Parameter Estimation for Stochastic Fractional Differential Equations

Piaopiao Zhou, Yue Hou and Chao Wei

Abstract—**This study is concerned with the maximum likelihood estimation (MLE) for stochastic differential equations (SDEs) driven by fractional Brownian motion (fBm). Firstly, we give the likelihood function and obtain the maximum likelihood estimator (MLEr). Then, we prove the strong consistency of the estimator. Finally, we derive the asymptotic normality of the estimation error.**

Index Terms—**SDEs; MLE; consistency; asymptotic normality; fBm**

I. Introduction

To estimate the parameters is very important for modeling the stochastic models and many scholars devoted to study this problem. For example, Ding ([2]) explored the properties of the least squares methods and the multi-innovation least squares methods. Farghali et al. ([5]) proposed generalized two-parameter estimators and an algorithm for the estimation of shrinkage parameters to combat multicollinearity in the multinomial logit regression model. Liu and Liu ([11]) used the principle of least squares between the uncertainty distribution and the empirical distribution of the observed data and estimated the unknown parameters in uncertain differential equation. Liu ([12]) proposed the moment estimation of uncertain regression model and used uncertain hypothesis to test the estimated uncertain regression model. Xu ([20]) offered a dynamical length stochastic gradient estimation technique to obtain more accurate parameter estimates by using dynamical length measured data from the step response. Yang et al. ([23]) focused on iterative parameter estimation methods for a nonlinear closed-loop system with an equation-error model for the open-loop part. Basit et al. ([1]) introduced a neural-network-based unified estimation framework to estimate the unknown nonlinear function in conjunction with the system state and unknown parameters. Lenzi et al. ([9]) used deep learning models to estimate parameters in statistical models when standard likelihood estimation methods are computationally infeasible. Wei ([18]) studied the parameter estimation for Ornstein-Uhlenbeck process driven by Liu process. Xu ([21]) provided a novel parameter estimation method for the systems with colored noises by using the filtering identification idea. Yang and Liu ([24]) used a method of moments to estimate unknown parameters in uncertain partial differential equations. Guo et al. ([7]) proposed three distributed-like algorithms for multivariate Gaussian mixture models. Wei et al. ([19]) used MLE to study the partially observed SDEs and gave the asymptotic properties of the estimators.

In view of recent empirical findings of long memory in finance, it becomes necessary to extend the diffusion models to processes having long-range dependence. One way is to use stochastic differential equations with fBm driving term, with Hurst index greater than $\frac{1}{2}$, the solution of which is called fractional diffusion. The fBm being not a Markov process and not a semimartingale, the classical Itô calculus is not applicable to develop its theory. In recent years, many scholars devoted themselves to the study of this problem. For instance, Dufitinema et al. ([4]) introduced the long-range dependent completely correlated mixed fBm. Feng et al. ([6]) proposed a general parameter estimation neural network to jointly identify the system parameters and the noise parameters of a stochastic differential equation driven by fBm from a short sample trajectory. Omari ([15]) dealed with the parameter estimation problem for an nth-order mixed fBm. Tuan et al. ([17]) investigated four problems for stochastic fractional pseudo-parabolic containing bounded and unbounded delays. Panunzi et al. ([16]) estimated the order of the fractional stochastic process on night-time continuously measured glycemia data. Han and Zhang ([8]) investigated the nonparametric Nadaraya-Watson estimator for the drift function of stochastic differential equations driven by fBm. Liu ([10]) considered the time discretization of fractional stochastic wave equation with Gaussian noise and derived the error estimates of the time discretization. Djerfi et al. ([3]) defined the MLEr of the drift parameter and provided a sufficient condition for the James-Stein type estimators. Yamagishi and Yoshida ([22]) constructed a theory of exponents based on a graphical representation of the structure of the functionals.

Although the problem of parameter estimation for SDEs driven by fBm has been developed in recent years, the strong consistency and asymptotic normality of estimators have been considered in few literature. Motivated by the above considerations, in this paper, we investigate the maximum likelihood estimation for SDEs driven by fBm. We give the likelihood function and obtain the expressions of MLEr and estimation error. We prove the strong consistency of MLEr and derive the asymptotic normality of estimation error. The structure of this paper is organized as follows. Section 2 gives the likelihood function and obtain the MLEr. Section 3 provide the strong consistency and asymptotic normality of estimator. The conclusion is given in Section 4.

II. Problem Formulation and Preliminaries

Let $(\Omega, F, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $\{F_t\}_{t\geq 0}$.

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The SDE driven by fractional Brownian Motion considered in this paper is described as follows:

$$\begin{cases} dX_t = \alpha a(t, X_t)dt + b(t, X_t)dW_t^H, \\ X_0 = \eta, \end{cases} \tag{1}$$

where α is an unknown parameter, η is a finite random variable, $W^H, H \in (\frac{1}{2}, 1)$ is the fractional Brownian motion.

We assume that the function $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are known and satisfy

Assumption 1: $|f(t,x)| + |g(t,x)| \le K(1 + |x|)$ for all $t \in [0,T]$,

Assumption 2: $|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le K(|x - y|)$ for all $t \in [0,T]$,
for some constant $K > 0$.

Remark 1: Under the conditions 1 and 2, it is known that there exists a unique solution of the SDE (1) (see([13])).

Define

$$M_t^* = \int_0^t k_*^t(s) dW_s^H, \tag{2}$$

where $k_*^t(s) = \tau_H^{-1}(s(t-s))^{\frac{1}{2}-H}$, $\tau_H = 2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})$.

Hence, M^* is a Gaussian martingale with variance function

$$\langle M^* \rangle_t = \frac{t^{2-2H}}{\lambda_H}, \tag{3}$$

where $\lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}$.

Then M_t^* can be rewritten as

$$M_t^* = \int_0^t f(s) dW_s, \tag{4}$$

where $f(s) = \sqrt{\frac{2(1-H)}{\lambda_H}} s^{\frac{1}{2}-H}$ and W_s is a standard Brownian motion.

Thus we have

$$\begin{aligned} Y_t &= \int_0^t \kappa(t,s) dX_s \\ &= \alpha \int_0^t \kappa(t,s) a(s,X_s) ds + \int_0^t \kappa(t,s) b(s,X_s) dW_s^H \\ &= \alpha \int_0^t \kappa(t,s) a(s,X_s) ds + \int_0^t f(s) dW_s. \end{aligned}$$

Consider the probability $\widetilde{P}$

$$\begin{aligned} \frac{d\widetilde{P}}{dP} &= \exp(-\alpha \int_0^T \frac{\kappa(T,t)a(t,X_t)}{f(t)} dW_t \\ &\quad - \frac{\alpha^2}{2} \int_0^T (\frac{\kappa(T,t)a(t,X_t)}{f(t)})^2 dt). \end{aligned}$$

According to the Girsanov's theorem, the following process is a Brownian motion

$$\widetilde{W}_t = W_t + \int_0^t \alpha \frac{\kappa(T,s)a(s,X_s)}{f(s)} ds. \tag{5}$$

Hence, we have

$$Y_T = \int_0^T f(t) d\widetilde{W}_t, \tag{6}$$

and

$$Y_T \sim N(0, \frac{T^{2-2H}}{\lambda_H}). \tag{7}$$

Therefore, it can be checked that the likelihood function is

$$\begin{aligned} \ell_T(\alpha) &= \alpha \int_0^T \frac{\kappa(T,s)a(s,X_s)}{f(s)} dW_s \\ &\quad + \frac{\alpha^2}{2} \int_0^T (\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2 ds \\ &= \alpha \int_0^T \frac{\kappa(T,s)a(s,X_s)}{(f(s))^2} dY_s \\ &\quad - \frac{\alpha^2}{2} \int_0^T (\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2 ds. \end{aligned}$$

Then, we can obtain the following maximum likelihood estimator (MLEr)

$$\widehat{\alpha}_T = \frac{\int_0^T \frac{\kappa(T,s)a(s,X_s)}{(f(s))^2} dY_s}{\int_0^T (\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2 ds}. \tag{8}$$

III. Main Results and Proofs

Lemma 1: ([14]) Let $M : [0,T] \times [0,T] \times \Omega \to \mathbb{R} \times \mathbb{R}$ be a $\mathcal{B}([0,T]) \otimes \mathcal{B}([0,T]) \otimes \mathcal{F}$ measurable process satisfying the following conditions:

$$(1) M(t,s) = 0 if s > t,$$

$$(2) M(t,s) is \mathcal{F}_s - adapted,$$

(3) There exists a positive random variable ε and $\beta \in (0,2]$ such that for all t, $r \in [0,T]$

$$\int_0^{r \wedge t} |M(t,s) - M(r,s)|^2 \le \varepsilon |t-r|^\beta.$$

Then for any θ, $0 < \theta \le 1 \wedge \beta$, there exist positive constants K_1 (depending only on θ),K_2,K_3, such that

$$\begin{aligned} &\mathbb{P}(\sup_{0 \le t \le T} |\int_0^t M(t,s) dW_s| > \eta, \\ &\|M\|_\infty \le K_M, \varepsilon \le C_M) \\ &\le \exp(-\frac{\eta^2}{TK_M^2 + T^\beta C_M} K_3), \end{aligned}$$

for any $\eta > 0$,$C_M \ge 0$ and $K_M \ge 0$ such that $\eta(T^{\beta-\theta}C_M + T^{1-\theta}K_M^2)^{-\frac{1}{2}} \ge K_1 \vee K_2((1+T)T^{\frac{\theta}{2}})$.

Theorem 1: Under conditions 1–2, as $T \to \infty$, the MLEr $\widehat{\alpha}_T$ is strong consistent, that is to say

$$\widehat{\alpha}_T \overset{a.s.}{\to} \alpha.$$

Proof: According to the expression of $\widehat{\alpha}_T$ and Equation (1), we have

$$\begin{aligned} \widehat{\alpha}_T &= \frac{\int_0^T \frac{\kappa(T,s)a(s,X_s)}{(f(s))^2} dY_s}{\int_0^T (\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2 ds} \\ &= \alpha + \frac{\int_0^T \frac{\kappa(T,s)a(s,X_s)}{f(s)} dW_s}{\int_0^T (\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2 ds}. \end{aligned}$$

Then, from Lemma 1, we have

$$\begin{aligned}\mathbb{P}(|\widehat{\alpha}_T-\alpha|>\eta) &= P(|\frac{\int_0^T \frac{\kappa(T,s)a(s,X_s)}{f(s)}dW_s}{\int_0^T(\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2ds}|>\eta)\\ &\le \mathbb{P}(|\frac{\int_0^T \frac{\kappa(T,s)a(s,X_s)}{f(s)}dW_s}{|}>\eta_1,\\ &|\int_0^T(\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2ds|<\eta_2)\\ &\le e^{-\frac{K\eta_1^2}{TK_M^2+T^\beta C_M}},\end{aligned}$$

where $\eta=\frac{\eta_1}{\eta_2}$, $\|\frac{\kappa(T,s)a(s,X_s)}{f(s)}\|_\infty\le K_M$, $\beta\in(0,2]$.

Thus, by using Borel-Cantelli lemma, when $T\to\infty$, we obtain that

$$\mathbb{P}(|\widehat{\alpha}_T-\alpha|>\eta)\to 0. \tag{9}$$

Then, we have

$$\widehat{\alpha}_T\stackrel{a.s.}{\to}\alpha. \tag{10}$$

The proof is complete. ■

Theorem 2: Under conditions 1–2, as $T\to\infty$,

$$\sqrt{Q_T}(\widehat{\alpha}_T-\alpha)\stackrel{d}{\to}N(0,1),$$

where

$$Q_T=\int_0^T(\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2ds.$$

Proof: Since

$$\sqrt{Q_T}(\widehat{\alpha}_T-\alpha)=\frac{\int_0^T\frac{\kappa(T,s)a(s,X_s)}{f(s)}dW_s}{\sqrt{\int_0^T(\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2ds}}. \tag{11}$$

It is known that the continuous semimartingale $\int_0^t h(t,s,X_s)dW_s$ has the decomposition as follows:

$$\begin{aligned}\int_0^t h(t,s,X_s)dW_s &= \int_0^t h(s,s,X_s)dW_s\\ &+\int_0^t(\int_0^r\frac{\partial h(r,u,X_u)}{\partial r}dW_u)dr.\end{aligned}$$

By using Skorohod embedding for continuous semimartingale, we have

$$\int_0^T h(T,s,X_s)dW_s=W^*(\int_0^T h^2(T,s,X_s)ds), \tag{12}$$

where W^* is some Brownian motion.

Define

$$h(T,s,x)=\frac{\kappa(T,s)a(s,X_s)}{f(s)}. \tag{13}$$

Then we have

$$\frac{\int_0^T\frac{\kappa(T,s)a(s,X_s)}{f(s)}dW_s}{\sqrt{\int_0^T(\frac{\kappa(T,s)a(s,X_s)}{f(s)})^2ds}}=\frac{W^*(Q_T)}{\sqrt{Q_T}}. \tag{14}$$

Since

$$\frac{W^*(Q_T)}{\sqrt{Q_T}}\stackrel{d}{\to}N(0,1), \tag{15}$$

when $T\to\infty$, we have

$$\sqrt{Q_T}(\widehat{\alpha}_T-\alpha)\stackrel{d}{\to}N(0,1). \tag{16}$$

The proof is complete. ■

IV. Example

Consider the Ornstein-Uhlenbeck process described by the following stochastic differential equation

$$dX_t=\alpha X_tdt+dW_t^H,\quad t\ge 0,\quad X_0=0, \tag{17}$$

where α is an unknown parameter, $W^H, H\in(\frac{1}{2},1)$ is the fractional Brownian motion.

Define

$$M_t^*=\int_0^t k_*^t(s)dW_s^H, \tag{18}$$

where $k_*^t(s)=\tau_H^{-1}(s(t-s))^{\frac{1}{2}-H}$, $\tau_H=2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})$.

Hence, M^* is a Gaussian martingale with variance function

$$\langle M^*\rangle_t=\frac{t^{2-2H}}{\lambda_H}, \tag{19}$$

where $\lambda_H=\frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}$.

Then M_t^* can be rewritten as

$$M_t^*=\int_0^t f(s)dW_s, \tag{20}$$

where $f(s)=\sqrt{\frac{2(1-H)}{\lambda_H}}s^{\frac{1}{2}-H}$ and W_s is a standard Brownian motion.

Define

$$Q_t=\frac{d\int_0^t k_*^t(s)X_sds}{d\nu_t}, \tag{21}$$

where

$$\nu_t=\lambda_H^{-1}t^{2-2H}. \tag{22}$$

Then, we can obtain that

$$Q_t=\frac{\lambda_H}{2(2-2H)}\{t^{2H-1}Z_t+\int_0^t r^{2H-1}dZ_s\}. \tag{23}$$

Define the process $Z=(Z_t,t\in[0,T])$ as follows:

$$Z_t=\int_0^t\tau_H^{-1}(s(t-s))^{\frac{1}{2}-H}dX_s. \tag{24}$$

Thus, we can get that Z is the fundamental semimartingale associated with the process X.

According to the Girsanov theorem, the Radon-Nikodym derivative of P_α^t with respect to P_0^t is

$$\ell_T(\alpha)=\frac{dP_\alpha^t}{dP_0^t}=\exp\{\alpha\int_0^T Q_tdZ_t-\frac{\alpha^2}{2}\int_0^T Q_t^2d\nu_t\}. \tag{25}$$

Then, the derivative of the likelihood function is

$$\frac{d\ell_T(\alpha)}{d\alpha}=\int_0^T Q_tdZ_t-\alpha\int_0^T Q_t^2d\nu_t. \tag{26}$$

Hence, the maximum likelihood estimator is

$$\widehat{\alpha}_T=\frac{\int_0^T Q_tdZ_t}{\int_0^T Q_t^2d\nu_t}. \tag{27}$$

It is easy to check that the coefficients of the Ornstein-Uhlenbeck process satisfy the assumptions 1-2. Therefore, the maximum likelihood estimator is strong consistent.

Remark 2: When the process is observed discretely, we consider the following contrast function

$$\rho_n(\alpha)=\sum_{i=1}^n\frac{|X_{t_i}-X_{t_{i-1}}-\alpha a(t_{i-1},X_{t_{i-1}})\Delta t_{i-1}|^2}{a^2(t_{i-1},X_{t_{i-1}})\Delta t_{i-1}},$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

Then, it is easy to obtain that

$$\widehat{\alpha}_n = \frac{\sum_{i=1}^{n} \frac{(X_{t_i}-X_{t_{i-1}})a(t_{i-1},X_{t_{i-1}})}{b^2(t_{i-1},X_{t_{i-1}})}}{\sum_{i=1}^{n} \frac{a^2(t_{i-1},X_{t_{i-1}})}{nb^2(t_{i-1},X_{t_{i-1}})}}.$$

We use the discrete sample $(X_{t_i})_{i=0,1,\ldots,n}$ to compute the estimator $\widehat{\alpha}_n$. In Table 1, $x_0 = 0.5$, the sample size from 100 to 500. In Table 2, $x_0 = 0.1$, the sample size from 1000 to 5000. In Table 3, $x_0 = 0.01$, the sample size from 10000 to 50000. These tables list the value of estimator "$\widehat{\alpha}_n$" and the absolute errors (AE)"$|\alpha_0 - \widehat{\alpha}_n|$".

These tables provide that when n is large enough, the estimator is very close to the true parameter value.

TABLE I
ESTIMATOR SIMULATION RESULTS OF α_0

True	Aver		AE
α_0	Size n	$\widehat{\alpha}_{n,\varepsilon}$	$\lvert\alpha_0 - \widehat{\alpha}_{n,\varepsilon}\rvert$
1	100	1.1873	0.1873
	300	1.1392	0.1392
	500	1.0915	0.0915
2	100	2.1736	0.1736
	300	2.1281	0.1281
	500	2.0843	0.0843

TABLE II
ESTIMATOR SIMULATION RESULTS OF α_0

True	Aver		AE
α_0	Size n	$\widehat{\alpha}_{n,\varepsilon}$	$\lvert\alpha_0 - \widehat{\alpha}_{n,\varepsilon}\rvert$
1	1000	1.1125	0.1125
	3000	1.0276	0.0276
	5000	1.0008	0.0008
2	1000	2.1209	0.1209
	3000	2.0315	0.0315
	5000	2.0007	0.0007

TABLE III
ESTIMATOR SIMULATION RESULTS OF γ_0

True	Aver		AE
γ_0	Size n	$\widehat{\gamma}_{n,\varepsilon}$	$\lvert\gamma_0 - \widehat{\gamma}_{n,\varepsilon}\rvert$
1	10000	0.9246	0.0754
	30000	1.0283	0.0283
	50000	1.0002	0.0002
2	10000	2.0512	0.0512
	30000	2.0194	0.0194
	50000	2.0003	0.0003

V. CONCLUSIONS

In this article, we have investigated the problem of MLE for SDEs driven by fBm. We have given the likelihood function by using the Girsanov's theorem. Then, we have obtained the expression of MLEr. Furthermore, we have derived the strong consistency and asymptotic normality of the estimator by utilizing exponential inequality for s-tochastic integrals and Skorohod embedding for continuous semimartingale. We will consider the estimation for partially observed SDEs driven by fBm in future works.

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