Zoran D. Mitrovic, "The Extended Cone b-Metric-like Spaces over Banach Algebra and Some Applications", Σ *Mathematics*, vol. 10, no. 1, 2022, DOI:10.3390/math10010149.

[31] Jerolina Fernandez, Neeraj Malviya and Santosh Kumar, "Extended N_b-cone metric spaces over Banach algebra with an application", *Topological Algebra and its Applications*, vol. 10, no. 1, pp. 94–102, 2022.

[32] Sanjay Kumar, Akash Singhal, Manish Sharma, "Common Fixed Point Theorems of Integral Type Contraction on Cone Metric Spaces and Applications", *Communications in Mathematics and Applications*, vol. 14, no. 2, pp. 503–510, 2023.

[33] Iqbal M. Batiha, Leila Ben Aoua, Taki-Eddine Oussaeif, Adel Ouannas, Shamseddin Alshorman, Iqbal H. Jebril, and Shaher Momani, "Common Fixed Point Theorem in Non-Archimedean Menger PM-Spaces Using CLR Property with Application to Functional Equations", *IAENG International Journal of Applied Mathematics*, vol. 53, no. 1, pp. 360-368, 2023.

[34] Hemavathy K., and Thalapathiraj S., "Common Fixed Point Theorems for 2-Metric Space using Various E.A Properties", *IAENG International Journal of Applied Mathematics*, vol. 53, no. 4, pp. 1308-1314, 2023.

[35] Satyaj Tiwari, Lokesh Kumar Dewangan, "Fixed Point and Common Fixed Point Theorems for Single Valued and Multivalued Mappings in Partial b-Metric Spaces", *Indian Journal of Science and Technology*, vol. 17, no. 1, pp. 59-64, 2024.

[36] Thokchom Chhatrajit Singh, "Some Theorems on Fixed Points in N-Cone Metric Spaces with Certain Contractive Conditions", *Engineering Letters*, vol. 32, no. 9, pp. 1833-1839, 2024.